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WHITE NOISE REPRESENTATION OF GAUSSIAN RANDOM FIELDS

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Abstract. We obtain a representation theorem for Banach space valued Gaussian random variables as integrals against a white noise. As a corollary we obtain necessary and sufficient conditions for the existence of a white noise representation for a Gaussian random field indexed by a measure space. We then show how existing theory for integration with respect to Gaussian processes indexed by $[0, 1]$ can be extended to Gaussian random fields indexed by measure spaces.

1. Introduction

Much of literature regarding the representation of Gaussian fields as integrals against white noise has focused on processes indexed by $\mathbb{R}$, in particular canonical representations (most recently see [8] and references therein) and Volterra processes (e.g. [1, 3]). An example of the use of such integral representations is the construction of a stochastic calculus for Gaussian processes admitting a white noise representation with a Volterra kernel (e.g. [1, 10]).

In this paper we study white noise representations for Gaussian random variables in Banach spaces, focusing in particular on Gaussian random fields indexed by a measure space. We show that the existence of a representation as an integral against a white noise on a Hilbert space H is equivalent to the existence of a version of the field whose sample paths lie almost surely in H. For example, a consequence of our results is that a centered Gaussian process Y_t indexed by $[0, 1]$ admits a representation

$$Y_t \stackrel{d}{=} \int_0^1 h(t, z) dW(z)$$

for some $h \in L^2([0, 1] \times [0, 1], d\nu \times d\nu)$, ν a measure on $[0, 1]$ and W the white noise on $L^2([0, 1], d\nu)$ if and only if there is a version of Y_t whose sample paths belong almost surely to $L^2([0, 1], d\nu)$.

The stochastic integral for Volterra processes developed in [10] depends on the existence of a white noise integral representation for the integrator. If there exists an integral representation for a given Gaussian field then the method in [10] can be extended to define a stochastic integral with respect to this field. We describe this

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extension for Gaussian random fields indexed by a measure space whose sample paths are almost surely square integrable.

Section 2 contains preliminaries we will need from Malliavin Calculus and the theory of Gaussian measures over Banach spaces. In section 3, Theorem 3.1 gives our abstract representation and Corollary 3.6 specializes to Gaussian random fields indexed by a Radon measure space. Section 4 contains the extension of results in [10].

2. Preliminaries

2.1. Malliavin calculus. We collect here only those parts of the theory that we will explicitly use, see [14].

Definition 2.1. Suppose we have a Hilbert space H. Then there exists a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a map $W : H \to L^2(\Omega, \mathbb{P})$ satisfying the following:

(1) $W(h)$ is a centered Gaussian random variable with $\mathbb{E}[W(h)^2] = \|h\|_H$
(2) $\mathbb{E}[W(h_1)W(h_2)] = \langle h_1, h_2\rangle_H$

This process is unique up to distribution and is called the *Isonormal* or *White Noise Process* on H.

The classical example is $H = L^2[0, 1]$ and $W(h)$ is the Wiener-Ito integral of $h \in L^2$.

Let $\mathcal{S}$ denote the set of random variables of the form

$$F = f(W(h_1), ..., W(h_n))$$

for some $f \in C^\infty(\mathbb{R}^n)$ such that f and all its derivatives have at most polynomial growth at infinity. For $F \in \mathcal{S}$ we define the derivative as

$$DF = \sum_1^n \partial_j f(W(h_1), ..., W(h_n))h_j.$$

We denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm induced by the inner product

$$\langle F, G\rangle_D = \mathbb{E}[FG] + \mathbb{E}[\langle DF, DG\rangle_H].$$

