Communications on Stochastic Analysis
Vol. 7, No. 1 (2013) 125-152

Serials Publications
www.serialspublications.com

MOMENTS FOR THE PARABOLIC ANDERSON MODEL: ON A RESULT BY HU AND NUALART

DANIEL CONUS

ABSTRACT. We consider the parabolic Anderson model $\partial_t u = Lu + u\dot{W}$, where L is the generator of a Lévy process and $\dot{W}$ is a white noise in time, possibly correlated in space. We present an alternate proof and an extension to a result by Hu and Nualart ([17]) giving explicit expressions for moments of the solution. We do not consider a Feynman-Kac representation, but rather make a recursive use of Itô's formula. Moments of solutions play a crucial role in understanding physical properties of solutions, such as intermittency.

1. Introduction

We consider the spatially continuous form of the parabolic Anderson model, namely the stochastic partial differential equation given by

$$\frac{\partial u}{\partial t}(t,x) = Lu(t,x) + u(t,x)\dot{W}(t,x), \qquad t > 0, x \in \mathbb{R}^d, \tag{1.1}$$

where L is the generator of a real-valued Lévy process $(X_t)_{t\geq 0}$. The noise $\dot{W}(t,x)$ is white in time and possibly correlated in space, with covariance function informally given by

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)f(x-y),$$

where f is a (possibly generalized) non-negative symmetric function on $\mathbb{R}^d \setminus \{0\}$. We consider a non-random, bounded and measurable initial condition $u_0 : \mathbb{R}^d \to \mathbb{R}$.

By the work of Dalang [9] (see also [14] for Lévy process generators and [11] for a general presentation of Walsh's approach for SPDEs), it is well-known that this equation admits a random-field solution $\{u(t,x) : t > 0, x \in \mathbb{R}^d\}$ such that $\sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}^d} \mathbb{E}[u(t,x)^2] < \infty$, provided that

$$\int_{\mathbb{R}} \frac{\mu(d\xi)}{1 + 2\,\mathrm{Re}\,\Psi(\xi)} < \infty,$$

where $\Psi(\xi) = \mathbb{E}[e^{i\xi X_1}]$ is the Lévy exponent of X and $\mu(d\xi)$ is the Fourier transform of f, usually called the *spectral measure* of the noise.

Equation (1.1) arises in different contexts. It is the continuous form of the parabolic Anderson model studied by Carmona and Molchanov ([4]). We also would like to mention the major role played by equation (1.1) in the study of the so-called

Received 2011-12-16; Communicated by F. Viens.
2000 *Mathematics Subject Classification.* Primary 60H15; Secondary 60H35.
Key words and phrases. Stochastic partial differential equations, moment formulae, parabolic Anderson model.

KPZ equation of physics ([18]). The connection between the stochatic heat equation and the KPZ equation, via a Hopf-Cole transformation has been informally known and studied in the past (see for instance [1], [2]) and has been recently made formal by Hairer [16]. In particular, intermittency and chaos properties of the solution to (1.1) are active topics of research. We refer to [7], [8], [14] for more details about these properties.

The purpose of this paper is to obtain explicit expressions for the moments of the solution u to (1.1). The study of moments of solutions to the parabolic Anderson model has been the subject to an extensive literature, initiated by Carmona and Molchanov [4]. They study (in the spatially discrete case) the asymptotic behavior of the moments as $t \to \infty$. In particular, they prove existence of the *moment Lyapunov exponents*, namely

$$\gamma(p) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[|u(t,x)|^p].$$

Existence and properties of these Lyapunov exponents play a major role in the study of the *intermittency* of the solution. Physically, this corresponds to the fact that the solution develops (as $t \to \infty$) very high peaks concentrated on small islands. Mathematically, the solution is *intermittent* if

$$\gamma(1) < \frac{\gamma(2)}{2} < \cdots < \frac{\gamma(p)}{p} < \cdots .$$

As a consequence, proving that a solution is intermittent requires a careful understanding of the moments of the solution and their behavior. A similar program as the one of Carmona and Molchanov [4] has been developed for the spatially continuous equation (1.1) by Foondun and Khoshnevisan [14].

More recently, the need for a careful understanding of moments of solutions to (1.1) has appeared, not only in the study of intermittency, but also in the effort to study intermittent-like behavior in finite time and chaotic properties of the solution (see for instance [7], [8]). It has become needed to not only understand the asymptotic behavior of the moments as $t \to \infty$, but also for a finite time t. Moreover, results in [7], [8] illustrate how different behaviors of the moments $\mathbb{E}[|u(t,x)|^p]$ as functions of p can lead to drastically different quantitative behaviors of the solution, typically as regards its largest values. These considerations motivate the search for explicit expressions for moments.

Explicit results for moments have already been obtained in the particular case where $L = \Delta$ is the Laplacian; the generator of Brownian Motion and $f = \delta_0$ the Dirac measure in [2]. A similar formula has been developped in [17] in the case where $L = \Delta$ and the noise is fractional in time, white in space. Both [2] and [17] use the Feynman-Kac representation of the solution to deduce a formula for the moments. In this paper, we will follow a different approach and obtain results through a direct computation based on an iterative use of Itô's formula. One motivation for the search of different methods to compute moments is, on the one hand, the objective of extending existing results to more general operators (Lévy generator rather than Δ) and noises that are not white in space, as we are able to do in this paper. Such models are considered in [7], [8] and their study exhibit properties that are different from the classical space-time white noise situation. Studying different operators and different noises helps to distinguish the role played by the operator or the noise when it comes to understand the physical behavior of the solution. The moments formulae of this paper are used in the proofs of the results of [7, 8]. On another hand, we would eventually be interested in handling equations where the multiplicative factor in front the noise, $\sigma(u)$, is not a linear function, in order to understand the role played by the nonlinerity in intermittency. Methods using Feynman-Kac representations have no chance of succeeeding in these situations, since such representations are not available. Some bounds are provided in [7], but obtaining explicit expressions is subject to ongoing research. We notice that in this paper we will only consider a noise that is white in time. This is necessary since the methods of this paper are strongly based on Itô's formula and require the underlying process to be a (semi-)martingale.

Section 2 below is a short reminder about the Parabolic Anderson Model, existence, uniqueness and series representation of the solution. We also recall a few results about Lévy processes. The moment formulae of Theorem 3.1 and its Corollary 3.2 constitute the main results of this paper. They are stated and proved in Section 3 in the case where the spatial covariance of the noise is a measurable function. Section 4 is devoted to the case of space-time white noise (Theorem 4.1). Section 5 is devoted to the formal proofs of some technical results.

2. Parabolic Anderson model

In this section, we are going to remind a few known results about the parabolic Anderson model and Lévy processes in general. Let's start by setting the framework in which we are going to work.

We remind that L denotes the generator of a symmetric Lévy process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$. For instance, one can consider $L = \Delta$, the Laplacian operator: it is the generator of Brownian Motion $(B_t)_{t\geq 0}$.

Let's assume first that the spatial covariance of the noise is a measurable function $f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$. We ask f to be locally integrable around 0. Let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of C^∞ functions with compact support on $\mathbb{R}^{d+1}$. Then $W = \{W(\phi), \phi \in \mathcal{D}(\mathbb{R}^{d+1})\}$ is a centered Gaussian noise with covariance functional given by

$$\mathbb{E}[W(\phi)W(\psi)] = \int_0^\infty dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \phi(t,x) f(x-y) \psi(t,y).$$

This noise can be extended to a worthy martingale measure in the sense of Walsh (see [10] and [19] for details). Hence, by the theory of Walsh [19], extended by Dalang [9], we can define stochastic integrals with respect to the noise W.

We notice that the covariance functional can be written as

$$\mathbb{E}[W(\phi)W(\psi)] = \int_0^\infty dt \int_{\mathbb{R}^d} dx\, f(x)(\phi(t,\cdot) * \tilde{\psi}(t,\cdot))(x), \tag{2.1}$$

where $*$ denotes spatial convolution and $\tilde{\psi}(t,x) = \psi(t,-x)$ for all $x \in \mathbb{R}^d$. Using the representation (2.1), we can then define the noise W in the case where f is a finite measure, using

$$\mathbb{E}[W(\phi)W(\psi)] = \int_0^\infty dt \int_{\mathbb{R}^d} f(dx)\,(\phi(t,\cdot) * \tilde{\psi}(t,\cdot))(x).$$

We will consider a random-field solution to (1.1), i.e. a jointly measurable stochastic process $(u(t,x))_{t\geq 0, x\in\mathbb{R}^d}$ such that

$$\sup_{0\leq t\leq T, x\in\mathbb{R}^d} \mathbb{E}[u(t,x)^2] < \infty,$$

for every fixed $T > 0$ and satsifying the mild-form equation

$$u(t,x) = (\tilde{p}_t * u_0)(x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) u(s,y) W(ds,dy),$$

where p_t is the fundamental solution of the homogeneous equation $\frac{\partial u}{\partial t}(t,x) = Lu(t,x)$, $\tilde{p}_t(x) = p_t(-x)$ for all $x \in \mathbb{R}^d$ and the stochastic integral is taken in the sense of Walsh [19].

Notice that since L is the generator of a Lévy process $(X_t)_{t\geq 0}$, the fundamental solution p_t corresponds to the law of X_t. We assume that the Lévy process X admits densities, so that p_t is a well-defined measurable function. An extension to the case where p_t is a measure would require to work with integrals in the spectral domain. Such an extension is not discussed here, but an insight can be found in [6, Section 6].

We have the following existence and uniqueness result.

Proposition 2.1. *Let μ be the spectral measure of the noise and Ψ the Lévy exponent of the process generated by L. If*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + 2\,\mathrm{Re}\,\Psi(\xi)} < \infty, \tag{2.2}$$

then there exists a unique random-field solution to (1.1).

For a proof of this result, we refer to [9], [12] and [19].

In the case where f is a measurable function, condition (2.2) implies that

$$E^X\left[\int_0^t ds\, f(X_s^{(1)} - X_s^{(2)})\right] = \int_0^t ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, p_s(x) f(x-y) p_s(y) < \infty,$$

where $X^{(1)}$ and $X^{(2)}$ are two independent copies of the Lévy process generated by L and E^X is the expectation with respect to these processes. This implies that the additive functional $A_t^f := \int_0^t f(\bar{X}_s)\,ds$ (associated to the function f) of the Lévy process $\bar{X} := X^{(1)} - X^{(2)}$ is well-defined. Since the Lévy process $\bar{X}$ is symmetric, its Lévy exponent is real given by $2\,\mathrm{Re}\,\Psi(\xi)$. A direct computation allows to prove that $A_t^f \in L^p(\Omega)$ for all $p \geq 1$.

In the case where $f = \delta_0$, we have $\mu(d\xi) = d\xi$ and (2.2) becomes a standard condition for existence of local times for the Lévy process $\bar{X}$. See [3, Theorem 1, p.126]. We denote by $(L_t^x, t > 0, x \in \mathbb{R}^d)$ the local times of the symmetrized process $\bar{X}$. We informally have

$$L_t^x = \int_0^t \delta_x(X_s^{(1)} - X_s^{(2)})\,ds.$$

We also refer to the paper by Foondun, Khoshnevisan and Nualart [15] on the connection between existence of local times for Lévy processes and existence of solutions to SPDE's driven by space-time white noise.

