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TWO-DIMENSIONAL MAGNETO-HYDRODYNAMIC SYSTEM WITH JUMP PROCESSES: WELL POSEDNESS AND INVARIANT MEASURES

UTPAL MANNA AND MANIL T. MOHAN

Abstract. In this work we prove the existence and uniqueness of the strong solution to the two-dimensional stochastic magneto-hydrodynamic system perturbed by Lévy noise. The local monotonicity arguments have been exploited in the proofs. The existence of a unique invariant measures has been proved using the exponential stability of solutions.

1. Introduction

Magneto-hydrodynamics (MHD) is the branch of fluid mechanics which studies the motion of electrically conducted fluid in the presence of magnetic field. MHD system consists of the Navier-Stokes equations, which describe the motion of a fluid in the electric field coupled with the Maxwell equations, which describe the motion of the fluid in the magnetic field (see Chandrasekhar [6]). The deterministic MHD system has been studied by mathematicians and physicists (e.g. Cabannes [5], Cowling [8], Ladyzhenskaya [20], Landau and Lifshitz [21], Sermange and Temam [33], and Temam [39]) over the past fifty years. Significant contributions in the mathematical study of stochastic MHD system are due to Sritharan and Sundar [34], Sundar [37], and Barbu and Da Prato [3]. The authors in [34] prove the well posedness of the martingale problem associated to two and three-dimensional MHD system perturbed by white noise. In [37], the author study the existence and uniqueness of solutions to the two-dimensional MHD system perturbed by more general noise, namely multiplicative noise and additive fractional Brownian noise. The authors in [3] establish the existence and uniqueness of an invariant measure via coupling methods developed by Odasso [26].

In this paper we consider the stochastic MHD system in the non-dimensional form. Let $G \subset \mathbb{R}^2$ be a bounded, open, simply-connected domain with a sufficiently smooth boundary ∂G, then

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \frac{1}{R_e}\Delta u - S(B \cdot \nabla)B + \nabla\left(p + \frac{S|B|^2}{2}\right)$$
$$= \sigma_1(u, B)\partial W_1(t) + \int_Z g_1(u, z)\tilde{N}_1(dt, dz) \text{ in } G \times (0, T),$$

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along with the Maxwell's equations

$$\frac{\partial B}{\partial t}+(u\cdot\nabla)B+\frac{1}{R_m}\text{curl}(\text{curl}\,B)-(B\cdot\nabla)u$$
$$=\sigma_2(u,B)\partial W_2(t)+\int_Z g_2(u,z)\tilde{N}_2(dt,dz)\text{ in }G\times(0,T),$$

where the terms and the initial and the boundary conditions are discussed in section 2. We reduce the above coupled equations into a single equation whose abstract form is given by

$$\mathrm{d}\mathbf{X}+[\mathbf{A}\mathbf{X}+\mathbf{B}(\mathbf{X})]\mathrm{d}t=\sigma(t,\mathbf{X})\mathrm{d}W(t)+\int_Z g(\mathbf{X},z)\tilde{N}(dt,dz)$$

with $\mathbf{X}(0)=\mathbf{X}_0$, where $\mathbf{X}$ denotes the transpose of (u,B). The operators $\mathbf{A}$ and $\mathbf{B}$ are defined in section 2. $W(t)$ is an H-valued Wiener process with positive symmetric trace class covariance operator Q. $\tilde{N}(dt,dz)=N(dt,dz)-\lambda(dz)dt$ is a compensated Poisson random measure(cPrm), where $N(dt,dz)$ denotes the Poisson random measure associated to Poisson point process $\mathbf{p}(t)$ on Z and $\lambda(dz)$ is a σ-finite Lévy measure on $(Z,\mathscr{B}(Z))$.

The construction of the paper is as follows. In the next section, we give the description of the deterministic MHD system and describe the functional setting of the problem. In section 3, we first formulate the abstract stochastic model and then prove certain a-priori energy estimates (both L^2 and L^p) with exponential weights. We then prove the existence and uniqueness of the strong solution exploiting certain monotonicity property of the operators. In section 4, existence of a unique invariant measures has been established using the tightness property and the exponential stability of solutions.

2. The Deterministic MHD System

In this section we first describe the deterministic MHD System with suitable functional setting for the model. Also we will discuss about the linear and nonlinear operators and their properties. Some of the results of this section are taken from Sritharan and Sundar [34], Sundar [37], and Temam [39]. Some results presented in this section are known but given for the sake of completeness.

2.1. The Description of the Deterministic MHD System. We consider the motion of a viscous incompressible and resistive fluid filling G, where $G\subset\mathbb{R}^2$ is a bounded, open, simply-connected, domain with a sufficiently smooth boundary ∂G. We fix a final time $T>0$. The motion described by the MHD system (see Temam [39]) in the non-dimensional form is given by

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u-\frac{1}{R_e}\Delta u-S(B\cdot\nabla)B+\nabla\left(p+\frac{S|B|^2}{2}\right)=f_1(t)\text{ in }G\times(0,T),\tag{2.1}$$

where p denotes the pressure of the fluid, $u(t,x)=(u_1(t,x),u_2(t,x))$ the velocity of the particle of fluid which is at the point x at time t, $B(t,x)=(B_1(t,x),B_2(t,x))$ the magnetic field at a point x at time t and R_e the Reynolds number. Let $S=\frac{M^2}{R_eR_m}$, where M is the Hartman number and R_m is the magnetic Reynolds number, Δ denotes the Laplace operator and ∇ the gradient. The Maxwell's equation is given by

$$\frac{\partial B}{\partial t} + (u\cdot\nabla)B + \frac{1}{R_m}\mathrm{curl}(\mathrm{curl}\, B) - (B\cdot\nabla)u = f_2(t) \text{ in } G\times(0,T), \tag{2.2}$$

where

$$\nabla\cdot u = 0 \text{ and } \nabla\cdot B = 0 \text{ in } G\times(0,T), \tag{2.3}$$

where $f_1(t)$ and $f_2(t)$ are the external forces acting on the system.

The above system of equations deduced from the coupled equation of Navier-Stokes and the Maxwell's equations. The Navier-Stokes part is

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u = \frac{j\times B}{\rho} + \frac{qE}{\rho} - \frac{\nabla p}{\rho} + \nu\Delta u,\ \ \nabla\cdot u = 0,$$

where E is the electric field, ρ is the mass density, $-q$ is the charge of an electron and ν is the viscosity. Let μ_0 be the permeability of the free space, ρ_c be the charge density and c be the speed of light, the Maxwell's part is

$$\frac{\partial B}{\partial t} = -\mathrm{curl}E,\ \ \mathrm{curl}B = \mu_0 j + \frac{1}{c^2}\frac{\partial E}{\partial t},\ \ \nabla\cdot B = 0,\ \ \nabla\cdot E = \rho_c.$$

But in most of the practical applications, only low-frequency behavior is studied and thus the term $\frac{q}{\rho}E$ is ignored from the Navier-Stokes part and the displacement current, $\frac{1}{c^2}\frac{\partial E}{\partial t}$ from the Maxwell's part. A resistive form of Ohm's law is used to eliminate E, the electric field from the equations, $j = \sigma(E + u\times B)$, where σ denotes the electric conductivity of the fluid. (For more details see Chapter 8 of Landau, Lifshitz, and Pitaevskii [22] and Section 7.7 of Jackson [17].) For a vector-valued function $u = (u_1, u_2)$ defined on G, $\mathrm{curl}\, u = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$. Let ϕ be a scalar-valued function defined on G, then $\mathrm{curl}\,\phi = \left(\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x}\right)$. Also we know that $\mathrm{curl}(\mathrm{curl}\, u) = \mathrm{grad}\,\mathrm{div}\, u - \Delta u$ is valid in the two dimensions. Now let us give the boundary conditions for the equations (2.1) and (2.2).

$$u(t,x) = 0 \text{ on } (0,T)\times\partial G, \tag{2.4}$$

$$B\cdot\hat{n} = 0 \text{ and } \mathrm{curl}\, B = 0 \text{ on } (0,T)\times\partial G. \tag{2.5}$$

The initial conditions are given by

$$u(x,0) = u_0(x), \quad B(x,0) = B_0(x) \text{ for all } x\in G. \tag{2.6}$$

Here $\hat{n}$ denotes the unit outward normal to ∂G. Also (2.4) is the nonslip condition and (2.5) describes a perfectly conducting wall. The system (2.1)-(2.6) needs to be written as an abstract evolution equation with

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \left[-\mathbf{A}\mathbf{X}(s) - \mathbf{B}\left(\mathbf{X}(s)\right)\right] \mathrm{d}s + \int_0^t \mathbf{f}(t)dt, \tag{2.7}$$

for all $t\in[0,T]$. In the above formulation, $\mathbf{X}$ denotes the transpose of (u, B) and $\mathbf{f}(t)$ is the transpose of $\{f_1(t), f_2(t)\}$.