We also define a directional derivative for $h \in H$ as

$$D_h F = \langle DF, h\rangle_H.$$

D is then a closed operator from $L^2(\Omega)$ to $L^2(\Omega, H)$ and $\mathrm{dom}(D) = \mathbb{D}^{1,2}$. Further, $\mathbb{D}^{1,2}$ is dense in $L^2(\Omega)$. Thus we can speak of the adjoint of D as an operator from $L^2(\Omega, H)$ to $L^2(\Omega)$. This operator is called the *divergence operator* and denoted by δ. Next, $\mathrm{dom}(\delta)$ is the set of all $u \in L^2(\Omega, H)$ such that there exists a constant c (depending on u) with

$$|\mathbb{E}[\langle DF, u\rangle_H]| \le c\|F\|$$

for all $F \in \mathbb{D}^{1,2}$. For $u \in \mathrm{dom}(\delta)$, $\delta(u)$ is characterized by

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_H]$$

for all $F \in \mathbb{D}^{1,2}$.

For examples and descriptions of the domain of δ see [14], section 1.3.1.

When we want to specify the isonormal process defining the divergence we write δ^W. We will also use the following notations interchangeably

$$\delta^W(u)\,, \int u\, dW.$$

2.2. Gaussian measures on Banach spaces. Here we collect the necessary facts regarding Gaussian measures on Banach spaces and related notions that we will use in what follows. For proofs and further details see e.g. [4, 12]. All Banach spaces are assumed real and separable throughout.

Definition 2.2. Let B be a Banach space. A probability measure μ on the Borel sigma field $\mathcal{B}$ of B is called *Gaussian* if for every $l \in B^*$ the random variable $l(x) : (B, \mathcal{B}, \mu) \to \mathbb{R}$ is Gaussian. The *mean* of μ is defined as

$$m(\mu) = \int_B x\, d\mu(x).$$

μ is called *centered* if $m(\mu) = 0$. The *(topological) support* of μ in B, denoted B_0, is defined as the smallest closed subspace of B with μ-measure equal to 1.

Suppose we have a probability space $(\Omega, \mathcal{F}, P)$ and a measurable map $X : \Omega \to B$, i.e. X is a B-valued random variable. Then we say μ is the *distribution* of X if $P(X^{-1}(A)) = \mu(A)$ for any Borel set $A \subset B$. Such an X always exists, for we can let X be the identity map on B as B is a probability space with measure μ.

The mean of a Guassian measure is always an element of B, and thus it suffices to consider only centered Gaussian measures as we can then acquire any Gaussian measure via a simple translation of a centered one. For the remainder of the paper all measures considered are centered.

Definition 2.3. The covariance of a Gaussian measure is the bilinear form $C_\mu : B^* \times B^* \to \mathbb{R}$ given by

$$C_\mu(k, l) = \mathbb{E}[k(X)l(X)] = \int_B k(x)l(x)d\mu(x).$$

Any Gaussian measure is completely determined by its covariance: if for two Gaussian measures μ, ν on B we have $C_\mu = C_\nu$ on $B^* \times B^*$ then $\mu = \nu$.

Remark 2.4. Every B valued random variable defined on a probability space $(\Omega, \mathbb{P})$ determines a probability measure on B, and vice versa. If we want to specify a random element x of B, then as a consequence of the canonical embedding of B into B^{**} it is enough to specify $x(l) \equiv l(x)$ for all $l \in B^*$. Thus the family of random variables $\{l(x)\}_{l \in B^*}$ completely determine μ. In other words, every Gaussian measure on B is the distribution of a Gaussian process indexed by B^* (in fact by the closed unit ball in B^*, [5]). We will make free use of these identifications without mention throughout.

If H is a Hilbert space then

$$C_\mu(f, g) = \mathbb{E}[\langle X, f\rangle\langle X, g\rangle] = \int_B \langle x, f\rangle\langle x, g\rangle d\mu(x)$$

defines a continuous, positive, symmetric bilinear form on $H \times H$ and thus determines a positive symmetric operator K_μ on H. K_μ is of trace class and is injective if and only if $\mu(H) = 1$. Conversely, any positive trace class operator on H uniquely determines a Guassian measure on H [6]. Whenever we consider a Gaussian measure μ over a Hilbert space H we can after restriction to a closed subspace assume $\mu(H) = 1$ and do so throughout.