We would like to point out that the proof of Proposition 2.1 shows that the solution u is given as the limit of a Picard iteration scheme. Namely, we set $u_0(t,x) = (\tilde{p}_t * u_0)(x)$ for all $t \geq 0, x \in \mathbb{R}^d$. Then, for all $n \geq 1$, we define recursively a sequence of stochastic process $(u_n(t,x))_{t\geq 0, x\in\mathbb{R}^d}$ by

$$u_n(t,x) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) u_{n-1}(s,y) W(ds,dy).$$

Then, $u(t,x) = \lim_{n\to\infty} u_n(t,x)$ in $L^2(\Omega)$, uniformly over $t \in [0,T]$ and $x \in \mathbb{R}^d$. One can show that the convergence also occurs in $L^p(\Omega)$ for all $p > 2$ (see [9] for details).

We are now going to show that the solution u to (1.1) can be written as a series of iterated integrals. This expansion, which corresponds to the Wiener-chaos expansion of u, will be the main tool used in order to obtain explicit expressions for the moments of u. This result is a direct consequence of the existence result and already appears in different contexts, among which [6] and [13] and the wide litterature of Malliavin Calculus.

Proposition 2.2. *Under the assumptions above, we can show that the solution u to (1.1) is given by*

$$u(t,x) = \sum_{n=0}^{\infty} v_n(t,x), \quad a.s.$$

where $v_0 = u_0$ is deterministic and the processes $(v_n(t,x) : t \geq 0, x \in \mathbb{R}^d)$ are defined recursively by

$$v_n(t,x) = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) v_{n-1}(s,y) W(ds,dy). \tag{2.3}$$

The series is convergent in $L^2(\Omega)$.

Proof. Fix $t \geq 0$ and $x \in \mathbb{R}^d$. Using the Picard iteration scheme, we have

$$u_n(t,x) = u_0(t,x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x) u_{n-1}(s,y) W(ds,dy)$$

As a consequence,

$$u_{n+1}(t,x) - u_n(t,x) = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(y-x)(u_n(s,y) - u_{n-1}(s,y)) W(ds,dy).$$

Hence, we set $v_n(t,x) = u_n(t,x) - u_{n-1}(t,x)$ for all $n \geq 1$. Then, (2.3) is satisfied for $n \geq 2$ and we have

$$u_n(t,x) = u_0(t,x) + \sum_{j=1}^{n} (u_j(t,x) - u_{j-1}(t,x)) = u_0(t,x) + \sum_{j=1}^{n} v_j(t,x) = \sum_{j=0}^{n} v_j(t,x),$$

provided we set $v_0 = u_0$, which implies that (2.3) is satisfied for $n = 1$ as well. Finally, taking the limit as n goes to ∞ establishes the result. ∎

Remark 2.3. The process v_n is a n-times iterated stochastic integral. As a consequence, (2.2) shows that u is given as a series of iterated stochastic integrals. This also helps to obtain explicit formulas for moments if one considers a hyperbolic equation, see [5] and [6].

Remark 2.4. The expansion obtained in Proposition 2.2 corresponds to the Wiener-chaos expansion of the solution $u(t,x)$ to (1.1). Indeed, since v_n is an n-th iterated stochastic integral, it belongs to the n-th Wiener chaos.

3. Moment formula for a measurable covariance.

In this section, we will consider the covariance $f : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$ to be a measurable function, locally integrable on $\mathbb{R}^d \setminus \{0\}$. For instance, one can consider bounded functions, such as $f(x) = e^{-\|x\|}$ or functions which are unbounded at $x = 0$, such as the Riesz kernels, given by $f(x) = \|x\|^{-\alpha}$ for $0 < \alpha < d \wedge 2$.

We are first going to establish explicit formulas for moments of the processes v_n. From this, we will use the series expansion stated in Proposition 2.2 to deduce a formula for the moments of the solution u.

First of all, we would like to point out that it is possible to write explicit expressions for moments of v_n in terms of the Fourier transform of p_t, since $\mathcal{F}p_t(\xi) = e^{t\Psi(\xi)}$. This is not precisely the formula that we would like to obtain here. For details, we refer to [5, Lemma 5.2], which mainly concerns the hyperbolic equation but applies without restrictions to the parabolic case.

We are going to use similar techniques as in the proof of [5, Lemma 5.2] but we will not write the integrals in spectral form and rather express those as functionals of the Lévy process associated to the operator L. This isn't possible in the hyperbolic case, since the Green function does not correspond to the probability distribution of a Markov process.

We are now ready to state our results. Theorem 3.1 below states the formula for the expectation of the product of u taken at different points in space. Corollary 3.2 gives the formula for a moment of order $p \geq 1$.

Theorem 3.1. *Let u denote the solution of (1.1) with operator L as given in Proposition 2.2. Let $t \geq 0$ and $x_1, \dots, x_p \in \mathbb{R}^d$. Then,*

$$E\left[\prod_{j=1}^{p} u(t,x_j)\right] = E^X_{x_1,\dots,x_p}\left[\prod_{i=1}^{p} u_0(X_t^{(i)}) \times \exp\left(\sum_{\substack{j,k=1\\ j<k}}^{p} \int_0^t dr f(X_r^{(j)} - X_r^{(k)})\right)\right]. \tag{3.1}$$

where the processes $(X_t^{(j)})_{t\geq 0}$ $(j = 1,\dots,p)$ are p independent copies of the Lévy process generated by the operator L and $E^X_{x_1,\dots,x_p}$ is the expectation with respect to the law of these processes conditionned such that $X_0^{(1)} = x_1, \dots, X_0^{(p)} = x_p$.

Corollary 3.2. *Let u denote the solution of (1.1) with operator L as given in Proposition 2.2. Let $t \geq 0$ and $x \in \mathbb{R}^d$. Then,*

$$E\left[u(t,x)^p\right] = E^X_x\left[\prod_{i=1}^{p} u_0(X_t^{(i)}) \times \exp\left(\sum_{\substack{j,k=1\\ j<k}}^{p} \int_0^t dr\, f(X_r^{(j)} - X_r^{(k)})\right)\right], \tag{3.2}$$

where the processes $(X_t^{(j)})_{t\geq 0}$ $(j=1,\dots,p)$ *are* p *independent copies of the Levy process generated by the operator* L *and* E_x^X *is the expectation with respect to the law of these processes conditionned such that* $X_0^{(1)} = \cdots = X_0^{(p)} = x$.

In order to prove Theorem 3.1, we need to go through a series of partial results, the most important of which is Proposition 3.4 below. First of all, let us define the following processes. For $x \in \mathbb{R}^d$, $t \geq 0$ and $s \leq t$, set

$$w_0(s;t,x) := v_0(t,x) = u_0(t,x) = (\tilde{p}_t * u_0)(x).$$

Then, for $n \geq 1$, $x \in \mathbb{R}^d$, $t \geq 0$ and $s \leq t$, we set

$$w_n(s;t,x) := \int_0^s \int_{\mathbb{R}^d} p_{t-r}(y-x) v_{n-1}(r,y) W(dr,dy). \tag{3.3}$$

Obviously, we have $v_n(t,x) = w_n(t;t,x)$. The point of this construction is that, by the defnition of Walsh stochastic integrals, the processes $s \mapsto w_n(s;t,x)$ are martingales for each fixed $t \geq 0$, $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$. This will allow us to use Itô's formula on the processes $s \mapsto w_n(s;t,x)$. We start with a lemma.

Lemma 3.3. *Let* $(X_t)_{t\geq 0}$ *be the Lévy process with generator* L, *then*

$$v_0(t,x+y) = E_y^X[u_0(x+X_t)],$$

where E_y^X *denotes the expectation with respect to the law of* X *under the condition* $X_0 = y$.

Proof. Recall that we assume that the process X admits densities. We have

$$v_0(t,x+y) = \int_{\mathbb{R}^d} dz\, p_t(z-x-y) u_0(z) = \int_{\mathbb{R}^d} dz\, p_t(z-y) u_0(x+z) = E_y^X[u_0(x+X_t)],$$

as p_t is the density of the process X_t starting at 0. $\square$

We can now turn to the first intermediate result of this section, a moment formula for the processes $(w_n)_{n\in\mathbb{N}}$.

Proposition 3.4. *Fix an integer* $m \geq 1$ *and* $n_1,\dots,n_m$ *positive integers. Fix* $x_1,\dots,x_m \in \mathbb{R}^d$ *and* $t_1,\dots,t_m \in \mathbb{R}_+$. *Let* $s \geq 0$ *such that* $s \leq t_1 \wedge \cdots \wedge t_m$. *Set* $N = \sum_{i=1}^m n_i$.

- *If* N *is odd, then* $E\left[\prod_{j=1}^m w_{n_j}(s;t_j,x_j)\right] = 0$.
- *If* N *is even, set* $n = \frac{N}{2}$. *Then,*

$$\begin{aligned} E&\left[\prod_{j=1}^m w_{n_j}(s;t_j,x_j)\right] \\ &= \sum_{\mathcal{P}(n_1,\dots,n_m)} E^X\left[\prod_{j=1}^m u_0(x_j + X_{t_j}^{(j)}) \right.\\ &\qquad \left. \times \int_0^s dr_1 \cdots \int_0^{r_{n-1}} dr_n \prod_{i=1}^n f(x_{p_i} + X_{t_{p_i}-r_i}^{(p_i)} - x_{q_i} - X_{t_{q_i}-r_i}^{(q_i)})\right], \end{aligned} \tag{3.4}$$

where $\mathcal{P}(n_1,\ldots,n_m)$ is the set of all orderings in pairs of different integers of the set $\mathcal{N}(n_1,\ldots,n_m)=\{1,\ldots,1,2,\ldots,2,\ldots,m,\ldots,m\}$ with n_i occurences of the integer i. More precisely,

$$\mathcal{P}(n_1,\ldots,n_m)= \\ \{((p_1,q_1),\ldots,(p_n,q_n)):\{p_1,q_1,\ldots,p_n,q_n\}=\mathcal{N}(n_1,\ldots,n_m), \\ \text{and } p_j<q_j,\forall j=1,\ldots,n\}.$$

The processes $(X_t^{(j)})_{t\geq 0}$ ($j=1,\ldots,m$) are m independent copies of the Lévy process generated by L, starting at 0 and E^X denotes expectation with respect to the joint law of these processes. If the set $\mathcal{P}(n_1,\ldots,n_m)=\emptyset$, then the expectation vanishes.

Proposition 3.4 is the main ingredient in order to obtain explicit expression for moments in the proof of Theorem 3.1 below.