2.2. Abstract Formulation of the Equations. In this section we collect the needed information for the operator formulation of the problem (2.1)-(2.6) as in Sermanage and Temam [33]. We consider the spaces $H = H_1 \times H_2$ and $V = V_1 \times V_2$, where

$$\begin{aligned} H_1 &= \left\{\phi \in \mathrm{L}^2(G) : \nabla \cdot \phi = 0, \ \ \phi \cdot \hat{n}\big|_{\partial G} = 0\right\}, H_2 = H_1, \\ V_1 &= \left\{\phi \in H_0^1(G) : \nabla \cdot \phi = 0\right\}, \ V_2 = \left\{\phi \in H^1(G) : \nabla \cdot \phi = 0, \phi \cdot \hat{n}\big|_{\partial G} = 0\right\}, \\ H^1 &= \left\{u \in \mathrm{L}^2(G) : \nabla u \in \mathrm{L}^2(G)\right\}, H_0^1 = \left\{u \in \mathrm{L}^2(G) : \nabla u \in \mathrm{L}^2(G), u\big|_{\partial G} = 0\right\}. \end{aligned}$$

Here H_1 and H_2 are equipped with the $\mathrm{L}^2(G)$ norm. Let us define the inner product on H by $[\mathbf{X}_1, \mathbf{X}_2] = (u_1, u_2)_{H_1} + S(B_1, B_2)_{H_2}$, where $\mathbf{X}_i = \{u_i, B_i\}$. Hence we get the inner product is equivalent to $(\mathbf{X}_1, \mathbf{X}_2)_H = (u_1, u_2)_{H_1} + (B_1, B_2)_{H_2}$ and the norm on H is given by $\|\mathbf{X}\|_H = \sqrt{(\mathbf{X}, \mathbf{X})_H}$. Also the space V_1 is endowed with the inner product given by $[[\phi, \psi]]_{V_1} = (\nabla\phi, \nabla\psi)_{\mathrm{L}^2(G)}$. The norm on V_1 is given by $\|\phi\|_{V_1} = \sqrt{[[\phi, \phi]]_{V_1}}$. Let us note that V_1 norm given here is equivalent to the usual $H^1(G)$ norm since $\|\nabla\phi\|_{\mathrm{L}^2(G)} \geq C\|\phi\|_{\mathrm{L}^2(G)}$ for a suitable C by the Poincaré inequality. Let V_2 be endowed with the inner product $[[\phi, \psi]]_{V_2} = (\mathrm{curl}\phi, \mathrm{curl}\psi)_{\mathrm{L}^2(G)}$ and the norm on V_2 is given by $\|\phi\|_{V_2} = \sqrt{[[\phi, \phi]]_{V_2}}$. From Proposition 1.8 and Lemma 1.6 of Temam [38], we conclude that the norm given by

$$\left\{\|\phi\|^2_{\mathrm{L}^2(G)} + \|\mathrm{curl}\,\phi\|^2_{\mathrm{L}^2(G)} + \|\phi \cdot \hat{n}\|^2_{H^{1/2}(\partial G)}\right\}^{1/2}$$

is equivalent to the $H^1(G)$ norm. Thus the space V is endowed with the scalar product: $[[\mathbf{X}_1, \mathbf{X}_2]] = [[u_1, u_2]]_{V_1} + S[[B_1, B_2]]_{V_2}$. Having defined the spaces H and V, we have the dense, continuous and compact embedding: $V \hookrightarrow H \equiv H' \hookrightarrow V'$.

2.3. The linear operator. In this section we define the operator $\mathbf{A}$ that appear in (2.7) and discuss its properties. $\mathbf{A}$ is defined through a bilinear coercive form.

Let us define a function $a : V \times V \to \mathbb{R}$ as follows:

$$a(\mathbf{X}_1, \mathbf{X}_2) = \frac{1}{R_e}[[u_1, u_2]]_{V_1} + \frac{S}{R_m}[[B_1, B_2]]_{V_2}. \tag{2.8}$$

Proposition 2.1. *The function $a(\cdot, \cdot)$ defined by* (2.8) *is continuous and coercive.*

Proof. Without loss of generality let us assume that R_e and R_m be equal to 1. Then there exists a constant k such that

$$\begin{aligned} |a(\mathbf{X}_1, \mathbf{X}_2)| &= \left|[[u_1, u_2]]_{V_1} + S[[B_1, B_2]]_{V_2}\right| \\ &\leq \|\nabla u_1\|_{\mathrm{L}^2(G)}\|\nabla u_2\|_{\mathrm{L}^2(G)} + S\|\mathrm{curl}\,B_1\|_{\mathrm{L}^2(G)}\|\mathrm{curl}\,B_2\|_{\mathrm{L}^2(G)} \\ &\leq k\Big(\|u_1\|_{H^1(G)}\|u_2\|_{H^1(G)} + S\|B_1\|_{H^1(G)}\|B_2\|_{H^1(G)}\Big) \\ &\leq k\Big(\|u_1\|^2_{H^1(G)} + \|B_1\|^2_{H^1(G)}\Big)^{1/2}\Big(\|u_2\|^2_{H^1(G)} + \|B_2\|^2_{H^1(G)}\Big)^{1/2} \\ &\leq k\|\mathbf{X}_1\|_V\|\mathbf{X}_2\|_V. \end{aligned}$$

Hence we obtain $|a(\mathbf{X}_1,\mathbf{X}_2)| \leq k\|\mathbf{X}_1\|_V\|\mathbf{X}_2\|_V$ for all $\mathbf{X}_1,\mathbf{X}_2 \in V$. For the coercive property we use the estimate $\frac{1}{k}\|u\|_{H^1(G)} \leq \|u\|_{\mathrm{L}^2(G)} \leq k\|u\|_{H^1(G)}$, consider

$$a(\mathbf{X},\mathbf{X}) = [[u,u]]_{V_1} + S[[B,B]]_{V_2} = \|\nabla u\|^2 + S\|\mathrm{curl}\, B\|^2$$
$$\geq c\Big(\|u\|^2_{H^1(G)} + \|B\|^2_{H^1(G)}\Big) = c\|\mathbf{X}\|^2_V.$$

Hence we get for a positive constant c, $a(\mathbf{X},\mathbf{X}) \geq c\|\mathbf{X}\|^2_V$ for all $\mathbf{X} \in V$. □

By Lax-Milgram Theorem, there exists an operator $\mathbf{A}: V \to V'$ such that

$$a(\mathbf{X},\mathbf{Y}) = \big(\mathbf{A}\mathbf{X},\mathbf{Y}\big)_{V\times V'} \text{ for all } \mathbf{X},\mathbf{Y} \in V. \tag{2.9}$$

Therefore $\mathbf{A}: V \to V'$ can be restricted to a self-adjoint operator $\mathbf{A}: \mathfrak{D}(\mathbf{A}) \to H$. Now we can write $\mathfrak{D}(\mathbf{A}) = \mathfrak{D}(\mathbf{A}_1) \times \mathfrak{D}(\mathbf{A}_2)$ where $\mathbf{A}_1, \mathbf{A}_2, \mathfrak{D}(\mathbf{A}_1), \mathfrak{D}(\mathbf{A}_2)$ are obtained as follows:

Let us consider the "Stokes" problem in G:

$$\frac{-1}{R_e}\Delta u + \nabla p = g \text{ with } \nabla\cdot u = 0 \text{ and } u\big|_{\partial G} = 0.$$

Let $\phi \in V_1$ and multiply the above equation with ϕ. Apply the integration by parts formula to obtain $\frac{1}{R_e}[[u,\phi]]_{V_1} = (g,\phi)$ for $\phi \in V_1$. Define $\mathbf{A}_1 u = \frac{1}{R_e}[[u,\cdot]]_{V_1} = g \in V_1'$. By the Cattabriga regularity theorem, we can conclude that $\mathfrak{D}(\mathbf{A}_1) = H^2(G)\cap V_1$ whenever $g \in H_1$.

Consider the elliptic problem in G for defining $\mathbf{A}_2$:

$$\frac{1}{R_m}\mathrm{curl}\,\mathrm{curl}\, B = g \text{ with } \nabla\cdot B; \quad B\cdot\hat{n}\big|_{\partial G} = 0; \quad \mathrm{curl}\ \mathrm{B}\big|_{\partial G} = 0,$$

so that $\mathbf{A}_2 B = \frac{1}{R_m}\mathrm{curl}\,\mathrm{curl}\ \mathrm{B} = g$. It is proved in [33] that if $g \in H$, then

$$\mathfrak{D}(\mathbf{A}_2) = \big\{u \in H^2(G) : \nabla\cdot u = 0, u\cdot\hat{n}\big|_{\partial G} = 0 \text{ and } \mathrm{curl}\ \mathrm{u}\big|_{\partial G} = 0\big\}.$$

2.4. The nonlinear operator. We next define $\mathbf{B}$ that figures in (2.7). Let us consider the trilinear form $b(\cdot,\cdot,\cdot): V\times V\times V \to \mathbb{R}$ defined by

$$b(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3) = \tilde{b}(u_1,u_2,u_3) - S\tilde{b}(B_1,B_2,u_3) + S\tilde{b}(u_1,B_2,B_3) - S\tilde{b}(B_1,u_2,B_3), \tag{2.10}$$

for all $\mathbf{X}_i = (u_i, B_i) \in V$ where $\tilde{b}(\cdot,\cdot,\cdot) : (H^1(G))^{\otimes 3} \to \mathbb{R}$ is defined by

$$\tilde{b}(\phi,\psi,\theta) = \left(\sum_{i,j=1}^{2}\int_G \phi_i \frac{\partial\psi_j}{\partial x_i}\theta_j \mathrm{d}x\right).$$

Proposition 2.2. *The function $b(\cdot,\cdot,\cdot)$ defined by* (2.10) *is continuous.*

Proof. In order to prove the continuity of b, we need to show the continuity of $\tilde{b}$. Let us first consider $\tilde{b}(\phi,\psi,\theta)$ and by using the Hölder's inequality, we have

$$|\tilde{b}(\phi,\psi,\theta)| = \left|\sum_{i,j}\int_G \phi_i\frac{\partial\psi_j}{\partial x_i}\theta_j\mathrm{d}x\right| \leq \|\phi\|_{\mathrm{L}^4(G)}\|\nabla\psi\|_{\mathrm{L}^2(G)}\|\theta\|_{\mathrm{L}^4(G)}.$$