We will denote by H_μ the Reproducing Kernel Hilbert Space (RKHS) associated to a Gaussian measure μ on B . There are various equivalent constructions of the RKHS. We follow [15] and refer the interested reader there for complete details.

For any fixed $l \in B^*$, $C_\mu(l, \cdot) \in B$ (this is a non-trivial result in the theory). Consider the linear span of these functions,

$$S = \operatorname{span}\{C_\mu(l, \cdot) \ : \ l \in B^*\}.$$

Define an inner product on S as follows: if $\phi(\cdot) = \sum_1^n a_i C_\mu(l_i, \cdot)$ and

$\psi(\cdot) = \sum_1^m b_j C_\mu(k_j, \cdot)$ then

$$\langle \phi, \psi \rangle_{H_\mu} \equiv \sum_1^n \sum_1^m a_i b_j C_\mu(l_i, k_j).$$

H_μ is defined to be the closure of S under the associated norm $\|\cdot\|_{H_\mu}$. This norm is stronger than $\|\cdot\|_B$, H_μ is a dense subset of B_0 and H_μ has the reproducing property with reproducing kernel $C_\mu(l, k)$:

$$\langle \phi(\cdot), C_\mu(l, \cdot) \rangle_{H_\mu} = \phi(l) \qquad \forall\, l \in B^*,\, \phi \in H_\mu.$$

Remark 2.5. Often one begins with a collection of centered random variables indexed by some set, $\{Y_t\}_{t \in T}$. For example suppose (T, ν) is a finite measure space. Then setting $K(s, t) = \mathbb{E}[Y_s Y_t]$ and supposing that application of Fubini-Tonelli is justified we have for $f, g \in L^2(T)$

$$\mathbb{E}[\langle Y, f \rangle \langle Y, g \rangle] = \int_T \int_T \mathbb{E}[Y_s, Y_t] f(s) g(t) d\nu d\nu = \langle K(s, t)(f), g \rangle$$

where we denote $\int_T K(s, t) f(s) d\nu(s)$ by $K(s, t)(f)$. If one verifies that this last operator is positive symmetric and trace class then the above collection $\{Y_t\}_{t \in T}$ determines a measure μ on $L^2(T)$ and the above construction goes through with $C_\mu(f, g) = \langle K(s, t)(f), g \rangle$ and the end result is the same with H_μ a space of functions over T.

Define H_X to be the closed linear span of $\{X(l)\}_{l \in B^*}$ in $L^2(\Omega, \mathbb{P})$ with inner product $\langle X(l), X(l') \rangle_{H_X} = C_\mu(l, l')$ (again for simplicity assume X is nondegenerate). From the reproducing property we can define a mapping R_X from H_μ to H_X given initially on S by

$$R_X(\sum_1^n c_k C_\mu(l_k, \cdot)) = \sum_1^k c_k X(l)$$

and extending to an isometry. This isometry defines the isonormal process on H_μ.

In the case that H is a Hilbert space and μ a Gaussian measure on H with covariance operator K it is known that $H_\mu = \sqrt{K}(H)$ with inner product $\langle \sqrt{K}(x), \sqrt{K}(y) \rangle_{H_\mu} = \langle x, y \rangle_H$.

It was shown in [11] that given a Banach space B there exists a Hilbert space H such that B is continuously embedded as a dense subset of H. Any Gaussian measure μ on B uniquely extends to a Gaussian measure μ_H on H. The converse question of whether a given Gaussian measure on H restricts to a Gaussian measure on B is far more delicate. There are some known conditions, e.g. [7]. The particular case when X is a metric space and $B = C(X)$ has been the subject of extensive research [13]. Let us note here however that either $\mu(B) = 0$ or $\mu(B) = 1$ (an extension of the classical zero-one law, see [4]).

From now on we will not distinguish between a measure μ on B and its unique extension to H when it is clear which space we are considering.