The formal proof of Proposition 3.4 looks technical, and we postpone the details to Section 5. However, we would like to provide a sketch of the ideas, which are rather simple. The proof is done by induction, first on the number m of terms in the product and, second, on the total order N of the terms. The order in both inductions is reduced step-by-step using Itô's formula recursively (also known as the *integration by parts formula* in this form) on the product of the martingales

$$s\mapsto w_{n_m}(s;t_m,x_m).$$

Hence, we are able to diminish N by a factor 2 at each step. At certain steps, one term of the product disappears, hence the induction on m as well. In order to prove each step, we use properties of Walsh integrals to write the moments as integrals and, then, transform the spatial integrals into expectations with respect to the Lévy processes, similarly as in the proof of Lemma 3.3. We can then combine the new expectation with the ones from previous steps together using the Markov Property for Lévy processes (Proposition 5.1). Eventually, we obtain the expressions in Proposition 3.4. To get a first glimpse at how these ideas combine to produce the result of Proposition 3.4, one can look at the proof for the case $m=2$ in Section 5. The induction step on m is more complicated to write, but is really treated similarly.

The recursive use of Itô's formula is similar to [6] (see also [5]), where the spectral representation of p is used. Here, we use the Lévy process X to express the integrals as expectations of additive functionals of the symmetrized Lévy process $\bar{X}$.

In Proposition 3.4, we assumed that the n_j's were all positive. But, in the general case, there might be some of the n_j's being equal to 0. In Proposition 3.5, we show that the same expression is valid in that case.

Proposition 3.5. *Fix an integer $M\geq 1$ and $n_1,\ldots,n_M$* non-negative *integers. Fix $x_1,\ldots,x_M\in\mathbb{R}^d$ and $t_1,\ldots,t_M\in\mathbb{R}_+$. Let $s\geq 0$ such that $s\leq t_1\wedge\cdots\wedge t_M$. Set $N=\sum_{i=1}^M n_i$.*

- *If N is odd, then $E\left[\prod_{j=1}^M w_{n_j}(s;t_j,x_j)\right]=0$.*

- *If N is even, set $n = \frac{N}{2}$. Then,*

$$
\begin{aligned}
&E\left[\prod_{j=1}^{M} w_{n_j}(s;t_j,x_j)\right] \\
&= \sum_{\mathcal{P}(n_1,\dots,n_M)} E^X\left[\prod_{j=1}^{M} u_0(x_j + X^{(j)}_{t_j}) \right.\\
&\qquad \left. \times \int_0^s dr_1 \cdots \int_0^{r_{n-1}} dr_n \prod_{i=1}^{n} f(x_{p_i} + X^{(p_i)}_{t_{p_i}-r_i} - x_{q_i} - X^{(q_i)}_{t_{q_i}-r_i})\right], \quad (3.5)
\end{aligned}
$$

where the notations are those of Proposition 3.4. The expectation vanishes if the set $\mathcal{P}(n_1,\dots,n_M) = \emptyset$.

Proof. The case where all $n_1,\dots,n_m$ are positive is proved in Proposition 3.4. Without loss of generality, suppose that there exists m such that $n_j = 0$ for all $m+1 \le j \le M$ and $n_j > 0$ for $j \le m$. Then, as w_0 is deterministic,

$$
\mathbb{E}\left[\prod_{j=1}^{M} w_{n_j}(s;t_j,x_j)\right] = \left(\prod_{k=m+1}^{M} w_0(s;t_k,x_k)\right) E\left[\prod_{j=1}^{m} w_{n_j}(s;t_j,x_j)\right].
$$

Now, we can use Proposition 3.4 to compute the expectation and use Lemma 3.3 to compute the product outside the expectation. We obtain

$$
\begin{aligned}
&\mathbb{E}\left[\prod_{j=1}^{M} w_{n_j}(s;t_j,x_j)\right] \\
&= \left(\prod_{k=m+1}^{M} E^X[u_0(x_k + X^{(k)}_{t_k})]\right) \sum_{\mathcal{P}(n_1,\dots,n_m)} E^X\left[\prod_{j=1}^{m} u_0(x_j + X^{(j)}_{t_j}) \right.\\
&\qquad \left. \times \int_0^s dr_1 \cdots \int_0^{r_{n-1}} dr_n \prod_{i=1}^{n} f(x_{p_i} + X^{(p_i)}_{t_{p_i}-r_i} - x_{q_i} - X^{(q_i)}_{t_{q_i}-r_i})\right] \\
&= \sum_{\mathcal{P}(n_1,\dots,n_m)} E^X\left[\prod_{j=1}^{M} u_0(x_j + X^{(j)}_{t_j}) \right.\\
&\qquad \left. \times \int_0^s dr_1 \cdots \int_0^{r_{n-1}} dr_n \prod_{i=1}^{n} f(x_{p_i} + X^{(p_i)}_{t_{p_i}-r_i} - x_{q_i} - X^{(q_i)}_{t_{q_i}-r_i})\right].
\end{aligned}
$$

The result is proved since $\mathcal{P}(n_1,\dots,n_M) = \mathcal{P}(n_1,\dots,n_m)$ when $n_{m+1} = \cdots = n_M = 0$. ☐

Now, we can simplify our expression by getting back to the processes v_n rather than w_n.

Corollary 3.6. *Fix an integer $m \ge 1$ and $n_1,\dots,n_m$ non-negative integers. Fix $x_1,\dots,x_m \in \mathbb{R}^d$ and $t \in \mathbb{R}_+$. Set $N = \sum_{i=1}^{m} n_i$.*

- *If N is odd, then* $E\left[\prod_{j=1}^m v_{n_j}(t,x_j)\right]=0$.
- *If N is even, set $n=\frac{N}{2}$. Then,*

$$\begin{aligned}E&\left[\prod_{j=1}^m v_{n_j}(t,x_j)\right]\\ &=\sum_{\mathcal{P}(n_1,\ldots,n_m)} E^X\left[\prod_{j=1}^m u_0(x_i+X_t^{(i)})\right.\\ &\qquad\left.\times\int_0^t dr_1\cdots\int_0^{r_{n-1}}dr_n\prod_{j=1}^n f(x_{p_i}+X_{t-r_i}^{(p_i)}-x_{q_i}-X_{t-r_i}^{(q_i)})\right],\end{aligned}\tag{3.6}$$

where the notations are those of Proposition 3.4. The expectation vanishes if the set $\mathcal{P}(n_1,\ldots,n_M)=\emptyset$.

Proof. Set $s=t_1=\cdots=t_m=t$ in Proposition 3.5. □

Remark 3.7. We would like to point out that another way to write down (3.6) is:

$$\begin{aligned}E&\left[\prod_{j=1}^m v_{n_j}(t,x_j)\right]\\ &=\sum_{\mathcal{P}(n_1,\ldots,n_m)} E^X_{x_1,\ldots,x_m}\left[\prod_{j=1}^m u_0(X_t^{(i)})\right.\\ &\qquad\left.\times\int_0^t dr_1\cdots\int_0^{r_{n-1}}dr_n\prod_{j=1}^n f(X_{t-r_i}^{(p_i)}-X_{t-r_i}^{(q_i)})\right],\end{aligned}$$

where $E^X_{x_1,\ldots,x_m}$ denotes the expectation with respect to the joint law of the processes $X^{(j)}$ $(j=1,\ldots,n)$ under the conditions $X_0^{(1)}=x_1,\ \ldots,\ X_0^{(m)}=x_m$.

We also recall the following result from Real Analysis. The proof is not very difficult, hence we leave it to the reader, since it is beyond the scope of this paper.

Lemma 3.8. *Let $g_1,\ldots,g_n$ be integrable functions on $\mathbb{R}_+$. Suppose that the n functions are divided in ℓ groups of respectively $k_1,\ldots,k_\ell$ identical functions ($k_1+\cdots+k_\ell=n$). Then, for all $t\geq 0$, the following result holds*

$$\sum_{\pi\in\mathcal{S}_n(k_1,\ldots,k_\ell)}\int_0^t dr_1\cdots\int_0^{r_{n-1}}dr_n\, g_{\pi(1)}(r_1)\cdots g_{\pi(n)}(r_n)=\frac{\prod_{i=1}^n\int_0^t dr\, g_i(r)}{k_1!\cdots k_\ell!},$$

where $\mathcal{S}_n(k_1,\ldots,k_\ell)$ is the set of all permutations of n objects divided in ℓ groups of respectively $k_1,\ldots,k_\ell$ identical objects. Namely, $|\mathcal{S}_n(k_1,\ldots,k_\ell)|=\frac{n!}{k_1!\cdots k_\ell!}$.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. By Proposition 2.2, we know that

$$u(t,x)=\sum_{n=0}^{\infty} v_n(t,x)$$

and the series converges in $L^p(\Omega)$, for all $p \geq 2$. Hence,

$$E\left[\prod_{j=1}^{p} u(t,x_j)\right]=\sum_{n_1=0}^{\infty}\cdots\sum_{n_p=0}^{\infty}\mathbb{E}\left[\prod_{j=1}^{p} v_{n_j}(t,x_j)\right]$$

Further, by Corollary 3.6 and distinguishing the vectors $(n_1,\ldots,n_p)$ depending on the sum of their components,

$$\begin{aligned}
&E\left[\prod_{j=1}^{p} u(t,x_j)\right]\\
&= \sum_{n_1=0}^{\infty}\cdots\sum_{n_p=0}^{\infty}\sum_{\mathcal{P}(n_1,\ldots,n_p)} \mathbf{1}_{\{\sum_{j=1}^{p} n_j \text{ is even.}\}} E^X\left[\prod_{j=1}^{p} u_0(x_j+X_t^{(j)})\right.\\
&\quad\left.\times\int_0^t dr_1\cdots\int_0^{r_{n-1}} dr_n \prod_{i=1}^{n} f(x_{p_i}+X_{t-r_i}^{(p_i)}-x_{q_i}-X_{t-r_i}^{(q_i)})\right]\\
&= \sum_{n=0}^{\infty}\sum_{\substack{n_1,\ldots,n_p=0\\ \sum_{j=1}^{p} n_j=2n}}^{\infty}\sum_{\mathcal{P}(n_1,\ldots,n_p)} E^X\left[\prod_{j=1}^{p} u_0(x_j+X_t^{(j)})\right.\\
&\quad\left.\times\int_0^t dr_1\cdots\int_0^{r_{n-1}} dr_n \prod_{i=1}^{n} f(x_{p_i}+X_{t-r_i}^{(p_i)}-x_{q_i}-X_{t-r_i}^{(q_i)})\right]\\
&= E^X\left[\prod_{j=1}^{p} u_0(x_j+X_t^{(j)})\times\sum_{n=0}^{\infty}\sum_{\substack{n_1,\ldots,n_p=0\\ \sum_{j=1}^{p} n_j=2n}}^{\infty}\sum_{\mathcal{P}(n_1,\ldots,n_p)}\right.\\
&\quad\left.\int_0^t dr_1\cdots\int_0^{r_{n-1}} dr_n \prod_{i=1}^{n} f(x_{p_i}+X_{t-r_i}^{(p_i)}-x_{q_i}-X_{t-r_i}^{(q_i)})\right].
\end{aligned}$$