Since $H^1(G) \subset \mathrm{L}^4(G)$, we get $|\tilde{b}(\phi,\psi,\theta)| \leq c\|\phi\|_{H^1}\|\psi\|_{H^1}\|\theta\|_{H^1}$. Now,

$$\begin{aligned}
&|b(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3)| \\
&\leq |\tilde{b}(u_1,u_2,u_3)| + S|\tilde{b}(B_1,B_2,u_3)| + S|\tilde{b}(u_1,B_2,B_3)| + S|\tilde{b}(B_1,u_2,B_3)| \\
&\leq c\Big(\|u_1\|_{H^1}\|u_2\|_{H^1}\|u_3\|_{H^1} + \|B_1\|_{H^1}\|B_2\|_{H^1}\|u_3\|_{H^1} \\
&\quad + \|u_1\|_{H^1}\|B_2\|_{H^1}\|B_3\|_{H^1} + \|B_1\|_{H^1}\|u_2\|_{H^1}\|B_3\|_{H^1}\Big) \\
&\leq c\Big(\|u_1\|^2_{H^1(G)} + \|B_1\|^2_{H^1(G)}\Big)^{1/2}\Big(\|u_2\|^2_{H^1(G)} + \|B_2\|^2_{H^1(G)}\Big)^{1/2} \\
&\qquad \Big(\|u_3\|^2_{H^1(G)} + \|B_3\|^2_{H^1(G)}\Big)^{1/2} \\
&\leq c\|\mathbf{X}_1\|_V\|\mathbf{X}_2\|_V\|\mathbf{X}_3\|_V,
\end{aligned}$$

since $\|\mathbf{X}\|_V$ is equivalent to $\Big(\|u\|^2_{H^1(G)} + \|B\|^2_{H^1(G)}\Big)^{1/2}$. □

Note that $\tilde{b}(\phi,\psi,\theta) = -\tilde{b}(\phi,\theta,\psi)$ if ψ or $\theta \in V_1$ and $\phi \in H_1$. Therefore we have $b(\mathbf{X},\mathbf{Y},\mathbf{Z}) = -b(\mathbf{X},\mathbf{Z},\mathbf{Y})$ by using (2.10). Also note that for all $\phi \in H^1(G)$ and $\psi \in V_1$, $\tilde{b}(\phi,\psi,\psi) = \sum_{i,j=1}^2 \int_G \phi_i \frac{\partial \psi_j}{\partial x_i}\psi_j \mathrm{d}x = 0$, by integration by parts. Therefore by (2.10), we get, $b(\mathbf{X},\mathbf{Y},\mathbf{Y}) = 0$. We now define $\mathbf{B} : V \times V \to V'$ as the continuous bilinear operator such that $b(\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3) = \big(\mathbf{B}(\mathbf{X}_1,\mathbf{X}_2),\mathbf{X}_3\big)$ for all $\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3 \in V$. Let the H-norm of $\mathbf{X}$ be denoted as $|\mathbf{X}|$ and the V-norm of $\mathbf{X}$ be denoted as $\|\mathbf{X}\|$. The existence of the operator $\mathbf{B}$ is justified by the Riesz representation theorem and let $\mathbf{B}(\mathbf{X})$ denote $\mathbf{B}(\mathbf{X},\mathbf{X})$. It is well known that for any $\mathbf{X} \in V$, $\|\mathbf{B}(\mathbf{X})\|_{V'} \leq C|\mathbf{X}|\|\mathbf{X}\|$. Since the constant C do not play a crucial role in this paper, we will set $C = 1$.

3. Stochastic MHD System with Lévy noise

3.1. Basic Concepts. In this sub-section definitions of Hilbert space valued Wiener processes and Lévy processes have been presented. Interested readers may look into Da Prato and Zabczyk [10], Applebaum [1], Peszat and Zabczyk [29], Kingman [18] for extensive study on the subject.

Definition 3.1. Let H be a Hilbert space. A stochastic process $\{W(t)\}_{0\leq t\leq T}$ is said to be an *H-valued $\mathscr{F}_t$-adapted Wiener process* with covariance operator Q if

(i) For each non-zero $h \in H$, $|Q^{1/2}h|^{-1}(W(t),h)$ is a standard one - dimensional Wiener process,
(ii) For any $h \in H, (W(t),h)$ is a martingale adapted to $\mathscr{F}_t$.

If W is a an H-valued Wiener process with covariance operator Q with $\mathrm{Tr}\, Q < \infty$, then W is a Gaussian process on H and $\mathbb{E}(W(t)) = 0$, Cov $(W(t)) = tQ$, $t \geq 0$. Let $H_0 = Q^{1/2}H$. Then H_0 is a Hilbert space equipped with the inner product $(\cdot,\cdot)_0$, $(u,v)_0 = \big(Q^{-1/2}u, Q^{-1/2}v\big)$, $\forall u,v \in H_0$, where $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$. Since Q is a trace class operator, the imbedding of H_0 in H is Hilbert-Schmidt. Let L_Q denote the space of linear operators S such that $SQ^{1/2}$ is a Hilbert-Schmidt operator from H to H.

Let $\mathbb{M}$ be the totality of non-negative (possibly infinite) integral valued measures on $(H, \mathscr{B}(H))$ and $\mathscr{B}_{\mathbb{M}}$ be the smallest σ-field on $\mathbb{M}$ with respect to which all $N \in \mathbb{M} \to N(B) \in \mathbb{Z}^+ \cup \{\infty\}, B \in \mathscr{B}(H)$, are measurable.

Definition 3.2. An $(\mathbb{M}, \mathscr{B}(\mathbb{M}))$-valued random variable N is called a *Poisson random measure*

(1) if for each $B \in \mathscr{B}(H), N(B)$ is Poisson distributed. i.e., $\mathbb{P}(N(B) = n) = \eta(B)e^{-\eta(B)}/n!, n = 0, 1, 2, \ldots,$ where $\eta(B) = \mathbb{E}(N(B)), B \in \mathscr{B}(H)$;
(2) if $B_1, B_2, \ldots, B_n \in \mathscr{B}(H)$ are disjoint, then $N(B_1), N(B_2), \ldots, N(B_n)$ are mutually independent.

A *point function* $\mathbf{p}$ on H is a mapping $\mathbf{p} : D_{\mathbf{p}} \subset (0, \infty) \to H$, where the domain $D_{\mathbf{p}}$ is a countable subset of $(0, \infty)$. $\mathbf{p}$ defines a counting measure $N_{\mathbf{p}}$ on $(0, \infty) \times H$ by $N_{\mathbf{p}}((0, t] \times Z, \omega) = \# \{s \in D_{\mathbf{p}}(\omega); s \leq t, \mathbf{p}(s, \omega) \in Z\}, t > 0, Z \in \mathscr{B}(H), \omega \in \Omega$. Let Π_H be the totality of point functions on H and $\mathscr{B}(\Pi_H)$ be the smallest σ-field on Π_H with respect to which all $\mathbf{p} \to N_{\mathbf{p}}((0, t] \times Z), t > 0, Z \in \mathscr{B}(H)$, are measurable.

Definition 3.3. A point process $\mathbf{p}$ on H is a $(\Pi_H, \mathscr{B}(\Pi_H))$-valued random variable, that is, a mapping $\mathbf{p} : \Omega \to \Pi_H$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ which is $\mathscr{F}/\mathscr{B}(\Pi_H)$-measurable. A point process $\mathbf{p}$ is called *Poisson point process* if $N_{\mathbf{p}}(\cdot)$, as defined above, is a Poisson random measure on $(0, \infty) \times H$.

Lévy process is a special type of Poisson random measure associated to a Poisson point process.

Definition 3.4. A càdlàg adapted process, $(\mathbf{L}_t)_{t \geq 0}$, is called a *Lévy process* if it has stationary independent increments and is stochastically continuous.

Let $(\mathbf{L}_t)_{t \geq 0}$ be a H-valued Lévy process. Hence, for every $\omega \in \Omega$, $\mathbf{L}_t(\omega)$ has countable number of jumps on $[0, t]$. Note that for every $\omega \in \Omega$, the jump $\triangle \mathbf{L}_t(\omega) = \mathbf{L}_t(\omega) - \mathbf{L}_{t-}(\omega)$ is a point function in $\mathscr{B}(H \backslash \{0\})$. Let us define $N(t, Z) = N(t, Z, \omega) = \# \{s \in (0, \infty) : \triangle \mathbf{L}_s(\omega) \in Z\}, t > 0, Z \in \mathscr{B}(H \backslash \{0\}), \omega \in \Omega$ as the *Poisson random measure associated with the Lévy process* $(\mathbf{L}_t)_{t \geq 0}$.

The differential form of the measure $N(t, Z, \omega)$ is written as $N(dt, dz)(\omega)$. We call $\tilde{N}(dt, dz) = N(dt, dz) - \lambda(dz)dt$ a *compensated Poisson random measure (cPrm)*, where $\lambda(dz)dt$ is known as *compensator* of the Lévy process $(\mathbf{L}_t)_{t \geq 0}$. Here dt denotes the Lebesgue measure on $\mathscr{B}(\mathbb{R}^+)$, and $\lambda(dz)$ is a σ-finite Lévy measure on $(Z, \mathscr{B}(Z))$.

Definition 3.5. A non-empty collection of sets $\mathscr{G}$ is a *semi-ring* if

(i) Empty set $\phi \in \mathscr{G}$;
(ii) If $A \in \mathscr{G}$, $B \in \mathscr{G}$, then $A \cap B \in \mathscr{G}$;
(iii) If $A \in \mathscr{G}$, $A \supset A_1 \in \mathscr{G}$, then $A = \cup_{k=1}^n A_k$, where $A_k \in \mathscr{G}$ for all $1 \leq k \leq n$ and A_k are disjoint sets.