3. White Noise Representation

3.1. The general case. The setting is the following: B is a Banach space densely embedded in some Hilbert space H (possibly with $B = H$), where H is identified with its dual, $H = H^*$. (A Hilbert space equal to its dual in this way is called a Pivot Space, see [2]). The choice of pivot space largely determines the analysis done on B and any other subspaces of H by determining duality in the following way: If μ is a Gaussian measure on B, then we have

$$H_\mu \subset B \subset H = H^* \subset B^* \subset H_\mu^*,$$

where the Riesz Map $\mathcal{R}$ taking H_μ to H_μ^* factors via any unitary map $L : H_\mu \to H$ as

$$\mathcal{R} = L^* L, \;\; H_\mu \xrightarrow{L} H = H^* \xrightarrow{L^*} H_\mu^*,$$

a fact we will make use of below.

The classical definition of canonical representation has no immediate analogue for fields not indexed by $\mathbb{R}$, but the notion of strong representation does. Let $L : H_\mu \to H$ be unitary. Then $W_X(h) = R_X(L^*(h))$ defines an isonormal process on H and $\sigma(\{W_X(h)\}_{h \in H}) = \sigma(H_X) = \sigma(\{X(l)\}_{l \in B^*})$ where the last equality follows from the density of H in B^*.

We now state our representation theorem.

Theorem 3.1. *Let B be a Banach space and μ a Gaussian measure on B. Then μ is the distribution of a random variable in B given as a white noise integral of the form*

$$X(l) = \int h(l) dW. \tag{3.1}$$

for some $h : B^* \to H$ and white noise process W over a Hilbert space H, where $h|_H$ is a Hilbert-Schmidt operator on H. Moreover, the representation is strong in the following sense: $\sigma(\{W(h)\}_{h \in H}) = \sigma(\{X(l)\}_{l \in B^*})$.

Proof. Let $B \subset H = H^*$ as above. Let W_X be the isonormal process constructed above and $C_\mu(l, k)$ the covariance of μ. Let L be a unitary map from H_μ to H and define the function $k_L(l) : B^* \to H$ by

$$k_L(l) \equiv L(C_\mu(l, \cdot)).$$

Consider the Gaussian random variable determined by

$$Y(l) \equiv \int k_L(l)dW_X.$$

We have

$$\mathbb{E}[Y(l_1), Y(l_2)] = \langle k_L(l_1), k_L(l_2)\rangle_H = \langle C_\mu(l_1, \cdot), C_\mu(l_2, \cdot)\rangle_{H_\mu} = C_\mu(l_1, l_2)$$

so that μ is the distribution of $Y(l)$ and

$$X(l) \stackrel{d}{=} \int k_L(l)dW_X.$$

It is clear that k_L is linear and if $C_\mu(h_1, h_2) = \langle K(h_1), h_2\rangle_H$, $h_1, h_2 \in H$, then from above

$$k_L^* k_L = K.$$

Because K is trace class this implies that k_L is Hilbert-Schmidt on H.

From the preceding discussion we have $\sigma(\{W_X(h)\}_{h\in H}) = \sigma(\{X(l)\}_{l\in B^*})$. □

Remark 3.2. While the statement of the above theorem is more general than is needed for most applications, this generality serves to emphasize that having a "factorable" covariance and thus an integral representation as above are basic properties of all Banach space valued Gaussian random variables.

Remark 3.3. The kernel $h(l)$ is unique up to unitary equivalence on H, that is if $L' = UL$ for some unitary U on H, then

$$\int h_{L'}(l)dW \stackrel{d}{=} \int U\left(h_L(l)\right)dW \stackrel{d}{=} \int h_L(l)dW.$$

Remark 3.4. In the proof above,

$$\langle k_L(l_1), k_L(l_2)\rangle_H = C_\mu(l_1, l_2) \tag{3.2}$$

is essentially the "canonical factorization" of the covariance operator given in [16], although in a slightly different form.