Now, let $d\mathcal{P}(n_1,\ldots,n_p)$ denote the set of pairings in $\mathcal{P}(n_1,\ldots,n_p)$ but without taking the order into account. (For instance,((1,2),(1,3)) and ((1,3),(1,2)) are different pairings in $\mathcal{P}(2,1,1)$, but are the same in $d\mathcal{P}(2,1,1)$.) We have

$$\begin{aligned} & E\left[\prod_{j=1}^p u(t,x_j)\right] \\ &= E^X\left[\prod_{j=1}^p u_0(x_j+X_t^{(j)}) \times \sum_{n=0}^\infty \sum_{\substack{n_1,\dots,n_p=0 \\ \sum_{j=1}^p n_j=2n}}^\infty \sum_{d\mathcal{P}(n_1,\dots,n_p)} \sum_{\substack{\text{all possible} \\ \text{orders of the pairing}}} \right. \\ & \qquad \left. \int_0^t dr_1 \cdots \int_0^{r_{n-1}} dr_n \prod_{i=1}^n f(x_{p_i}+X_{t-r_i}^{(p_i)} - x_{q_i} - X_{t-r_i}^{(q_i)})\right]. \end{aligned}$$

Then, by Lemma 3.8, we have

$$\begin{aligned} & E\left[\prod_{j=1}^p u(t,x_j)\right] \\ &= E^X\left[\prod_{j=1}^p u_0(x_j+X_t^{(j)}) \times \sum_{n=0}^\infty \sum_{\substack{n_1,\dots,n_p=0 \\ \sum_{j=1}^p n_j=2n}}^\infty \right. \\ & \qquad \left. \cdots \sum_{d\mathcal{P}(n_1,\dots,n_p)} \frac{1}{k_1!\cdots k_\ell!} \prod_{i=1}^n \int_0^t dr\, f(x_{p_i}+X_{t-r}^{(p_i)} - x_{q_i} - X_{t-r}^{(q_i)})\right] \\ &= E^X\left[\prod_{j=1}^p u_0(x_j+X_t^{(j)}) \times \sum_{n=0}^\infty \sum_{\substack{n_1,\dots,n_p=0 \\ \sum_{j=1}^p n_j=2n}}^\infty \right. \\ & \qquad \left. \cdots \sum_{d\mathcal{P}(n_1,\dots,n_p)} \frac{1}{k_1!\cdots k_\ell!} \prod_{i=1}^n \int_0^t dr\, f(x_{p_i}+X_{r}^{(p_i)} - x_{q_i} - X_{r}^{(q_i)})\right], \end{aligned}$$

where $k_1,\dots,k_\ell$ are the numbers of identical pairs coming in the pairing considered. Now, using the fact that

$$\begin{aligned} & \bigcup_{\substack{n_1,\dots,n_p=0 \\ \sum_{j=1}^p n_j=2n}}^\infty d\mathcal{P}(n_1,\dots,n_p) \\ &= \Big\{((p_1,q_1),\dots,(p_n,q_n)) : p_j < q_j; p_1,\dots,p_n,q_1,\dots,q_n \le p, \\ & \qquad \text{without taking the order into account}\Big\} \end{aligned}$$

and that there are $\frac{n!}{k_1!\cdots k_\ell!}$ ways to order a particular pairing, we have

$$\begin{aligned}
& E\left[\prod_{j=1}^{p} u(t,x_j)\right] \\
&= E^X\left[\prod_{j=1}^{p} u_0(x_j+X_t^{(j)}) \right.\\
&\quad \left. \times \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{p_1,q_1=1\\ p_1<q_1}}^{p}\cdots\sum_{\substack{p_n,q_n=1\\ p_n<q_n}}^{p}\prod_{i=1}^{n}\int_0^t dr f(x_{p_i}+X_r^{(p_i)}-x_{q_i}-X_r^{(q_i)})\right] \\
&= E^X\left[\prod_{j=1}^{p} u_0(x_j+X_t^{(j)}) \right.\\
&\quad \left. \times \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{\substack{p_1,q_1=1\\ p_1<q_1}}^{p}\int_0^t dr f(x_{p_1}+X_r^{(p_1)}-x_{q_1}-X_r^{(q_1)})\right)^n\right] \\
&= E^X\left[\prod_{j=1}^{p} u_0(x_j+X_t^{(j)}) \right.\\
&\quad \left. \times \exp\left(\sum_{\substack{p_1,q_1=1\\ p_1<q_1}}^{p}\int_0^t dr f(x_{p_1}+X_r^{(p_1)}-x_{q_1}-X_r^{(q_1)})\right)\right].
\end{aligned}$$

After conditionning on $X_0^{(1)}=x_1,\ldots,X_0^{(p)}=x_p$, the result is proved. □

Proof of Corollary 3.2. Take $x_1=\cdots=x_p=x$ in Theorem 3.1. □

4. Space-time white noise and Lévy local times.

In this section, we will consider the case where the covariance $f=\delta_0$ is the Dirac measure at 0. Hence, the noise W is a space-time white noise and is defined as a centered Gaussian noise $\{W(\phi):\phi\in\mathcal{D}(\mathbb{R}^{1+1})\}$ with covariance functional given by

$$\begin{aligned}
E[W(\phi)W(\psi)] &= \int_0^\infty dt\int_{\mathbb{R}}\delta_0(dx)\,(\phi(t,\cdot)*\tilde{\psi}(t,\cdot))(x)\\
&= \int_0^\infty dt\int_{\mathbb{R}} d\xi\,\mathcal{F}\phi(t,\xi)\overline{\mathcal{F}\psi(t,\xi)}.
\end{aligned}$$

We notice that no solution exists in dimension $d\geq 2$ for space-time white noise, hence we restrict our attention to the case $d=1$. The moment formula is given by Theorem 4.1 below.

Theorem 4.1. *Let u denote the solution of (1.1) with operator L, driven by a space-time white-noise. Let $t\geq 0$ and $x\in\mathbb{R}$. Let $(X_t^{(j)})_{t\geq 0}$ $(j=1,\ldots,p)$ be p*

independent copies of the Lévy process generated by L. Let $L_t^0(j,k)$ be the local time at 0 of the process $X^{(j)} - X^{(k)}$. Informally,

$$L_t^0(j,k) = \int_0^t dr\, \delta_0(X_r^{(j)} - X_r^{(k)}).$$

Then,

$$E\left[u(t,x)^p\right] = E^X\left[\prod_{i=1}^p u_0(x+X_t^{(i)}) \times \exp\left(\sum_{\substack{j,k=1\\ j<k}}^p L_t^0(j,k)\right)\right]. \tag{4.1}$$

In order to prove Theorem 4.1, we would like to apply the moment formulae of Section 3. Therefore, we need to smooth the covariance of the noise through a convolution procedure. Namely, let $\varphi : \mathbb{R} \to \mathbb{R}$ be the standard gaussian kernel. Namely,

$$\varphi(x) = (2\pi)^{-1/2} \exp(-x^2/2).$$

Let $\varphi_\varepsilon(x) := \varepsilon^{-1/2}\varphi(x/\sqrt{\varepsilon})$. Notice that the noise W can be extended to the space of rapidly decreasing C^∞-functions. Hence, we can define a noise $W_\varepsilon = \{W_\varepsilon(\phi) : \phi \in \mathcal{D}(\mathbb{R}^{1+1})\}$ by

$$W_\varepsilon(\phi) := W(\phi * \varphi_\varepsilon), \tag{4.2}$$

where $*$ stands for spatial convolution. The noise W_ε is well-defined on the same probability space as the noise W. Now, notice that

$$\begin{aligned} E[W_\varepsilon(\phi)W_\varepsilon(\psi)] &= E[W(\phi * \varphi_\varepsilon)W(\psi * \varphi_\varepsilon)] \\ &= \int_0^\infty dt \int_{\mathbb{R}} \delta_0(dx)((\phi(t,\cdot) * \varphi_\varepsilon) * (\tilde{\psi}(t,\cdot) * \varphi_\varepsilon))(x) \\ &= \int_0^\infty dt \int_{\mathbb{R}} \delta_0(dx)(\phi(t,\cdot) * \tilde{\psi}(t,\cdot) * \varphi_{2\varepsilon})(x) \\ &= \int_0^\infty dt \int_{\mathbb{R}} dx\, \varphi_{2\varepsilon}(x)(\phi(t,\cdot) * \tilde{\psi}(t,\cdot))(x). \end{aligned}$$

Hence, the noise W_ε is a Gaussian noise with covariance informally given by

$$\mathbb{E}[\dot{W}_\varepsilon(t,x)\dot{W}_\varepsilon(s,y)] = \delta_0(t-s)\varphi_{2\varepsilon}(x-y).$$

Since φ_ε is a bounded covariance function, we can apply the results of Section 3 to the solution u^ε of (1.1) with the noise W_ε.

Informally, $\varphi_{2\varepsilon}$ converges to δ_0 as $\varepsilon \to 0$. Hence, W_ε is actually an approximation of the noise W, in which we have turned the covariance measure into a bounded function. We state two results which make this convergence formal.

Proposition 4.2. *Let $Z = \{Z(t,x), t > 0, x \in \mathbb{R}^d\}$ be a Walsh-integrable stochastic process with respect to space-time white noise W. Then, for all $p \geq 1$, $\int_0^t \int_{\mathbb{R}} Z(s,y)W_\varepsilon(ds,dy)$ converges in $L^p(\Omega)$ to $\int_0^t \int_{\mathbb{R}} Z(s,y)W(ds,dy)$ uniformly in $t \in [0,T]$ as $\varepsilon \to 0$.*

Proof. For a noise with covariance f, let $Q_W([0,t]\times A\times B) := t\int_{\mathbb{R}} f(dx)(\mathbf{1}_A * \tilde{\mathbf{1}_B})(x)$ be the covariation measure of the extension of W as a martingale measure in the sense of Walsh [19]. Let $\phi,\psi\in\mathcal{D}(\mathbb{R})$. We have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2}\phi(x)\psi(y)Q_{W_\varepsilon}([0,t]\times dx\times dy)-\int_{\mathbb{R}^2}\phi(x)\psi(y)Q_W([0,t]\times dx\times dy)\right|\\
&\quad= t\left|\int_{\mathbb{R}} dx\,\varphi_{2\varepsilon}(x)(\phi*\tilde{\psi})(x)-\int_{\mathbb{R}}\delta_0(dx)(\phi*\tilde{\psi})(x)\right|\\
&\quad= t\left|\int_{\mathbb{R}} d\xi\,\mathcal{F}\varphi_{2\varepsilon}(\xi)\mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}-\int_{\mathbb{R}} d\xi\mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)}\right|\\
&\quad\le t\int_{\mathbb{R}} d\xi|\mathcal{F}\phi(\xi)||\mathcal{F}\psi(\xi)||1-e^{-\frac{\varepsilon\xi^2}{2}}|\longrightarrow 0,
\end{aligned}$$

as $\varepsilon\to 0$. This shows that the covariation measure of the martingale measure W_ε converges to the covariation measure of W as $\varepsilon\to 0$. Using the results of Walsh [19] and Dalang [9], this is enough to ensure the convergence of the stochastic integral in $L^p(\Omega)$ for all $p\ge 2$. □

Proposition 4.3. *The solution to* (1.1) *with noise* W_ε *converges to the solution to* (1.1) *with noise* W *as* $\varepsilon\to 0$ *in* $L^p(\Omega)$, *uniformly on* $t\in[0,T]$ *and* $x\in\mathbb{R}$.