Definition 3.6. Let H and F be separable Hilbert spaces. Let $F_t := \mathscr{B}(H) \otimes \mathscr{F}_t$ be the product σ-algebra generated by the semi-ring $\mathscr{B}(H) \times \mathscr{F}_t$ of the product sets $Z \times F$, $Z \in \mathscr{B}(H)$, $F \in \mathscr{F}_t$ (where $\mathscr{F}_t$ is the filtration of the additive process

$(\mathbf{L}_t)_{t\geq 0})$. Let $T>0$, define

$$\mathbb{H}(Z) = \big\{g : \mathbb{R}^+ \times Z \times \Omega \to F, \text{ such that g is } F_T/\mathscr{B}(F) \ \textit{measurable and} \\ g(t,z,\omega) \ \textit{is} \ \mathscr{F}_t - \textit{adapted} \ \forall z \in Z, \forall t \in (0,T]\big\}.$$

For $p \geq 1$, let us define,

$$\mathbb{H}^p_\lambda([0,T]\times Z; F) = \left\{ g \in \mathbb{H}(Z) : \int_0^T \int_Z \mathbb{E}[\|g(t,z,\omega)\|^p_F]\lambda(dz)dt < \infty \right\}.$$

For more details see Mandrekar and Rüdiger [23].

Remark 3.7. Let us define $\mathscr{D}([0,T];H)$ as the space of all cádlág paths from $[0,T]$ into H, where H is a Hilbert space.

Lemma 3.8. *(Kunita's Inequality) Let us consider the stochastic differential equations driven by Lévy noise of the form*

$$\boldsymbol{X}(t) = \int_0^t b(\boldsymbol{X}(s))\mathrm{d}s + \int_0^t \sigma(s,\boldsymbol{X}(s))\mathrm{d}W(s) + \int_0^t\int_Z g(\boldsymbol{X}(s-),z)\tilde{N}(ds,dz).$$

Then for all $p \geq 2$, there exists $C(p,t) > 0$ such that for each $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t_0\leq s\leq t} |\boldsymbol{X}(s)|^p\right] \\
&\leq C(p,t)\left\{ \mathbb{E}|\boldsymbol{X}(0)|^p + \mathbb{E}\left(\int_0^t |b(\boldsymbol{X}(r))|^p \mathrm{d}r\right) + \mathbb{E}\left[\int_0^t |\sigma(r,\boldsymbol{X}(r))|^p\mathrm{d}r\right]\right. \\
&\quad \left. + \mathbb{E}\left[\int_0^t \left(\int_Z |g(\boldsymbol{X}(r-),z)|^2\lambda(dz)\right)^{p/2}\mathrm{d}r\right] + \mathbb{E}\left[\int_0^t\int_Z |g(\boldsymbol{X}(r-),z)|^p\lambda(dz)dr\right]\right\}.
\end{aligned}$$

For proof see Corollary 4.4.24 of [1].

3.2. Formulation of the Model. The stochastic MHD system perturbed by Lévy noise is given by

$$\begin{aligned}
&\frac{\partial u}{\partial t} + (u\cdot\nabla)u - \frac{1}{R_e}\Delta u - S(B\cdot\nabla)B + \nabla\left(p + \frac{S|B|^2}{2}\right) \\
&\quad = \sigma_1(u,B)\partial W_1(t) + \int_Z g_1(u,z)\tilde{N}_1(dt,dz) \quad \text{in} \quad G\times(0,T), \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial B}{\partial t} + (u\cdot\nabla)B + \frac{1}{R_m}\mathrm{curl}(\mathrm{curl}\, B) - (B\cdot\nabla)u \\
&\quad = \sigma_2(u,B)\partial W_2(t) + \int_Z g_2(u,z)\tilde{N}_2(dt,dz) \quad \text{in} \quad G\times(0,T), \qquad (3.2)
\end{aligned}$$

where Z is a measurable space on H (recall that $H = H_1 \times H_2$). The processes W_1 and W_2 are independent Wiener processes and $\tilde{N}_1(dt,dz) = N_1(dt,dz) - \lambda_1(\mathrm{d}z)\mathrm{d}t$ and $\tilde{N}_2(dt,dz) = N_2(dt,dz) - \lambda_2(\mathrm{d}z)\mathrm{d}t$ are compensated Poisson random measures.

Next we introduce the operators $\sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ and $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}$ and the vectors $W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$, $\tilde{N} = \begin{pmatrix} \tilde{N}_1 \\ \tilde{N}_2 \end{pmatrix}$ and $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$. The noise term in its integral form is given by $\int_0^T \sigma(r,\mathbf{X}(r))\mathrm{d}W(r) + \int_0^t \int_Z g(\mathbf{X}(r),z)\tilde{N}(dr,dz)$, where W is an H-valued Wiener process with a nuclear covariance form Q and $\tilde{N}(dt,dz)$ is a compensated Poisson random measure. By the above description and the abstract formulation of the model, (3.1)-(3.2) in the integral form as

$$\mathbf{X}(t) = \mathbf{X}(0) + \int_0^t \left[-\mathbf{A}\mathbf{X}(s) - \mathbf{B}\left(\mathbf{X}(s)\right)\right] \mathrm{d}s + \int_0^t \sigma(s,\mathbf{X}(s))\mathrm{d}W(s) + \int_0^t \int_Z g(\mathbf{X}(s-),z)\tilde{N}(ds,dz). \tag{3.3}$$

Note that in the above formulation $\mathbf{X}$ denotes the transpose of (u,B).

3.3. Hypothesis and Local Monotonicity. Assume that σ and g satisfy the following hypothesis of joint continuity, Lipschitz condition and linear growth.

Hypothesis 3.9. *The main hypothesis is the following,*

(H.1) *The function* $\sigma \in C([0,T]\times V; L_Q(H_0;H))$*, and* $g \in \mathbb{H}^2_\lambda([0,T]\times Z;H)$.

(H.2) *For all* $t \in (0,T)$*, there exists a positive constant* K *such that for all* $\boldsymbol{X} \in H$,

$$|\sigma(t,\boldsymbol{X})|^2_{L_Q} + \int_Z |g(\boldsymbol{X},z)|^2_H \lambda(dz) \le K(1+|\boldsymbol{X}|^2).$$

(H.3) *For all* $t \in (0,T)$*, there exists a positive constant* $L<1$ *such that for all* $\boldsymbol{X}, \boldsymbol{Y} \in H$,

$$|\sigma(t,\boldsymbol{X}) - \sigma(t,\boldsymbol{Y})|^2_{L_Q} + \int_Z |g(\boldsymbol{X},z) - g(\boldsymbol{Y},z)|^2_H \lambda(dz) \le L|\boldsymbol{X}-\boldsymbol{Y}|^2.$$

Remark 3.10. From the above hypothesis, we have

$$\sigma_i \in C([0,T]\times V_i; L_Q(H_{i_0};H_i)) \text{ and } g_i \in \mathbb{H}^2_{\lambda_i}([0,T]\times Z_i;H_i) \text{ for } i=1,2.$$

Remark 3.11. Notice that, $Q: H \to H$ is a trace class covariance (nuclear) operator and hence compact. So $H_0 = Q^{1/2}H$ is a separable Hilbert space and the imbedding of H_0 in H is Hilbert-Schmidt. Let $\{e_n\}_{n=1}^\infty$ be the eigenfunctions of Q (may not be complete). Then $Qe_n = \beta_n e_n$, where each β_n is positive real and $\sum_n \beta_n < \infty$. Let $\{h_m\}$, with $h_m = \sqrt{\beta_m}e_m, m=1,2,\cdots$ be orthonormal basis in H_0 (see section 4.1, chapter 4 of Da Prato and Zabczyk [10]). Then,

$$\begin{aligned}
|\sigma(t,\mathbf{X})|^2_{\mathrm{L}_Q} &= \sum_{m,n=1}^\infty |(\sigma h_m, e_n)|^2 = \sum_{m,n=1}^\infty \beta_m |(\sigma e_m, e_n)|^2 \\
&= \sum_{n=1}^\infty (\sigma Q^{1/2}e_n, \sigma Q^{1/2}e_n) = \sum_{n=1}^\infty (Q^{1/2}\sigma^*\sigma Q^{1/2}e_n, e_n) \\
&= \mathrm{Tr}((\sigma Q^{1/2})^*(\sigma Q^{1/2})) = \mathrm{Tr}((\sigma Q^{1/2})(\sigma Q^{1/2})^*) = \mathrm{Tr}(\sigma Q \sigma^*).
\end{aligned}$$

Here we have used the property that, since $\sigma Q^{1/2}$ is a Hilbert-Schmidt operator, $\mathrm{Tr}((\sigma Q^{1/2})^*(\sigma Q^{1/2})) = \mathrm{Tr}((\sigma Q^{1/2})(\sigma Q^{1/2})^*)$.

The next lemma shows that the sum of linear and non-linear operator is locally monotone in the L^4-ball.