Remark 3.5. Put somewhat loosely, what we have shown is that every Gaussian random variable in a Hilbert space H is the solution to the equation

$$L(X) = W$$

for some closed unbounded operator L on H with inverse given by a Hilbert-Schmidt operator on H.

3.2. Gaussian random fields. The proof of Theorem 3.1 has the following corollary for Gaussian random fields:

Corollary 3.6. *Let X be a Hausdorff space, ν a positive Radon measure on the Borel sets of X and $H = L^2(X, d\nu)$. If $\{B_x\}$ is a collection of centered Gaussian random variables indexed by X, then $\{B_x\}$ has a version with sample paths belonging almost surely H if and only if*

$$B_x \stackrel{d}{=} \int h(x, \cdot)dW \tag{3.3}$$

for some $h : X \to H$ *such that the operator* $K(f) \equiv \int_X h(x,z)f(z)d\nu(z)$ *is Hilbert-Schmidt. In this case (3.2) takes the form*

$$\mathbb{E}[B_x B_y] = \int_X h(x,z)h(y,z)d\nu(z).$$

In other words, the field B_x determines a Gaussian measure on $L^2(X, d\nu)$ if and only if B_x admits an integral representation (3.3).

3.3. Some consequences and examples. In principle, all properties of a field are determined by its integral kernel. Without making an exhaustive justification of this statement we give some examples:

In Corollary 3.6 above, being the kernel of a Hilbert-Schmidt operator, $h \in L^2(X \times X, d\nu \times d\nu)$. This means that we can approximate h by smooth kernels (supposing these are available). If we assume $h(x, \cdot)$ is continuous as a map from X to H, i.e.,

$$\lim_{x \to y} \|h(x,\cdot) - h(y,\cdot)\|_H = 0$$

for each $y \in X$, and let $h_n \in C^\infty(X)$, $h_n \overset{H}{\to} h$ it follows that $\|h_n(x,\cdot) - h(x,\cdot)\|_H \to 0$ pointwise, so that if

$$B_x^n = \int h_n(x,\cdot)dW$$

we have

$$\mathbb{E}[B_x^n B_y^n] \to \mathbb{E}[B_x B_y]$$

pointwise. This last condition is equivalent to

$$B^n \overset{d}{\to} B$$

and we can approximate in distribution any field over X with a continuous (as above) kernel by fields with smooth kernels.

The kernel of a field indexed by, say, a compact subset $D \subset \mathbb{R}^d$ describes its local structure [9]: The limit in distribution of

$$\lim_{\substack{r_n \to 0 \\ c_n \to 0}} \frac{Y(t + c_n x) - Y(t)}{r_n}$$

is

$$\lim_{\substack{r_n \to 0 \\ c_n \to 0}} \int \frac{h(t + c_n x) - h(t)}{r_n} dW$$

where h is the integral kernel of Y in $H = L^2(D, \nu)$ for some measure ν on D, and this last limit is determined by the limit in H of

$$\lim_{\substack{r_n \to 0 \\ c_n \to 0}} \frac{h(t + c_n x) - h(t)}{r_n}.$$

The representation theorem yields a simple proof of the known series expansion using the RKHS. The setting is the same as in Theorem 3.1.

Proposition 3.7. *Let $Y(l)$ be a centered Gaussian random variable in a Hilbert space H with integral kernel $h(l)$. Let $\{e_k\}_1^\infty$ be a basis for H. Then there exist i.i.d. standard normal random variables $\{\xi_k\}$ such that*

$$Y(l) = \sum_1^\infty \xi_k \Phi_k(l)$$

where $\Phi_k(l) = \langle h(l), e_k \rangle_H$ and the series converges in $L^2(\Omega)$ and a.s.