Proof. This is a consequence of Proposition 4.2 and of the existence result (Proposition 2.1) using the Picard iteration scheme. □

In order to be able to state and prove the moment result in the case where $f=\delta_0$, we need one more result about additive functionals of Lévy processes. We remind that the existence and uniqueness result ensures that

$$\int_{\mathbb{R}^d}\frac{d\xi}{1+2\operatorname{Re}\Psi(\xi)}<\infty.$$

This implies that the symmetrized Lévy process $(\bar{X}_t)_{t\ge 0}$ admits local times, where the process $\bar{X}$ is defined by $\bar{X}_t := X_t^{(1)}-X_t^{(2)}$ for all $t\ge 0$, for $X^{(1)}$ and $X^{(2)}$ two independent copies of the process generated by L (see Section 2 and [3, Theorem 1, p.126]). Let L_t^x denote the local time at x of $\bar{X}$. Then, for any non-negative bounded function g, we have

$$\int_0^t g(\bar{X}_s)\,ds=\int_{\mathbb{R}^d} g(x)L_t^x\,dx,\qquad a.s. \tag{4.3}$$

([3, p.126]). We have the following approximation result.

Proposition 4.4. *Let* $(\bar{X}_t)_{t\ge 0}$ *be a Lévy process which admits local times* $(L_t^x : t>0, x\in\mathbb{R})$. *Then, for all* $p\ge 1$,

$$\mathbb{E}\left[\left(\exp\left(\int_0^t\varphi_{2\varepsilon}(\bar{X}_s)\,ds\right)-\exp(L_t^0)\right)^p\right]\to 0,$$

as $\varepsilon\to 0$.

Proof. We know by [3, Prop.4, p.130], that the local time L_t^0 has finite exponential moments. Hence, it suffices to prove that

$$\mathbb{E}\left[\left(\exp\left(\int_0^t \varphi_{2\varepsilon}(\bar{X}_s)ds - L_t^0\right) - 1\right)^p\right] \to 0,$$

as $\varepsilon \to 0$. By (4.3) applied to $g \equiv 1$, we know that $x \mapsto L_t^x$ is in $L^1(\mathbb{R})$ a.s. and it admits a Fourier transform $\hat{L}_t$. Using (4.3) again,

$$\begin{aligned}\left|\int_0^t \varphi_{2\varepsilon}(X_s)ds - L_t^0\right| &= \left|\int_{\mathbb{R}} \varphi_{2\varepsilon}(x) L_t^x\,dx - \int_{\mathbb{R}} L_t^x\,\delta_0(dx)\right| \\ &= \int_{\mathbb{R}} |\mathcal{F}\varphi_{2\varepsilon}(\xi) - 1| \hat{L}_t(\xi)\,d\xi \\ &\leq 2\int_{\mathbb{R}} \hat{L}_t(\xi)\,d\xi = 2L_t^0, \end{aligned} \tag{4.4}$$

Now, since $|\mathcal{F}\varphi_{2\varepsilon}(\xi) - 1|$ converges to 0 pointwise as $\varepsilon \to 0$, the result follows from the bound (4.4) and the Dominated Convergence Theorem, since L_t^0 has finite exponential moments. □

We are now ready to prove the moment formula in the case where $f = \delta_0$.

Proof of Theorem 4.1. Let u_ε denote the solution to (1.1) with noise W_ε defined in (4.2). By Corollary 3.2, we know that

$$E\left[u_\varepsilon(t,x)^p\right] = E^X\left[\prod_{i=1}^p u_0(x + X_t^{(i)}) \times \exp\left(\sum_{\substack{j,k=1\\ j<k}}^p \int_0^t dr\,\varphi_{2\varepsilon}(X_r^{(j)} - X_r^{(k)})\right)\right]. \tag{4.5}$$

Moreover, we know by Proposition 4.3 that $E[u(t,x)^p] = \lim_{\varepsilon\to 0} E[u_\varepsilon(t,x)^p]$. Also, by Proposition 4.4, we know that for any choice of $j,k \in \{1,\ldots,p\}$,

$$\exp\left(\int_0^t \varphi_{2\varepsilon}(X_s^{(j)} - X_s^{(k)})\,ds\right) \quad \stackrel{L^p(\Omega)}{\longrightarrow} \quad \exp(L_t^0(j,k)),$$

for any $p \geq 1$. This implies that the right-hand side of (4.5) converges to the right-hand side of (4.1) as $\varepsilon \to 0$. The result is proved by taking the limit as $\varepsilon \to 0$ in (4.5). □

Remark 4.5. In the case where $L = \Delta$ and X is a Brownian Motion, Theorem 4.1 corresponds to the first part of Theorem 5.3 of [17]. Theorem 3.1 doesn't require the use of a Feynman-Kac type representation for the solution, but uses the Wiener-Chaos expansion instead. Theorem 4.1 is also slightly more general in the sense that it considers generators of Lévy processes and not only Δ, although it only handles noises which are white in time. Indeed, this method has very little chance to apply to fractional noises in time, since they do not involve semi-martingales and this would prevent us from using Itô's formula. However, an advantage of this method is that there is a chance that could provide explicit expressions for equations with a nonlinear function multiplying the noise, typically polynomials. This is subject to ongoing research. We would like to notice that the approach used in [17] can also be used to obtain the generalized Theorem 3.1 in the case of generators of Lévy processes, rather than the Laplacian Δ.

5. Proof of Proposition 3.4

We first remind the Markov property for Lévy processes in the form that we are going to use in this section.

Proposition 5.1 (Markov Property). *Let $(X_t)_{t\geq 0}$ be a Lévy process with values in $\mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}$ a bounded continuous function. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by X. Then, the following Markov property holds*

$$E^X_{X_{t-s}}[g(X_s)] = E^X_0[g(X_t) \mid \mathcal{F}_{t-s}],$$

where $0 \leq s \leq t$ and E^X_y denotes the expectation with respect to the law of X under the condition $X_0 = y$.

We refer to [3] for more details about the Markov property for Lévy processes. In order to prove Proposition 3.4 below, we will need a Markov property stated in a slightly different way. Namely, we will need to consider a functional of two independent processes conditionned at two different times.

Lemma 5.2 (Markov Property). *Let $(X^{(1)}_t)_{t\geq 0}$ and $(X^{(2)}_t)_{t\geq 0}$ be two independent Lévy processes with values in $\mathbb{R}^d$. Let $(\mathcal{F}^{(i)}_t)_{t\geq 0}$ be the filtration generated by $X^{(i)}$ ($i = 1, 2$). Let $g : (\mathbb{R}^d)^n \times (\mathbb{R}^d)^m \to \mathbb{R}$ be a bounded continuous function. Then, for all r_1, r_2, $t_1, \ldots, t_n$, $s_1, \ldots, s_m \in \mathbb{R}_+$, we have*

$$\begin{aligned} &E^X_{X^{(1)}_{r_1}, X^{(2)}_{r_2}}\left[g(X^{(1)}_{t_1}, \ldots, X^{(1)}_{t_n}, X^{(2)}_{s_1}, \ldots, X^{(2)}_{s_m})\right] \\ &\quad = \; E^X_{0,0}\left[g(X^{(1)}_{t_1+r_1}, \ldots, X^{(1)}_{t_n+r_1}, X^{(2)}_{s_1+r_2}, \ldots, X^{(2)}_{s_m+r_2}) \mid \mathcal{F}^{(1)}_{r_1} \otimes \mathcal{F}^{(2)}_{r_2}\right], \end{aligned}$$

where $E^X_{x_1,x_2}$ denotes the expectation with respect to the joint law of $X^{(1)}, X^{(2)}$ under the condition $X^{(1)}_0 = x_1, X^{(2)}_0 = x_2$.

Proof. Without loss of generality, we can show the result when $n = m = 1$. The general case works the same way. As the processes $X^{(1)}$ and $X^{(2)}$ are independent,

$$E^X[Y \mid \mathcal{F}^{(1)}_{r_1} \otimes \mathcal{F}^{(2)}_{r_2}] = E^{X^{(1)}}[E^{X^{(2)}}[Y \mid \mathcal{F}^{(2)}_{r_2}] \mid \mathcal{F}^{(1)}_{r_1}].$$

Hence, by the Markov property for $X^{(2)}$ first and then for $X^{(1)}$, we have

$$\begin{aligned} &E^X_{0,0}\left[g(X^{(1)}_{t+r_1}, X^{(2)}_{s+r_2}) \mid \mathcal{F}^{(1)}_{r_1} \otimes \mathcal{F}^{(2)}_{r_2}\right] \\ &\quad = \; E^{X^{(1)}}_0\left[E^{X^{(2)}}_0[g(X^{(1)}_{t+r_1}, X^{(2)}_{s+r_2}) \mid \mathcal{F}^{(2)}_{r_2}] \mid \mathcal{F}^{(1)}_{r_1}\right] \\ &\quad = \; E^{X^{(1)}}_0\left[E^{X^{(2)}}_{X^{(2)}_{r_2}}[g(X^{(1)}_{t+r_1}, X^{(2)}_s)] \mid \mathcal{F}^{(1)}_{r_1}\right] \\ &\quad = \; E^{X^{(1)}}_{X^{(1)}_{r_1}}\left[E^{X^{(2)}}_{X^{(2)}_{r_2}}[g(X^{(1)}_t, X^{(2)}_s)]\right] \\ &\quad = \; E^X_{X^{(1)}_{r_1}, X^{(2)}_{r_2}}[g(X^{(1)}_t, X^{(2)}_s)]. \end{aligned}$$

□

Remark 5.3. We notice that the use of the Markov property above is completely formal when the function g is bounded. In the proofs below, we may want to consider the Markov property for a function that is unbounded at $x = 0$. In that case, we can consider the function

$$g_n(x) := g(x) \wedge n,$$

apply the Markov property and then pass to the limit as $n \to \infty$. This procedure will be valid in the proofs below since $E[(\int_0^t f(\bar{X}_s)\, ds)^p] < \infty$ for all $p \geq 1$ by the assumption of the existence result (Proposition 2.1). We will use this procedure throughout the proof below without developping the details.

Proof of Proposition 3.4. As mentioned in Section 3, the proof of Proposition 3.4 uses a double induction, both on the number of terms m and the order $N = \sum_{i=1}^m n_i$. We decomposed the proof below in different cases. We start the proof with the case $m = 1$ which is handled as a particular case.

Case $m = 1$. In that case, by the martingale property of the stochastic integral, we have $E[w_{n_1}(s; t, x)] = 0$ for all $x \in \mathbb{R}^d$, $0 \leq s \leq t$ and this proves the result if n_1 is odd. If n_1 is even, the left-hand side of (3.4) vanishes. Moreover, for $n_1 \in \mathbb{N}$, the set $\mathcal{N}(n_1)$ is formed with n_1 occurences of the integer 1. It is not possible to find any pairing of different integers from this set and, hence, $\mathcal{P}(n_1) = \emptyset$ and the right-hand side of (3.4) vanishes as well.

We are now going to prove the result by induction on m. We first consider the case $m = 2$, for which we are going to obtain the result by induction on $N = \sum_{i=1}^2 n_i$. The smallest possible value of N is $N = 2$.