Lemma 3.12. *For a given $r > 0$, let B_r denote the $L^4(G)$ ball in V, i.e., $B_r = \{\boldsymbol{Y} \in V : \|\boldsymbol{Y}\|_{L^4(G)} \leq r\}$. Define the non-linear operator F on V by $F(\boldsymbol{X}) = -\boldsymbol{A}\boldsymbol{X} - \boldsymbol{B}(\boldsymbol{X})$. Then the pair $\left(F, \sigma + \int_Z g(\cdot, z)\lambda(dz)\right)$ is monotone in B_r, i.e., for any $\boldsymbol{X} \in V$ and $\boldsymbol{Y} \in B_r$, if $\boldsymbol{W}$ denotes $\boldsymbol{X} - \boldsymbol{Y}$,*

$$\big(F(\boldsymbol{X}) - F(\boldsymbol{Y}), \boldsymbol{W}\big) - Cr^4|\boldsymbol{W}|^2 + |\sigma(t, \boldsymbol{X}) - \sigma(t, \boldsymbol{Y})|^2_{L_Q} + \int_Z |g(\boldsymbol{X}, z) - g(\boldsymbol{Y}, z)|^2 \lambda(dz) \leq 0. \quad (3.4)$$

Proof. First, it is clear that $(\mathbf{AW}, \mathbf{W}) = \|\mathbf{W}\|^2$, as

$$(\mathbf{AW}, \mathbf{W}) = a((\mathbf{W}, \mathbf{W})) = [[v, v]]_{V_1} + S[[B, B]]_{V_2} = [[\mathbf{W}, \mathbf{W}]]_V = \|\mathbf{W}\|^2.$$

Using the bilinearity of the operator $\mathbf{B}$, we get

$$\begin{aligned}\big(\mathbf{B}(\mathbf{X}, \mathbf{W}), \mathbf{Y}\big) &= b(\mathbf{X}, \mathbf{W}, \mathbf{Y}) = -b(\mathbf{X}, \mathbf{Y}, \mathbf{W}) = -b(\mathbf{X}, \mathbf{Y}, \mathbf{W}) - b(\mathbf{X}, \mathbf{W}, \mathbf{W}) \\ &= -b(\mathbf{X}, \mathbf{Y} + \mathbf{W}, \mathbf{W}) = -b(\mathbf{X}, \mathbf{X}, \mathbf{W}) = -\big(\mathbf{B}(\mathbf{X}), \mathbf{W}\big).\end{aligned}$$

Hence, $(\mathbf{B}(\mathbf{X}), \mathbf{W}) = -\left(\mathbf{B}(\mathbf{X}, \mathbf{W}), \mathbf{Y}\right)$. Also, we have,

$$(\mathbf{B}(\mathbf{Y}, \mathbf{W}), \mathbf{X}) = b(\mathbf{Y}, \mathbf{W}, \mathbf{X}) = -b(\mathbf{Y}, \mathbf{Y}, \mathbf{X}) = -b(\mathbf{Y}, \mathbf{Y}, \mathbf{W}) = -\left(\mathbf{B}(\mathbf{Y}), \mathbf{W}\right).$$

Using the two equations above, one obtains

$$\begin{aligned}\big(\mathbf{B}(\mathbf{X}) - \mathbf{B}(\mathbf{Y}), \mathbf{W}\big) &= -b(\mathbf{X}, \mathbf{W}, \mathbf{Y}) + b(\mathbf{Y}, \mathbf{W}, \mathbf{X}) \\ &= b(\mathbf{Y}, \mathbf{W}, \mathbf{W}) - b(\mathbf{W}, \mathbf{W}, \mathbf{Y}) = -\big(\mathbf{B}(\mathbf{W}), \mathbf{Y}\big).\end{aligned}$$

Using Hölder's inequality, Young's inequality and Sobolev embedding theorem, we have for any $\varepsilon > 0$,

$$\begin{aligned}\left|\big(\mathbf{B}(\mathbf{X}) - \mathbf{B}(\mathbf{Y}), \mathbf{W}\big)\right| &= \left|-\big(\mathbf{B}(\mathbf{W}), \mathbf{Y}\big)\right| = |b(\mathbf{W}, \mathbf{W}, \mathbf{Y})| \\ &\leq \|\mathbf{W}\|_{\mathrm{L}^4(G)}\|\mathbf{W}\|\|\mathbf{Y}\|_{\mathrm{L}^4(G)} \\ &\leq \|\mathbf{W}\|^{3/2}|\mathbf{W}|^{1/2}\|\mathbf{Y}\|_{\mathrm{L}^4(G)} \\ &\leq \varepsilon\|\mathbf{W}\|^2 + C_\varepsilon|\mathbf{W}|^2\|\mathbf{Y}\|^4_{\mathrm{L}^4(G)}.\end{aligned}$$

Here $C_\varepsilon > 0$ is a constant that depends on ε and denote it by C. Using the definition of the operator F yields

$$\begin{aligned}\big(F(\mathbf{X}) - F(\mathbf{Y}), \mathbf{W}\big) &= -\big(\mathbf{AW}, \mathbf{W}\big) - \big(\mathbf{B}(\mathbf{X}) - \mathbf{B}(\mathbf{Y}), \mathbf{W}\big) \\ &\leq (-1 + \varepsilon)\|\mathbf{W}\|^2 + C|\mathbf{W}|^2 r^4.\end{aligned} \quad (3.5)$$

But $V \subset H$ implies $(1 - \varepsilon)|\mathbf{W}|^2 \leq (1 - \varepsilon)\|\mathbf{W}\|^2$. Therefore one can have,

$$\big(F(\mathbf{X}) - F(\mathbf{Y}), \mathbf{W}\big) + (1 - \varepsilon)|\mathbf{W}|^2 - C|\mathbf{W}|^2 r^4 \leq 0.$$

But from the Hypothesis 3.9, $L < 1$, choose ε such that $\varepsilon < (1 - L)$, one obtains,

$$\big(F(\mathbf{X}) - F(\mathbf{Y}), \mathbf{W}\big) + L|\mathbf{W}|^2 - C|\mathbf{W}|^2 r^4 \leq 0.$$

Using condition $(H.3)$, we get (3.4). □

3.4. Energy estimate and existence theory. In this section we will find suitable energy estimates and prove the existence and uniqueness of stong solution of the MHD system. For that we first define the Galerkin approximations of the MHD system. Let $H_n := \text{span}\ \{e_1, e_2, \cdots, e_n\}$ where $\{e_j\}$ is any fixed orthonormal basis in H with each $e_j \in \mathfrak{D}(\mathbf{A})$. Let P_n denote the orthogonal projection of H to H_n. Define $\mathbf{X}_n = P_n\mathbf{X}$. Let $W_n = P_nW$. Let $\sigma_n = P_n\sigma$ and $\int_Z g^n(\cdot,z)\tilde{N}(dt,dz) = P_n \int_Z g(\cdot,z)\tilde{N}(dt,dz)$, where $g^n = P_n g$. Define $\mathbf{X}_n$ as the solution of the following stochastic differential equation in the variational form such that for each $\mathbf{v} \in H_n$ and with $\mathbf{X}_n(0) = P_n\mathbf{X}(0)$,

$$\mathrm{d}(\mathbf{X}_n(t), \mathbf{v}) = (F(\mathbf{X}_n(t)), \mathbf{v})\mathrm{d}t + (\sigma_n(t, \mathbf{X}_n(t))\mathrm{d}W_n(t), \mathbf{v}) + \int_Z \big(g^n(\mathbf{X}_n(t-), z), \mathbf{v}\big)\tilde{N}(dt, dz). \quad (3.6)$$

Theorem 3.13. *Under the above mathematical setting, let $\boldsymbol{X}(0)$ be $\mathscr{F}_0$ - measurable, $\sigma_n \in C([0,T]\times V; L_Q(H_0; H))$, $g^n \in \mathbb{H}^2_\lambda([0,T]\times Z; H)$ and let $\mathbb{E}|\boldsymbol{X}(0)|^2 < \infty$. Let $\boldsymbol{X}_n(t)$ denote the unique strong solution to finite system of equations* (3.6) *in $\mathscr{D}([0,T], H_n)$. Then with K as in condition (H.2), the following estimates hold: For all $0 \le t \le T$,*

$$\mathbb{E}|\boldsymbol{X}_n(t)|^2 + 2\int_0^t \mathbb{E}\|\boldsymbol{X}_n(s)\|^2\mathrm{d}s \le \left(1 + KTe^{KT}\right)\left(\mathbb{E}|\boldsymbol{X}(0)|^2 + KT\right) \ \textit{and} \quad (3.7)$$

$$\mathbb{E}\left[\sup_{0\le t\le T}|\boldsymbol{X}_n(t)|^2\right] + 4\int_0^T \mathbb{E}\|\boldsymbol{X}_n(t)\|^2\mathrm{d}t \le C\left(\mathbb{E}|\boldsymbol{X}(0)|^2, K, T\right). \quad (3.8)$$

Also for any $K > \delta > 0$,

$$\mathbb{E}|\boldsymbol{X}_n(t)|^2e^{-\delta t} + 2\int_0^t \mathbb{E}\|\boldsymbol{X}_n(s)\|^2e^{-\delta s}\mathrm{d}s \le C\left(\mathbb{E}|\boldsymbol{X}(0)|^2, K, \delta, T\right) \ \textit{and} \quad (3.9)$$

$$\mathbb{E}\left[\sup_{0\le t\le T}|\boldsymbol{X}_n(t)|^2e^{-\delta t}\right] + 4\int_0^T \mathbb{E}\|\boldsymbol{X}_n(t)\|^2e^{-\delta t}dt \le C\left(\mathbb{E}|\boldsymbol{X}(0)|^2, K, \delta, T\right). \quad (3.10)$$

We need the following assumption to get the p-th moment estimate for the stochastic MHD system with Lévy noise.