Proof. For each l

$$h(l) \stackrel{H}{=} \sum_1^\infty \Phi_k(l) e_k.$$

We have

$$Y(l) \stackrel{L^2}{=} \int \sum_1^\infty \Phi_k(l) e_k dW \stackrel{L^2}{=} \sum_1^\infty \Phi_k(l) \xi_k$$

where $\{\xi_k\} = \left\{ \int e_k dW \right\}$ are i.i.d. standard normal as $\int dW$ is unitary from H to $L^2(\Omega)$. As $\{\Phi_k(l)\} \in l^2(\mathbb{N})$ the series converges a.s. by the martingale convergence theorem. □

4. Stochastic Integration

Combined with Theorem 3.1 above, [10] furnishes a theory of stochastic integration for a large class of Gaussian random fields. In particular, as above, suppose $(X, d\nu)$ is a (positive) Radon measure space and B_x a centered Gaussian random field indexed by X. They by Corollary 3.6 if the sample paths of B_x belong to $L^2(X, d\nu)$ almost surely then we can define a stochastic integral with respect to B_x using [10] as follows:

Denote by μ the distribution of $\{B_x\}$ in $H = L^2(X, d\nu)$ and as above the RKHS of B_x by $H_\mu \subset H$. Let

$$B_x = \int h(x, \cdot) dW$$

and $L^*(f) = \int h(x, y) f(y) d\nu(y)$. Then $L^* : H \to H_\mu$ is an isometry and the map $v \mapsto R_B(L^*(v)) \equiv W(v) : H \to H_B$ (H_B is the closed linear span of $\{B_x\}$ as defined in sec. 2) defines an isonormal process on H. Denote this particular process by W in what follows.

First note that as $H_\mu = L^*(H)$ and L is unitary, it follows immediately that $\mathbb{D}^{1,2}_{H_\mu} = L^*(\mathbb{D}^{1,2}_H)$ where we use the notation in [14, 10] and the subscript indicates the underlying Hilbert space.

The following proof from [10] carries over directly: For a smooth variable $F(h) = f(B(L^*(h_1)), ..., B(L^*(h_n)))$ we have

$$\begin{aligned}
\mathbb{E}\langle D^B(F), u\rangle_{H_\mu} &= \mathbb{E}\langle \sum_1^n f'(B(L^*(h_1), ..., B(L^*(h_n))L^*(h_k), u\rangle_{H_\mu} \\
&= \mathbb{E}\langle \sum f'(B(L^*(h_1), ..., B(L^*(h_n))h_k, L(u)\rangle_H \\
&= \mathbb{E}\langle \sum f'(W(h_1), ..., W(h_n))h_k, L(u)\rangle_H \\
&= \mathbb{E}\langle D^W(F), L(u)\rangle_H
\end{aligned}$$

which establishes

$$\mathrm{dom}(\delta^B) = L^*(\mathrm{dom}(\delta^W))$$

and

$$\int L^*(u)dB = \int u\, dW \qquad \forall\, u \in \mathrm{dom}(\delta^W)$$

The series approximation in [10] also extends directly to this setting:

Theorem 4.1. *If $\{\Phi_k\}$ is a basis of H_μ then there exists i.i.d. standard normal $\{\xi_k\}$ such that:*

(1) *If $f \in H$ then*

$$\int L^*(f)dB = \sum_1^\infty \langle L^*(f), \Phi_k\rangle_{H_\mu} \xi_k \quad a.s.$$

(2) *If $u \in \mathbb{D}^{1,2}_{H_\mu}$ then*

$$\int u dB = \sum_1^\infty (\langle u, \Phi_k\rangle_{H_\mu} - \langle D^B_{\Phi_k} u, \Phi_k\rangle_{H_\mu}) \quad a.s.$$

Proof. The proof of (1) and (2) follows that in [10].

□

Remark 4.2. For our purposes the method of approximation via series expansions above seems most appropriate. However in [1] a Riemann sum approximation is given under certain regularity hypotheses on the integral kernel of the process, and this could be extended in various situations as well.

Remark 4.3. The availability of the kernel above suggests the method in [1] whereby conditions are imposed on the kernel in order to prove an Itô Formula as promising for extension to more general settings.

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