Case $m = 2$, $N = 2$. If $m = 2$ and $N = 2$, then we must have $n_1 = 1, n_2 = 1$. In that case, using the definition of w_1 and the properties of Walsh stochastic integrals, we have

$$\begin{aligned} & E[w_1(s; t_1, x_1) w_1(s; t_2, x_2)] \\ & \quad = \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, p_{t_1 - r}(y - x_1) f(y - z) p_{t_2 - r}(z - x_2) v_0(r, y) v_0(r, z). \end{aligned}$$

Using the fact that p_t is the density of the Lévy process $(X_t)_{t \geq 0}$ starting at 0 and Lemma 3.3, we have

$$\begin{aligned} & E[w_1(s; t_1, x_1) w_1(s; t_2, x_2)] \\ & \quad = \int_0^s dr E^X[f(x_1 + X^{(1)}_{t_1 - r} - x_2 - X^{(2)}_{t_2 - r}) v_0(r, x_1 + X^{(1)}_{t_1 - r}) v_0(r, x_2 + X^{(2)}_{t_2 - r})] \\ & \quad = \int_0^s dr E^X \Big[f(x_1 + X^{(1)}_{t_1 - r} - x_2 - X^{(2)}_{t_2 - r}) \\ & \qquad \qquad \times E^X_{X^{(1)}_{t_1 - r}}[u_0(x_1 + X^{(1)}_r)] E^X_{X^{(2)}_{t_2 - r}}[u_0(x_2 + X^{(2)}_r)] \Big]. \end{aligned}$$

Now, let $(\mathcal{F}^{(i)}_t)_{t \geq 0}$ denote the filtration generated by $X^{(i)}$. By the Markov properties (Theorem 5.1) for the processes $X^{(1)}$ and $X^{(2)}$ and their independence, we have

$$
\begin{aligned}
&E[w_1(s;t_1,x_1)w_1(s;t_2,x_2)] \\
&= \int_0^s drE^X\Big[f(x_1+X^{(1)}_{t_1-r}-x_2-X^{(2)}_{t_2-r}) \\
&\qquad \times E^X[u_0(x_1+X^{(1)}_{t_1}) \mid \mathcal{F}^{(1)}_{t_1-r}]E^X[u_0(x_2+X^{(2)}_{t_2}) \mid \mathcal{F}^{(2)}_{t_2-r}]\Big] \\
&= \int_0^s drE^X\Big[f(x_1+X^{(1)}_{t_1-r}-x_2-X^{(2)}_{t_2-r}) \\
&\qquad \times E^X[u_0(x_1+X^{(1)}_{t_1})u_0(x_2+X^{(2)}_{t_2}) \mid \mathcal{F}^{(1)}_{t_1-r}\otimes\mathcal{F}^{(2)}_{t_2-r}]\Big] \\
&= \int_0^s drE^X\Big[E^X[f(x_1+X^{(1)}_{t_1-r}-x_2-X^{(2)}_{t_2-r}) \\
&\qquad \times u_0(x_1+X^{(1)}_{t_1})u_0(x_2+X^{(2)}_{t_2}) \mid \mathcal{F}^{(1)}_{t_1-r}\otimes\mathcal{F}^{(2)}_{t_2-r}]\Big] \\
&= \int_0^s drE^X\Big[f(x_1+X^{(1)}_{t_1-r}-x_2-X^{(2)}_{t_2-r})u_0(x_1+X^{(1)}_{t_1})u_0(x_2+X^{(2)}_{t_2})\Big],
\end{aligned}
$$

because $f(x_1+X^{(1)}_{t_1-r}-x_2-X^{(2)}_{t_2-r})$ is $\mathcal{F}^{(1)}_{t_1-r}\otimes\mathcal{F}^{(2)}_{t_2-r}$ measurable. Finally, by Fubini's theorem,

$$
\begin{aligned}
&E[w_1(s;t_1,x_1)w_1(s;t_2,x_2)] \\
&= E^X\left[u_0(x_1+X^{(1)}_{t_1})u_0(x_2+X^{(2)}_{t_2})\int_0^s drf(x_1+X^{(1)}_{t_1-r}-x_2-X^{(2)}_{t_2-r})\right].
\end{aligned}
$$

As $\mathcal{N}(1,1)=\{1,2\}$, $\mathcal{P}(1,1)=\{(1,2)\}$ and the result is proved for $m=2$, $N=2$.

Let us now consider the case $m=2$, $N=3$.

Case $m=2$, $N=3$. We must have $n_1=2, n_2=1$ (or the other way around). In that case, using the definitions of w_1 and w_2 and the properties of Walsh stochastic integrals, we have

$$
\begin{aligned}
&E[w_2(s;t_1,x_1)w_1(s;t_2,x_2)] \\
&= \int_0^s dr\int_{\mathbb{R}^d}dy\int_{\mathbb{R}^d}dz\, p_{t_1-r}(y-x_1)f(y-z)p_{t_2-r}(z-x_2)E[v_1(r,y)]v_0(r,z) \\
&= 0,
\end{aligned}
$$

as $E[v_1(r,y)]=0$ for all $r\geq 0$, $y\in\mathbb{R}^d$. The result is proved for $m=2$, $N=3$.

Let us now proceed by induction on the order N.

Case $m=2$, induction on N. Assume that the result is proved for $m=2$ and all $N\leq M-1$. Let $n_1,n_2\in\mathbb{N}$ such that $n_1+n_2=M$. Then, again by the definition of w_n and the properties of Walsh stochastic integrals, we have

$$\begin{aligned} &E[w_{n_1}(s;t_1,x_1)w_{n_2}(s;t_2,x_2)] \\ &\quad = \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, p_{t_1-r}(y-x_1)f(y-z)p_{t_2-r}(z-x_2) \\ &\qquad\qquad \times E[v_{n_1-1}(r,y)v_{n_2-1}(r,z)]. \end{aligned}$$

If M is odd, then $(n_1-1)+(n_2-1)=M-2$ is odd as well and

$$E[v_{n_1-1}(r,y)v_{n_2-1}(r,z)]=0$$

for all $r\geq 0$, $y,z\in\mathbb{R}^d$ by the induction assumption applied with $t_1=t_2=r$, $x_1=y$, $x_2=z$.

If M is even, then $(n_1-1)+(n_2-1)=M-2$ is even, and we can write

$$E[v_{n_1-1}(r,y)v_{n_2-1}(r,z)]$$

using the induction assumption with $t_1=t_2=r$, $x_1=y$, $x_2=z$. First, using the fact that p_t is the density of X, we have

$$\begin{aligned} &E[w_{n_1}(s;t_1,x_1)w_{n_2}(s;t_2,x_2)] \\ &\quad = \int_0^s dr \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dz\, p_{t_1-r}(y-x_1)f(y-z)p_{t_2-r}(z-x_2) \\ &\qquad\qquad \times E[v_{n_1-1}(r,y)v_{n_2-1}(r,z)] \\ &\quad = \int_0^s dr E^X\Big[f(x_1+X^{(1)}_{t_1-r}-x_2-X^{(2)}_{t_2-r}) \\ &\qquad \times \Big(E[w_{n_1-1}(r;r,x_1+y)w_{n_2-1}(r;r,x_2+z)]\Big|_{y=X^{(1)}_{t_1-r},z=X^{(2)}_{t_2-r}}\Big)\Big]. \end{aligned} \tag{5.1}$$

But, as $\mathcal{N}(n_1-1,n_2-1)$ is composed of n_1-1 occurences of 1 and n_2-1 occurences of 2, the only element of $\mathcal{P}(n_1-1,n_2-1)$ is $((1,2),\dots,(1,2))$. As a consequence, by the induction assumption,

$$\begin{aligned} &E[w_{n_1-1}(r;r,x_1+y)w_{n_2-1}(r;r,x_2+z)] \\ &\quad = E^X\Bigg[u_0(x_1+y+X^{(1)}_r)u_0(x_2+z+X^{(2)}_r) \\ &\qquad \times \int_0^r dr_1\cdots\int_0^{r_{n-2}} dr_{n-1}\prod_{j=1}^{n-1} f(x_1+y+X^{(1)}_{r-r_j}-x_2-z-X^{(2)}_{r-r_j})\Bigg], \end{aligned}$$

with $n = M/2$, and

$$
\begin{aligned}
& E[w_{n_1-1}(r;r,x_1+y)w_{n_2-1}(r;r,x_2+z)] \\
&= E^X_{y,z}\Bigg[u_0(x_1+X^{(1)}_r)u_0(x_2+X^{(2)}_r) \\
&\quad \times \int_0^r dr_1\cdots\int_0^{r_{n-2}} dr_{n-1}\prod_{j=1}^{n-1} f(x_1+X^{(1)}_{r-r_j}-x_2-X^{(2)}_{r-r_j})\Bigg] \\
&= \int_0^r dr_1\cdots\int_0^{r_{n-2}} dr_{n-1} E^X_{y,z}\Bigg[u_0(x_1+X^{(1)}_r)u_0(x_2+X^{(2)}_r) \\
&\quad \times \prod_{j=1}^{n-1} f(x_1+X^{(1)}_{r-r_j}-x_2-X^{(2)}_{r-r_j})\Bigg].
\end{aligned}
$$

Further, by Lemma 5.2,

$$
\begin{aligned}
& E[w_{n_1-1}(r;r,x_1+y)w_{n_2-1}(r;r,x_2+z)]\Big|_{y=X^{(1)}_{t_1-r},z=X^{(2)}_{t_2-r}} \\
&= \int_0^r dr_1\cdots\int_0^{r_{n-2}} dr_{n-1} E^X_{X^{(1)}_{t_1-r},X^{(2)}_{t_2-r}}\Bigg[u_0(x_1+X^{(1)}_r)u_0(x_2+X^{(2)}_r) \\
&\quad \times \prod_{j=1}^{n-1} f(x_1+X^{(1)}_{r-r_j}-x_2-X^{(2)}_{r-r_j})\Bigg] \\
&= \int_0^r dr_1\cdots\int_0^{r_{n-2}} dr_{n-1} E^X\Bigg[u_0(x_1+X^{(1)}_{t_1})u_0(x_2+X^{(2)}_{t_2}) \\
&\quad \times \prod_{j=1}^{n-1} f(x_1+X^{(1)}_{t_1-r_j}-x_2-X^{(2)}_{t_2-r_j})\Bigg|\,\mathcal{F}^{(1)}_{t_1-r}\otimes\mathcal{F}^{(2)}_{t_2-r}\Bigg]. \qquad (5.2)
\end{aligned}
$$

Replacing (5.2) in (5.1), we obtain

$$
\begin{aligned}
& E[w_{n_1}(s;t_1,x_1)w_{n_2}(s;t_2,x_2)] \\
&= \int_0^s dr E^X\Bigg[f(x_1+X^{(1)}_{t_1-r}-x_2-X^{(2)}_{t_2-r})\int_0^r dr_1\cdots\int_0^{r_{n-2}} dr_{n-1} \\
&\quad \times E^X\Bigg[u_0(x_1+X^{(1)}_{t_1})u_0(x_2+X^{(2)}_{t_2}) \\
&\quad\quad \times \prod_{j=1}^{n-1} f(x_1+X^{(1)}_{t_1-r_j}-x_2-X^{(2)}_{t_2-r_j})\Bigg|\,\mathcal{F}^{(1)}_{t_1-r}\otimes\mathcal{F}^{(2)}_{t_2-r}\Bigg]\Bigg].
\end{aligned}
$$

Using Fubini's theorem, the fact that $f(x_1+X^{(1)}_{t_1-r}-x_2-X^{(2)}_{t_2-r})$ is $\mathcal{F}^{(1)}_{t_1-r}\otimes\mathcal{F}^{(2)}_{t_2-r}$-measurable and renumbering the integration variables, we obtain

$$\begin{aligned}
&E[w_{n_1}(s;t_1,x_1)w_{n_2}(s;t_2,x_2)]\\
&= \int_0^s dr_1\cdots\int_0^{r_{n-1}} dr_n E^X\Big[u_0(x_1+X^{(1)}_{t_1})u_0(x_2+X^{(2)}_{t_2})\\
&\qquad\times\prod_{j=1}^n f(x_1+X^{(1)}_{t_1-r_j}-x_2-X^{(2)}_{t_2-r_j})\Big]\\
&= E^X\Big[u_0(x_1+X^{(1)}_{t_1})u_0(x_2+X^{(2)}_{t_2})\\
&\qquad\times\int_0^s dr_1\cdots\int_0^{r_{n-1}} dr_n\prod_{j=1}^n f(x_1+X^{(1)}_{t_1-r_j}-x_2-X^{(2)}_{t_2-r_j})\Big].
\end{aligned}$$

As the only element in the set $\mathcal{P}(n_1,n_2)$ is $((1,2),\dots,(1,2))$, this proves the result in the case where $m=2$.