Assumption 3.14. *Let $p \ge 2$. Then, for all $t \in (0,T)$, there exists a positive constant K_1 such that for all $\boldsymbol{X} \in H$,*

$$|\sigma(t, \boldsymbol{X}(t))|^p_{\mathrm{L}_Q} + \int_Z |g(\boldsymbol{X}(t), z)|^p_H\lambda(dz) \le K_1\left(1 + |\boldsymbol{X}(t)|^p\right). \quad (3.11)$$

Theorem 3.15. *Under the above mathematical setting, let $p \ge 2$, $\boldsymbol{X}(0)$ be $\mathscr{F}_0$ - measurable, $\sigma_n \in C([0,T]\times V; L_Q(H_0; H))$, $g^n \in \mathbb{H}^p_\lambda([0,T]\times Z; H)$ and let $\mathbb{E}|\boldsymbol{X}(0)|^p < \infty$. Let $\boldsymbol{X}_n(t)$ denote the unique strong solution to finite system of equations (3.6) in $\mathscr{D}([0,T], H_n)$. Then with K as in condition (H.2) and K_1 as in Assumption 3.14, the following estimates hold:*

$$\mathbb{E}(\sup_{0\le t\le T}|\boldsymbol{X}_n(t)|^p) + p\mathbb{E}\int_0^T \|\boldsymbol{X}_n(t)\|^2|\boldsymbol{X}_n(t)|^{p-2}dt \le C\left(\mathbb{E}|\boldsymbol{X}(0)|^p, K, K_1, p, T\right). \quad (3.12)$$

Proof. Define $\tau_N = \inf\left\{t : |\mathbf{X}_n(t)|^p + \int_0^t \|\mathbf{X}_n(s)\|^p \mathrm{d}s > N\right\}$ as the stopping time. We have to find the p^{th} moment estimates for the above system (for similar formulation see Theorem 4.4 of [2]). For this, let us take the function $f(x) = |x|^p$ and apply the Itô's formula (see Theorem 5.1, chapter II of [16], Theorem 4.4.7 of [1], Theorem 4.4 of [31]) to the process $\mathbf{X}_n(t)$ to obtain,

$$\begin{aligned}
&|\mathbf{X}_n(t\wedge\tau_N)|^p + p\int_0^{t\wedge\tau_N} \|\mathbf{X}_n(s)\|^2|\mathbf{X}_n(s)|^{p-2}\mathrm{d}s \\
&= |\mathbf{X}(0)|^p + p\int_0^{t\wedge\tau_N} |\mathbf{X}_n(s)|^{p-2}\left(\sigma_n(s,\mathbf{X}_n(s))\mathrm{d}W(s), \mathbf{X}_n(s)\right) \\
&+\frac{p(p-1)}{2}\int_0^{t\wedge\tau_N} |\mathbf{X}_n(s)|^{p-2}\mathrm{Tr}(\sigma_n(s,\mathbf{X}_n(s))Q\sigma_n^*(s,\mathbf{X}_n(s)))\mathrm{d}s + I
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
I \;=\; &\int_0^{t\wedge\tau_N}\int_Z \left[p|\mathbf{X}_n(s-)|^{p-2}\left(g^n(\mathbf{X}_n(s-),z), \mathbf{X}_n(s-)\right)\right]\tilde{N}(ds,dz) \\
&+\int_0^{t\wedge\tau_N}\int_Z \Big[|\mathbf{X}_n(s-)+g^n(\mathbf{X}_n(s-),z)|^p - |\mathbf{X}_n(s-)|^p \\
&\qquad -p|\mathbf{X}_n(s-)|^{p-2}\left(g^n(\mathbf{X}_n(s-),z), \mathbf{X}_n(s-)\right)\Big]N(\mathrm{d}s,\mathrm{d}z).
\end{aligned}$$

We take the term $\frac{p(p-1)}{2}\int_0^{t\wedge\tau_N} |\mathbf{X}_n(s)|^{p-2}\mathrm{Tr}(\sigma_n(s,\mathbf{X}_n(s))Q\sigma_n^*(s,\mathbf{X}_n(s)))\mathrm{d}s$ from (3.13) and apply Young's inequality to obtain,

$$\begin{aligned}
&\frac{p(p-1)}{2}\int_0^{t\wedge\tau_N} |\mathbf{X}_n(s)|^{p-2}\mathrm{Tr}(\sigma_n(s,\mathbf{X}_n(s))Q\sigma_n^*(s,\mathbf{X}_n(s)))\mathrm{d}s \\
&\leq \frac{p}{2}\int_0^{t\wedge\tau_N} \|\mathbf{X}_n(s)\|^2|\mathbf{X}_n(s)|^{p-2}\mathrm{d}s + C_1(p)\int_0^{t\wedge\tau_N} |\sigma_n(s,\mathbf{X}_n(s))|^p\mathrm{d}s,
\end{aligned} \tag{3.14}$$

where $C_1(p) = (p-1)^{p/2}\left(\frac{p-2}{p}\right)^{\frac{p-2}{2}}$. Now applying (3.14) in (3.13) to obtain,

$$\begin{aligned}
&|\mathbf{X}_n(t\wedge\tau_N)|^p + \frac{p}{2}\int_0^{t\wedge\tau_N} \|\mathbf{X}_n(s)\|^2|\mathbf{X}_n(s)|^{p-2}\mathrm{d}s \\
&\leq |\mathbf{X}(0)|^p + p\int_0^{t\wedge\tau_N} |\mathbf{X}_n(s)|^{p-2}\left(\sigma_n(s,\mathbf{X}_n(s))\mathrm{d}W(s), \mathbf{X}_n(s)\right) \\
&\quad +C_1(p)\int_0^{t\wedge\tau_N} |\sigma_n(s,\mathbf{X}_n(s))|^p\mathrm{d}s + I.
\end{aligned} \tag{3.15}$$

Let us take expectation on both sides of the inequality (3.15). By using Hypothesis (H.2) (linear growth property), we have $\mathbb{E}\int_0^{t\wedge\tau_N} |\mathbf{X}_n(s)|^2|\sigma_n(t,\mathbf{X}_n(t))|^2_{\mathrm{L_Q}}\mathrm{d}s < \infty$ and $\mathbb{E}\int_0^{t\wedge\tau_N}\int_Z |\mathbf{X}_n(t-)|^2|g^n(\mathbf{X}_n(t-),z)|^2\lambda(dz)\mathrm{d}s < \infty$. Then, the processes $\int_0^{t\wedge\tau_N} |\mathbf{X}_n(s)|^{p-2}\left(\sigma_n(s,\mathbf{X}_n(s))\mathrm{d}W(s), \mathbf{X}_n\right)$ and $\int_0^{t\wedge\tau_N}\int_Z \Big[|\mathbf{X}_n(s)+g^n(\mathbf{X}_n(s),z)|^p - |\mathbf{X}_n(s)|^p\Big]\tilde{N}(ds,dz)$ are martingales having zero averages (see Proposition 4.10 of Rüdiger [30]). The second term in I can be estimated by the following Taylor's formula (there exits a constant $C_p \geq 0 (p \geq 2)$),

$$||\mathbf{X}_n + h|^p - |\mathbf{X}_n|^p - p|\mathbf{X}_n|^p(\mathbf{X}_n, h)| \leq C_p(|\mathbf{X}_n|^{p-2}|h|^2 + |h|^p), \ \ \forall \mathbf{X}_n, h \in H_n)$$

and Hypothesis 3.9. Then on taking $N \to \infty$, $t \wedge \tau_N \to t$, one can show that,

$$\mathbb{E}\left[|\mathbf{X}_n(t)|^p + p\int_0^t \|\mathbf{X}_n(s)\|^2 |\mathbf{X}_n(s)|^{p-2} \mathrm{d}s\right] \leq C\left(\mathbb{E}|\mathbf{X}(0)|^p, K, K_1, p, T\right).$$

Let us take the supremum from $0 \leq t \leq T \wedge \tau_N$ and then taking the expectation on the inequality (3.15), one gets,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N} |\mathbf{X}_n(t)|^p + \frac{p}{2}\int_0^{T\wedge\tau_N} \|\mathbf{X}_n(t)\|^2 |\mathbf{X}_n(t)|^{p-2}\mathrm{d}t\right] \\
&\leq \mathbb{E}\left[|\mathbf{X}(0)|^p\right] + p\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}\left|\int_0^t |\mathbf{X}_n(s)|^{p-2}\left(\sigma_n(s,\mathbf{X}_n(s))\mathrm{d}W(s), \mathbf{X}_n(s)\right)\right|\right] \\
&\quad + C_1(p)\mathbb{E}\left(\int_0^{T\wedge\tau_N} |\sigma_n(s,\mathbf{X}_n(s))|^p \mathrm{d}s\right) + \mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N} I\right]. \qquad (3.16)
\end{aligned}$$

Take the term $p\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}\left|\int_0^t |\mathbf{X}_n(s)|^{p-2}\left(\sigma_n(s,\mathbf{X}_n(s))\mathrm{d}W(s), \mathbf{X}_n(s)\right)\right|\right]$ from the inequality (3.16) and apply Burkholder-Davis-Gundy inequality and Young's inequality to get,

$$\begin{aligned}
&p\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}\left|\int_0^t |\mathbf{X}_n(s)|^{p-2}\left(\sigma_n(s,\mathbf{X}_n(s))\mathrm{d}W(s), \mathbf{X}_n(s)\right)\right|\right] \\
&\leq p\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N} |\mathbf{X}_n(t)|^{p-1}\left(\int_0^{T\wedge\tau_N} |\sigma_n(s,\mathbf{X}_n(s))|^2\mathrm{d}s\right)^{1/2}\right] \qquad (3.17) \\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N} |\mathbf{X}_n(t)|^p\right] + (2(p-1))^{p-1}\mathbb{E}\left(\int_0^{T\wedge\tau_N} |\sigma_n(s,\mathbf{X}_n(s))|^2\mathrm{d}s\right)^{p/2}.
\end{aligned}$$