We will now prove the general formula by induction on the number of terms in the product m. We assume that the result is true for any number of terms in the product, up to $m-1$, and consider the case with m terms. Again, we are going to prove the result by induction on N. The smallest possible value is $N=m$.

Induction on m, case $N=m$. If we have m terms and $N=m$, then we must have $n_1=\cdots=n_m=1$. In that case, the process $s\mapsto w_1(s;t,x)$ is a martingale for all $t\geq 0$, $x\in\mathbb{R}^d$. We apply Itô's formula with the function $h(x_1,\cdots,x_m)=\prod_{j=1}^m x_j$, then take an expectation, which cancels the martingale term. We are only left with the expectation of the quadratic variation term. (For details, one can refer to [6, Proof of Lemma 6.2].) We finally obtain

$$\begin{aligned}
&E\left[\prod_{j=1}^m w_1(s;t_j,x_j)\right]\\
&= \sum_{i=1}^m\sum_{j=1}^{i-1}\int_0^s dr\int_{\mathbb{R}^d}dy\int_{\mathbb{R}^d}dz\,p_{t_i-r}(y-x_i)f(y-z)p_{t_j-r}(z-x_j)\\
&\qquad\times v_0(r,y)v_0(r,z)\times E\left[\prod_{\substack{k=1\\k\neq i,j}}^m w_1(r;t_k,x_k)\right]\\
&= \sum_{i=1}^m\sum_{j=1}^{i-1}\int_0^s drE^X\Big[f(x_i+X^{(i)}_{t_i-r}-x_j-X^{(j)}_{t_j-r})\\
&\qquad\times v_0(r,x_i+X^{(i)}_{t_i-r})v_0(r,x_j+X^{(j)}_{t_j-r})\Big]\times E\left[\prod_{\substack{k=1\\k\neq i,j}}^m w_1(r;t_k,x_k)\right].
\end{aligned}$$

We now handle the first term in the time integral as in the case $m = 2$ using Lemma 3.3. The second term is the expectation of a product of $m - 2$ terms. Hence, we can use the induction assumption on m to express it. If m is odd, then the second expectation in the integral vanishes and the whole expression as well. If m is even, we can use (3.4). Let $\tilde{\mathcal{P}}_{i,j}$ denote the set of ordered pairs of different integers of the set $\mathcal{N}(n_1, \dots, n_m)$, from which we delete the occurence of each i and j. We finally obtain

$$
\begin{aligned}
& E\left[\prod_{j=1}^{m} w_1(s; t_j, x_j)\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{i-1}\int_0^s dr E^X[u_0(x_i + X_{t_i}^{(i)})u_0(x_j + X_{t_j}^{(j)})f(x_i + X_{t_i-r}^{(i)} - x_j - X_{t_j-r}^{(j)})] \\
&\quad \times \sum_{\tilde{\mathcal{P}}_{i,j}} E^X\left[\prod_{\substack{k=1\\k\neq i,j}}^{m} u_0(x_k + X_{t_k}^{(k)})\int_0^r dr_1 \cdots \int_0^{r_{n-2}} dr_{n-1}\right. \\
&\qquad\qquad \left.\times \prod_{\ell=1}^{n-1} f(x_{p_\ell} + X_{t_{p_\ell}-r_\ell}^{(p_\ell)} - x_{q_\ell} - X_{t_{q_\ell}-r_\ell}^{(q_\ell)})\right],
\end{aligned}
$$

where $n = m/2$. By Fubini's theorem, a renumbering of the integration variables and the fact that

$$
\bigcup_{i=1}^{m}\bigcup_{j=i+1}^{m}\bigcup_{p\in\tilde{\mathcal{P}}_{i,j}}\{((i,j),p)\} = \mathcal{P}(\underbrace{1,\dots,1}_{m \text{ times}}),
$$

the result is proved for $N = m$.

Now, we consider the case where $N = m + 1$.

Induction on m, case $N = m + 1$. Without loss of generality, we can suppose that $n_1 = 2$ and $n_2 = \cdots = n_m = 1$. In that case, Itô's formula shows that

$$
\begin{aligned}
& E\left[w_2(s; t_1, x_1)\prod_{j=2}^{m} w_1(s; t_j, x_j)\right] \\
&= \sum_{i=2}^{m}\sum_{j=2}^{i-1}\int_0^s dr\int_{\mathbb{R}^d} dy\int_{\mathbb{R}^d} dz\, p_{t_i-r}(y - x_i)f(y - z)p_{t_j-r}(z - x_j) \\
&\qquad \times v_0(r, y)v_0(r, z) \times E\left[w_2(r; t_1, x_1)\prod_{\substack{k=2\\k\neq i,j}}^{m} w_1(r; t_k, x_k)\right]
\end{aligned}
$$

$$+\sum_{j=2}^{m}\int_0^s dr\int_{\mathbb{R}^d}dy\int_{\mathbb{R}^d}dz\, p_{t_1-r}(y-x_1)f(y-z)p_{t_j-r}(z-x_j)v_0(r,z)$$
$$\times E\left[w_1(r;r,y)\prod_{\substack{k=2\\k\neq j}}^{m}w_1(r;t_k,x_k)\right],$$

and

$$\begin{aligned}
&E\left[w_2(s;t_1,x_1)\prod_{j=2}^{m}w_1(s;t_j,x_j)\right]\\
&= \sum_{i=2}^{m}\sum_{j=2}^{i-1}\int_0^s drE^X\Big[f(x_i+X^{(i)}_{t_i-r}-x_j-X^{(j)}_{t_j-r})\\
&\quad\times v_0(r,x_1+X^{(i)}_{t_i-r})v_0(r,x_j+X^{(j)}_{t_j-r})\Big]E\left[w_2(r;t_1,x_1)\prod_{\substack{k=2\\k\neq i,j}}^{m}w_1(r;t_k,x_k)\right]\\
&\quad+\sum_{j=2}^{m}\int_0^s drE^X\Bigg[f(x_1+X^{(1)}_{t_1-r}-x_j-X^{(j)}_{t_j-r})v_0(r,x_j+X^{(j)}_{t_j-r})\\
&\quad\times\left(E\left[w_1(r;r,y)\prod_{\substack{k=2\\k\neq j}}^{m}w_1(r;t_k,x_k)\right]\Bigg|_{y=X^{(1)}_{t_1-r}}\right)\Bigg].
\end{aligned}\tag{5.3}$$

First, we can see that if m is even, then $N=m+1$ is odd. In that case, the last expectation in the first term in (5.3) corresponds to the case $m-2$, $N=m-1$ and hence vanishes by induction. The last expectation in brackets in the second term of (5.3) corresponds to the case $m-1$, $N=m-1$ and vanishes as well. Hence, the result is true if m is even. Now, if m is odd, we can handle the first term above with Lemma 3.3 and the induction assumption, since the second expectation does not depend on $X^{(i)}$ and $X^{(j)}$. For the second term, we first use the induction assumption, then Lemma 5.2 (Markov property) and Fubini's theorem. The arguments are analogous to those in the case $m=2$ and we skip the details. This proves the result for $N=m+1$.

Now, we conclude the proof by proving the result for m terms by induction on N. This will prove the induction step on m and conclude the proof.

Induction on m, induction on N. Suppose that the result is true for all $N\leq M-1$ and pick $n_1,\dots,n_m\in\mathbb{N}$ such that $\sum_{i=1}^{m}n_i=M$. By Itô's formula, we have

$$
\begin{aligned}
&E\left[\prod_{j=1}^{m} w_{n_j}(s;t_j,x_j)\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{i-1}\int_0^s dr\int_{\mathbb{R}^d} dy\int_{\mathbb{R}^d} dz\, p_{t_i-r}(y-x_i)f(y-z)p_{t_j-r}(z-x_j) \\
&\qquad \times E\left[w_{n_i-1}(r;r,y)w_{n_j-1}(r;r,z)\left(\prod_{\substack{k=1\\k\neq i,j}}^{m} w_{n_k}(r;t_k,x_k)\right)\right]. \qquad (5.4)
\end{aligned}
$$

Now, if $N=\sum_{j=1}^m n_j$ is odd, then

$$
n_i-1+n_j-1+\sum_{\substack{k=1\\k\neq i,j}}^{m} n_k = N-2
$$

is odd as well and the expectation above vanishes by induction. The result is proved for N odd. If N is even, then $N-2$ is even as well and we can use the induction assumption to compute the expectation in (5.4). We obtain

$$
\begin{aligned}
&E\left[\prod_{j=1}^{m} w_{n_j}(s;t_j,x_j)\right] \\
&= \sum_{i=1}^{m}\sum_{j=1}^{i-1}\int_0^s dr E^X\left[f(x_i+X^{(i)}_{t_i-r}-x_j-X^{(j)}_{t_i-r})g(X^{(i)}_{t_i-r},X^{(j)}_{t_j-r})\right], \qquad (5.5)
\end{aligned}
$$

where

$$
g(y,z) = E\left[w_{n_i-1}(r;r,x_i+y)w_{n_j-1}(r;r,x_j+z)\left(\prod_{\substack{k=1\\k\neq i,j}}^{m} w_{n_k}(r;t_k,x_k)\right)\right].
$$

Let

$$
\tilde{x}_k = \begin{cases} x_k & \text{if } k\neq i,j \\ x_i+y & \text{if } k=i \\ x_j+z & \text{if } k=j \end{cases} \qquad \text{and} \qquad \tilde{t}_k = \begin{cases} t_k & \text{if } k\neq i,j \\ r & \text{if } k=i,j. \end{cases}
$$

Then, by the induction assumption with $(\tilde{t}_1, \ldots, \tilde{t}_m)$ and $(\tilde{x}_1, \ldots, \tilde{x}_m)$,

$$
\begin{aligned}
&g(y,z)\\
&\quad= E\left[w_{n_1-1}(r;r,x_1+y)w_{n_2-1}(r;r,x_2+z)\left(\prod_{\substack{k=1\\k\neq i,j}}^m w_{n_k}(r;t_k,x_k)\right)\right]\\
&\quad= \sum_{\substack{\mathcal{P}(n_1,\ldots,n_i-1,\ldots,\\ n_j-1,\ldots,n_m)}} E^X\left[u_0(x_i+y+X_r^{(i)})u_0(x_j+z+X_r^{(j)})\prod_{\substack{k=1\\k\neq i,j}}^m u_0(x_k+X_{t_k}^{(k)})\right.\\
&\qquad\left.\times\int_0^r dr_1\cdots\int_0^{r_{n-2}} dr_{n-1}\prod_{\ell=1}^{n-1} f(\tilde{x}_{p_\ell}+X^{(p_\ell)}_{\tilde{t}_{p_\ell}-r_\ell}-\tilde{x}_{q_\ell}-X^{(q_\ell)}_{\tilde{t}_{q_\ell}-r_\ell})\right],\\
&\quad= \sum_{\substack{\mathcal{P}(n_1,\ldots,n_i-1,\ldots,\\ n_j-1,\ldots,n_m)}} \int_0^r dr_1\cdots\int_0^{r_{n-2}} dr_{n-1}E^X_{y,z}\left[u_0(x_i+X_r^{(i)})u_0(x_j+X_r^{(j)})\right.\\
&\qquad\left.\times\prod_{\substack{k=1\\k\neq i,j}}^m u_0(x_k+X_{t_k}^{(k)})\prod_{\ell=1}^{n-1} f(x_{p_\ell}+X^{(p_\ell)}_{\tilde{t}_{p_\ell}-r_\ell}-x_{q_\ell}-X^{(q_\ell)}_{\tilde{t}_{q_\ell}-r_\ell})\right].
\end{aligned}
$$

Further, by Lemma 5.2,

$$
\begin{aligned}
&g(X^{(i)}_{t_i-r},X^{(j)}_{t_j-r})\\
&\quad= \sum_{\substack{\mathcal{P}(n_1,\ldots,n_i-1,\ldots,\\ n_j-1,\ldots,n_m)}} \int_0^r dr_1\cdots\int_0^{r_{n-2}} dr_{n-1}\\
&\qquad E^X_{X^{(i)}_{t_i-r},X^{(j)}_{t_j-r}}\left[u_0(x_i+X_r^{(i)})u_0(x_j+X_r^{(j)})\times\prod_{\substack{k=1\\k\neq i,j}}^m u_0(x_k+X_{t_k}^{(k)})\right.\\
&\qquad\left.\times\prod_{\ell=1}^{n-1} f(x_{p_\ell}+X^{(p_\ell)}_{\tilde{t}_{p_\ell}-r_\ell}-x_{q_\ell}-X^{(q_\ell)}_{\tilde{t}_{q_\ell}-r_\ell})\right]\\
&\quad= \sum_{\substack{\mathcal{P}(n_1,\ldots,n_i-1,\ldots,\\ n_j-1,\ldots,n_m)}} \int_0^r dr_1\cdots\int_0^{r_{n-2}} dr_{n-1}E^X\left[\prod_{k=1}^m u_0(x_k+X_{t_k}^{(k)})\right.\\
&\qquad\left.\times\prod_{\ell=1}^{n-1} f(x_{p_\ell}+X^{(p_\ell)}_{t_{p_\ell}-r_\ell}-x_{q_\ell}-X^{(q_\ell)}_{t_{q_\ell}-r_\ell})\,\middle|\,\mathcal{F}^{(i)}_{t_i-r}\otimes\mathcal{F}^{(j)}_{t_j-r}\right],
\end{aligned}
\tag{5.6}
$$

because $\tilde{t}_{p_\ell} - r_\ell + t_{p_\ell} - r = t_{p_\ell} - r_k$, whenever $p_\ell = i$ or j. Replacing (5.6) in (5.5), we obtain

$$\begin{aligned} &E\left[\prod_{j=1}^{m} w_{n_j}(s;t_j,x_j)\right] \\ &= \sum_{i=1}^{m}\sum_{j=1}^{i-1}\int_0^s dr E^X \Bigg[f(x_i + X^{(i)}_{t_i-r} - x_j - X^{(j)}_{t_j-r}) \\ &\quad \times \sum_{\substack{\mathcal{P}(n_1,\dots,n_i-1,\dots,\\ n_j-1,\dots,n_m)}} \int_0^r dr_1 \cdots \int_0^{r_{n-2}} dr_{n-1} E^X \Bigg[\prod_{k=1}^{m} u_0(x_k + X^{(k)}_{t_k}) \\ &\quad \times \prod_{\ell=1}^{n-1} f(x_{p_\ell} + X^{(p_\ell)}_{t_{p_\ell}-r_\ell} - x_{q_\ell} - X^{(q_\ell)}_{t_{q_\ell}-r_\ell}) \Bigg| \mathcal{F}^{(i)}_{t_i-r} \otimes \mathcal{F}^{(j)}_{t_j-r}\Bigg]\Bigg]. \end{aligned}$$

Using Fubini's theorem, the fact that $f(x_i + X^{(i)}_{t_i-r} - x_j - X^{(j)}_{t_j-r})$ is $\mathcal{F}^{(i)}_{t_i-r} \otimes \mathcal{F}^{(j)}_{t_j-r}$-measurable, renumbering the integration variables and the fact that

$$\bigcup_{i=1}^{m}\bigcup_{j=i+1}^{m}\bigcup_{\substack{p\in\mathcal{P}(n_1,\dots,n_i-1,\dots,\\ n_j-1,\dots,n_m)}} \{((i,j),p)\} = \mathcal{P}(n_1,\dots,n_m),$$

we obtain

$$\begin{aligned} &E\left[\prod_{j=1}^{m} w_{n_j}(s;t_j,x_j)\right] \\ &= \sum_{\mathcal{P}(n_1,\dots,n_m)} \int_0^s dr_1 \cdots \int_0^{r_{n-1}} dr_n \; E^X \Bigg[\prod_{k=1}^{m} u_0(x_k + X^{(k)}_{t_k}) \\ &\qquad \times \prod_{\ell=1}^{n} f(x_{p_\ell} + X^{(p_\ell)}_{t_{p_\ell}-r_\ell} - x_{q_\ell} - X^{(q_\ell)}_{t_{q_\ell}-r_\ell})\Bigg] \\ &= \sum_{\mathcal{P}(n_1,\dots,n_m)} E^X \Bigg[\prod_{k=1}^{m} u_0(x_k + X^{(k)}_{t_k}) \times \int_0^s dr_1 \cdots \int_0^{r_{n-1}} dr_n \\ &\qquad \times \prod_{\ell=1}^{n} f(x_{p_\ell} + X^{(p_\ell)}_{t_{p_\ell}-r_\ell} - x_{q_\ell} - X^{(q_\ell)}_{t_{q_\ell}-r_\ell})\Bigg]. \end{aligned}$$

One should notice that in the case where one of the n_j's is equal to 1, then the argument is similar but slightly different. Indeed, $w_{n_j-1} = w_0$ and, hence, w_{n_j-1} comes out of the expectation. We can still apply the induction assumption for the expectation, but we also have to apply Lemma 3.3 for the additional w_0 outside the expectation. The result is proved. □

Acknowledgment. I would like to thank an anonymous referee, Frederi Viens and Davar Khoshnevisan for their valuable suggestions which improved this paper a lot.

References

1. Balazs, M. & Quastel, J. & Seppäläinen, T.: Scaling exponent for the Hopf-Cole solution of KPZ/Stochastic Burgers equation. *J. Amer. Math. Soc.* **24** (2011), 683–708.
2. Bertini L. & Cancrini N.: The stochastic heat equation: Feynman-Kac formula and intermittence. *J. Stat. Phys.* **78**, n5-6 (1995) 1377–1401.
3. Bertoin, J.: *Lévy Processes.* Cambridge University Press, 1996.
4. Carmona, R. A. & Molchanov, S.A.: Parabolic Anderson Problem and Intermittency. *Mem. Am. Math. Soc.* **518** (1994).
5. Conus, D.: The non-linear stochastic wave equation in high dimensions: existence, Hölder-continuity and Itô-Taylor expansion. *Ph.D Thesis #4265*, EPFL Lausanne (2008).
6. Conus, D. & Dalang, R.C.: The non-linear stochastic wave equation in high dimensions. *Electron. J. Probab.* **13** (2008), 629–670.
7. Conus, D. & Joseph, M. & Khoshnevisan, D.: On the chaotic character of the stochastic heat equation, before the onset of intermittency. *Ann. Probab.*, to appear.
8. Conus, D. & Joseph, M. & Khoshnevisan, D. & Shiu, S.-Y.: On the chaotic character of the stochastic heat equation, II. *Probab. Theory Rel.*, to appear.
9. Dalang, R.C.: Extending martingale measure stochastic integral with applications to spatially homogeneous spde's. *Electron. J. Probab.* **4** (1999), 1–29.
10. Dalang, R.C. & Frangos, N.E.: The stochastic wave equation in two spatial dimensions. *Ann. Probab.* **26**, n1 (1998), 187–212.
11. Dalang, R.C. & Khohsnevisan, D. & Mueller, C. & Nualart, D. & Xiao, Y.: A Minicourse in Stochastic Partial Differential Equations. *Lect. Notes Math.*, **1962** (2006), Springer.
12. Dalang, R.C. & Mueller, C.: Some non-linear s.p.d.e's that are second order in time. *Electron. J. Probab.* **8** (2003), 1–21.
13. Dalang, R.C. & Mueller, C. & Tribe, R.: A Feynman-Kac-type formula for the deterministic and stochastic wave equations and other p.d.e.'s. *T. Am. Math. Soc.* **360**, n9 (2008), 4681–4703.
14. Foondun, M. & Khoshnevisan, D.: Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron. J. Probab.* **14** (2009), 548–568.
15. Foondun, M. & Khoshnevisan, D. & Nualart, E.: A local-time correspondence for stochastic partial differential equations. *T. Am. Math. Soc.*, **363**, n5 (2011), 2481–2515.
16. Hairer, M.: Solving the KPZ equation. *Ann. Math.* (2013), to appear.
17. Hu, Y. & Nualart, D.: Stochastic heat equation driven by fractional noise and local time. *Probab. Theory Rel.*, **143**, n1-2 (2009), 285–328.
18. Kardar, M. & Parisi, G. & Zhang, Y.-C.: Dynamic scaling of growing interfaces. *Phys. Rev. Lett.* **56** (1986), 889–892.
19. Walsh, J.B.: An Introduction to Stochastic Partial Differential Equations. In : *Ecole d'Et de Probabilits de St-Flour, XIV (1984).* Lecture Notes in Mathematics **1180** (1986) Springer, 265–439.

Daniel Conus: Mathematics Department, Lehigh University, 14 East Packer Avenue, Bethlehem, PA 18109, USA.
E-mail address: daniel.conus@lehigh.edu
URL: http://www.lehigh.edu/∼dac311