Let us now take the term $\mathbb{E}\left(\int_0^{T\wedge\tau_N} |\sigma_n(s,\mathbf{X}_n(s))|^2\mathrm{d}s\right)^{p/2}$ from the inequality (3.17) and apply Hölder's inequality with $\frac{p}{p-2}$ and $\frac{p}{2}$ as the exponents to get,

$$\mathbb{E}\left(\int_0^{T\wedge\tau_N} |\sigma_n(s,\mathbf{X}_n(s))|^2\mathrm{d}s\right)^{p/2} \leq T^{\frac{p-2}{2}}\mathbb{E}\left(\int_0^{T\wedge\tau_N} |\sigma_n(s,\mathbf{X}_n(s))|^p\mathrm{d}s\right). \quad (3.18)$$

Use (3.18) in (3.17), then (3.16) reduces to,

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N} |\mathbf{X}_n(t)|^p\right] + \frac{p}{2}\mathbb{E}\left(\int_0^{T\wedge\tau_N} \|\mathbf{X}_n(s)\|^2 |\mathbf{X}_n(s)|^{p-2}\mathrm{d}s\right) \\
&\leq \mathbb{E}|\mathbf{X}(0)|^p + C_1(p,T)\mathbb{E}\left(\int_0^{T\wedge\tau_N} |\sigma_n(s,\mathbf{X}_n(s))|^p\mathrm{d}s\right) + \mathbb{E}\left(\sup_{0\leq t\leq T\wedge\tau_N} I\right), \quad (3.19)
\end{aligned}$$

where $C_1(p,T) = \left[(2(p-1))^{p-1}T^{\frac{p-2}{2}} + C_1(p)\right]$. Let us take stochastic integral as $\mathbf{Y}_n(t) = \int_0^t \int_Z g^n(\mathbf{X}_n(s-), z)\tilde{N}(ds, dz)$ and apply Itô's lemma to the function $|x|^p$

for the process $\mathbf{Y}_n(t)$. Then apply Kunita's inequality (see Lemma 3.8, for more details see Theorem 4.4.23 and Corollary 4.4.24 of [1]), we have,

$$\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}|I(t)|\right]\leq C_2(p,T)\left\{\mathbb{E}\left[\int_0^{T\wedge\tau_N}\left(\int_Z|g^n(\mathbf{X}_n(s-),z)|^2\lambda(dz)\right)^{p/2}\mathrm{d}s\right]\right.$$
$$\left.+\mathbb{E}\left[\int_0^{T\wedge\tau_N}\int_Z|g^n(\mathbf{X}_n(s-),z)|^p\lambda(dz)\mathrm{d}s\right]\right\}. \tag{3.20}$$

Apply Assumption 3.14 on the term $C_1(p,T)\mathbb{E}\left(\int_0^{T\wedge\tau_N}|\sigma_n(s,\mathbf{X}_n(s))|^p\mathrm{d}s\right)$, then

$$C_1(p,T)\mathbb{E}\int_0^{T\wedge\tau_N}|\sigma_n(s,\mathbf{X}_n(s))|^p\mathrm{d}s$$
$$\leq C_1(K_1,p,T)T+C_1(K_1,p,T)\mathbb{E}\int_0^{T\wedge\tau_N}\sup_{0\leq s\leq t}|\mathbf{X}_n(s)|^p\mathrm{d}s. \tag{3.21}$$

In the inequality (3.20), let us denote the RHS as I_1. Apply Hypothesis 3.9 and the Assumption 3.14 and using the estimate $(a+b)^p\leq 2^{p-1}(a^p+b^p)$ to obtain,

$$|I_1|\leq C_2(p,T)\left\{\mathbb{E}\left[\int_0^{T\wedge\tau_N}\left(K(1+|\mathbf{X}_n(s)|^2)\right)^{p/2}\mathrm{d}s\right]\right.$$
$$\left.+\mathbb{E}\left[\int_0^{T\wedge\tau_N}K_1\left(1+|\mathbf{X}_n(s)|^p\right)\mathrm{d}s\right]\right\}$$
$$\leq C_3(K,K_1,p,T)T+C_3(K,K_1,p,T)\mathbb{E}\int_0^{T\wedge\tau_N}\sup_{0\leq s\leq t}|\mathbf{X}_n(s)|^p\mathrm{d}t, \tag{3.22}$$

where $C_3(K,K_1,p,T)=\left(C_2(p,T)K^{p/2}2^{p/2-1}+K_1\right)$. Hence we have,

$$\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}|\mathbf{X}_n(t)|^p\right]+p\mathbb{E}\left(\int_0^{T\wedge\tau_N}\|\mathbf{X}_n(s)\|^2|\mathbf{X}_n(s)|^{p-2}\mathrm{d}s\right) \tag{3.23}$$
$$\leq 2\mathbb{E}|\mathbf{X}(0)|^p+C_4(K,K_1,p,T)T+C_4(K,K_1,p,T)\mathbb{E}\left(\int_0^{T\wedge\tau_N}\sup_{0\leq s\leq t}|\mathbf{X}_n(s)|^p\mathrm{d}t\right),$$

where $C_4(K,K_1,p,T)=2\left[C_1(K_1,p,T)+C_3(K,K_1,p,T)\right]$. In particular, we have,

$$\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}|\mathbf{X}_n(t)|^p\right]\leq 2\mathbb{E}|\mathbf{X}(0)|^p+C_4(K,K_1,p,T)T$$
$$+C_4(K,K_1,p,T)\mathbb{E}\int_0^{T\wedge\tau_N}\sup_{0\leq s\leq t}|\mathbf{X}_n(s)|^p\mathrm{d}t.$$

Applying Gronwall's inequality to get,

$$\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}|\mathbf{X}_n(t)|^p\right]\leq\left(2\mathbb{E}|\mathbf{X}(0)|^p+C_4(K,K_1,p,T)T\right)e^{C_4(K,K_1,p,T)T}. \tag{3.24}$$

By applying (3.24) in (3.23) and taking $N\to\infty$, $T\wedge\tau_N\to T$, we get (3.12). □

Theorem 3.16. *Under the above mathematical setting, let $p \geq 2$, $\boldsymbol{X}(0)$ be $\mathscr{F}_0$ - measurable, $\sigma_n \in C([0,T] \times V; L_Q(H_0; H))$, $g^n \in \mathbb{H}^p_\lambda([0,T] \times Z; H)$ and let $\mathbb{E}|\boldsymbol{X}(0)|^p < \infty$. Let $\boldsymbol{X}_n(t)$ denote the unique strong solution to finite system of equations (3.6) in $\mathscr{D}([0,T], H_n)$. Then with K as in condition (H.2) and K_1 as in Assumption 3.14, the following estimate hold for given $\delta > 0$ and for all $0 \leq t \leq T$,*

$$\mathbb{E} \sup_{0\leq t\leq T} |\boldsymbol{X}_n(t)|^p e^{-\delta t} + (p+2\delta)\mathbb{E}\int_0^T |\boldsymbol{X}_n(t)|^p e^{-\delta t} dt \leq C(\mathbb{E}|\boldsymbol{X}(0)|^2, K, K_1, \delta, p, T).$$

Proof. The proof the Theorem can be obtained by applying Itô's formula to the function $e^{-\delta t}|x|^p$ by taking the process $x = \mathbf{X}_n(t)$ and following Theorem 3.15. □

Definition 3.17. A *strong solution* $\mathbf{X}$ is defined on a given probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ as a $\mathrm{L}^2(\Omega; \mathrm{L}^\infty(0,T;H) \cap \mathrm{L}^2(0,T;V) \cap \mathscr{D}(0,T;H))$ valued adapted process which satisfies the stochastic MHD system

$$\begin{aligned} \mathrm{d}\mathbf{X} + \big[\mathbf{A}\mathbf{X} + \mathbf{B}(\mathbf{X})\big]\mathrm{d}t &= \sigma(t,\mathbf{X})\mathrm{d}W(t) + \int_Z g(\mathbf{X},z)\tilde{N}(dt,dz) \qquad (3.25)\\ \mathbf{X}(0) &= \mathbf{X}_0 \in H, \end{aligned}$$

in the weak sense and also the energy inequalities in Theorem 3.13.

Monotonicity arguments were first used by Krylov and Rozovskii[19] to prove the existence and uniqueness of the strong solutions for a wide class of stochastic evolution equations (under certain assumptions on the drift and diffusion coefficients), which in fact is the refinement of the previous results by Pardoux[28, 27] (also see Mativier[25]) and also the generalization of the results by Bensoussan and Temam[4]. Menaldi and Sritharan[24] further developed this theory for the case when the sum of the linear and nonlinear operators are locally monotone. Since Lévy noise appears in this paper, the proof of existence and uniqueness is given in complete form.

Theorem 3.18. *Let $\boldsymbol{X}(0)$ be $\mathscr{F}_0$ measurable and $\mathbb{E}|\boldsymbol{X}_0|^2 < \infty$. The diffusion coefficient satisfies the conditions (H.1)-(H.3). Then there exists a unique strong solution $\boldsymbol{X}(t,x,w)$ with the regularity*

$$\boldsymbol{X} \in \mathrm{L}^2(\Omega; \mathrm{L}^\infty(0,T;H) \cap \mathrm{L}^2(0,T;V) \cap \mathscr{D}(0,T;H))$$

satisfying the stochastic MHD system (3.25) *and a priori bounds in Theorem 3.13.*

Proof. **Part I (Existence).** Using the a priori estimate in the Theorem 3.13, it follows from the Banach-Alaoglu theorem that along a subsequence, the Galerkin approximations $\{\mathbf{X}_n\}$ have the following limits:

$$\begin{aligned} &\mathbf{X}_n \longrightarrow \mathbf{X} \quad \text{weak star in } \mathrm{L}^2(\Omega; \mathrm{L}^\infty(0,T;H)) \cap \mathrm{L}^2(\Omega; \mathrm{L}^2(0,T;V)),\\ &F(\mathbf{X}_n) \longrightarrow F_0 \quad \text{weakly in } \mathrm{L}^2(\Omega; \mathrm{L}^2(0,T;V')),\\ &\sigma_n(\cdot,\mathbf{X}_n) \longrightarrow S \quad \text{weakly in } \mathrm{L}^2(\Omega; \mathrm{L}^2(0,T;\mathrm{L}_Q)),\\ &g(\mathbf{X}_n,\cdot) \longrightarrow G \quad \text{weakly in } \mathbb{H}^2_\lambda([0,T]\times Z; H). \qquad (3.26) \end{aligned}$$

The assertion of the second statement holds since $F(\mathbf{X}_n)$ is bounded in $\mathrm{L}^2(\Omega; \mathrm{L}^2(0,T;V'))$. Likewise since diffusion coefficient has the linear growth property and $\mathbf{X}_n$ is bounded in $\mathrm{L}^2(0,T;V)$ uniformly in n, the last two statements hold. Then $\mathbf{X}$ satisfies the Itô differential

$$\mathrm{d}\mathbf{X}(t) = F_0(t)\mathrm{d}t + S(t)\mathrm{d}W(t) + \int_Z G(t)\tilde{N}(dt, dz) \text{ weakly in } \mathrm{L}^2(\Omega; \mathrm{L}^2(0,T;V')).$$

Then as Theorem 3.13, one can prove that, $\mathbb{E}\left[\sup_{0\le t\le T} |\mathbf{X}(t)|^2 + \int_0^T \|\mathbf{X}(t)\|^2 \mathrm{d}t\right] \le C\left(\mathbb{E}|\mathbf{X}(0)|^2, K, T\right)$. Then by Gyöngy and Krylov [15], we have $\mathbf{X}$ is an H-valued cádlág $\mathscr{F}_t$-adapted process satisfying

$$\begin{aligned}
|\mathbf{X}(t)|^2 &= |\mathbf{X}(0)|^2 + 2\int_0^t (F_0(s), \mathbf{X}(s))\mathrm{d}s + 2\int_0^t (S(s)\mathrm{d}W(s), \mathbf{X}(s)) \\
&+ \int_0^t |S|^2_{\mathrm{L}_Q}\mathrm{d}s + 2\int_0^t \int_Z (G(s), \mathbf{X}(s))\tilde{N}(ds, dz) + \int_0^t \int_Z |G(s)|^2 N(ds, dz).
\end{aligned}$$

Let us set, $r(t) := 2C\int_0^t \|\mathbf{v}(s)\|^4_{\mathrm{L}^4}\mathrm{d}s$, where $\mathbf{v}(\omega, t, x)$ is any adapted process in $\mathrm{L}^\infty(\Omega\times(0,T); H)$. Apply Itô's Lemma to the function $2e^{-r(t)}|x|^2$ and to the process $\mathbf{X}_n(t)$, integrating from $0 \le t \le T$ and taking expectation, one obtains,

$$\begin{aligned}
&\mathbb{E}\left[e^{-r(T)}|\mathbf{X}_n(T)|^2 - |\mathbf{X}_n(0)|^2\right] \\
&= \mathbb{E}\left[\int_0^T e^{-r(t)}\big(2F(\mathbf{X}_n(t)) - \dot{r}(t)\mathbf{X}_n(t), \mathbf{X}_n(t)\big)\mathrm{d}t\right] \\
&\quad + \mathbb{E}\int_0^T e^{-r(t)}|\sigma_n(t, \mathbf{X}_n(t))|^2_{\mathrm{L}_Q}\mathrm{d}t + \mathbb{E}\int_0^T e^{-r(t)}\int_Z \left|g^n(\mathbf{X}_n(t), z)\right|^2\lambda(dz)\mathrm{d}t \\
&\quad + 2\mathbb{E}\int_0^T e^{-r(t)}\big(\sigma_n(t, \mathbf{X}_n(t))\mathrm{d}W(t), \mathbf{X}_n(t)\big) \\
&\quad + 2\mathbb{E}\int_0^T e^{-r(t)}\int_Z \big(\mathbf{X}_n(t-), g^n(\mathbf{X}_n(t-), z)\big)\tilde{N}(dt, dz).
\end{aligned}$$

The last two terms of the above inequality are martingales having zero averages. Then by the lower semi-continuity property of the L^2-norm and strong convergence of the initial data, we obtain,

$$\begin{aligned}
&\liminf_n \mathbb{E}\left[\int_0^T e^{-r(t)}\big(2F(\mathbf{X}_n(t)) - \dot{r}(t)\mathbf{X}_n(t), \mathbf{X}_n(t)\big)\mathrm{d}t\right. \\
&\quad \left. + \int_0^T e^{-r(t)}|\sigma_n(t, \mathbf{X}_n(t))|^2_{\mathrm{L}_Q}\mathrm{d}t + \int_0^T e^{-r(t)}\int_Z \left|g^n(\mathbf{X}_n(t), z)\right|^2\lambda(dz)\mathrm{d}t\right] \\
&\quad = \liminf_n \mathbb{E}\left[e^{-r(T)}|\mathbf{X}_n(T)|^2 - |\mathbf{X}_n(0)|^2\right] \ge \mathbb{E}\left[e^{-r(T)}|\mathbf{X}(T)|^2 - |\mathbf{X}(0)|^2\right] \\
&\quad = \mathbb{E}\left[\int_0^T e^{-r(t)}\big(2F_0(t) - \dot{r}(t)\mathbf{X}(t), \mathbf{X}(t)\big)\mathrm{d}t + \int_0^T e^{-r(t)}|S|^2_{\mathrm{L}_Q}\mathrm{d}t\right. \\
&\qquad\qquad \left. + \int_0^T e^{-r(t)}\int_Z |G|^2\lambda(dz)\mathrm{d}t\right]. \qquad (3.27)
\end{aligned}$$

Now by monotonicity property from Lemma 3.12 (by choosing $\mathbf{X} = \mathbf{X}_n(t)$ and $\mathbf{Y} = \mathbf{v}(t)$ in (3.4)), we have,

$$
\begin{aligned}
&2\mathbb{E}\left[\int_0^T e^{-r(t)}\big(F(\mathbf{X}_n(t)) - F(\mathbf{v}(t)), \mathbf{X}_n(t) - \mathbf{v}(t)\big)\mathrm{d}t\right] \\
&-\mathbb{E}\left[\int_0^T e^{-r(t)}\dot{r}(t)|\mathbf{X}_n(t) - \mathbf{v}(t)|^2\mathrm{d}t\right] \\
&+\mathbb{E}\left[\int_0^T e^{-r(t)}|\sigma_n(t, \mathbf{X}_n(t)) - \sigma_n(t, \mathbf{v}(t))|^2_{\mathrm{L}_Q}\mathrm{d}t\right] \\
&+\mathbb{E}\left[\int_0^T e^{-r(t)}\int_Z |g^n(\mathbf{X}_n(t), z) - g^n(\mathbf{v}(t), z)|^2\lambda(dz)\mathrm{d}t\right] \leq 0.
\end{aligned}
$$

On rearranging the terms, we get,

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-r(t)}\big(2F(\mathbf{X}_n(t)) - \dot{r}(t)\mathbf{X}_n(t), \mathbf{X}_n(t)\big)\mathrm{d}t\right. \\
&\left.+\int_0^T e^{-r(t)}|\sigma_n(t, \mathbf{X}_n(t))|^2_{\mathrm{L}_Q}\mathrm{d}t + \int_0^T e^{-r(t)}\int_Z |g(\mathbf{X}_n(t), z)|^2\lambda(dz)\mathrm{d}t\right] \\
&\leq \mathbb{E}\left[\int_0^T e^{-r(t)}\big(2F(\mathbf{X}_n(t)) - \dot{r}(t)(2\mathbf{X}_n(t) - \mathbf{v}(t)), \mathbf{v}(t)\big)\mathrm{d}t\right] \\
&+\mathbb{E}\left[\int_0^T e^{-r(t)}\big(2F(\mathbf{v}(t)), \mathbf{X}_n(t) - \mathbf{v}(t)\big)\mathrm{d}t\right] \\
&+\mathbb{E}\left[\int_0^T e^{-r(t)}\big(2\sigma_n(t, \mathbf{X}_n(t)) - \sigma_n(t, \mathbf{v}(t)), \sigma_n(t, \mathbf{v}(t))\big)_{\mathrm{L}_Q}\mathrm{d}t\right] \\
&+\mathbb{E}\left[\int_0^T e^{-r(t)}\int_Z \big(2g^n(\mathbf{X}_n(t), z) - g^n(\mathbf{v}(t), z), g^n(\mathbf{v}(t), z)\big)\lambda(dz)\mathrm{d}t\right].
\end{aligned}
$$

Taking limit in n, using the result from (3.27), we have,

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T e^{-r(t)}\big(2F_0(t) - \dot{r}(t)\mathbf{X}(t), \mathbf{X}(t)\big)\mathrm{d}t + \int_0^T e^{-r(t)}|S|^2_{\mathrm{L}_Q}\mathrm{d}t\right. \\
&\left.+\int_0^T e^{-r(t)}\int_Z |G|^2\lambda(dz)\mathrm{d}t\right] \\
&\leq \mathbb{E}\left[\int_0^T e^{-r(t)}\big(2F_0(t) - \dot{r}(t)(2\mathbf{X}(t) - \mathbf{v}(t)), \mathbf{v}(t)\big)\mathrm{d}t\right] \\
&+\mathbb{E}\left[\int_0^T e^{-r(t)}\big(2F(\mathbf{v}(t)), \mathbf{X}(t) - \mathbf{v}(t)\big)\mathrm{d}t\right]
\end{aligned}
$$

$$+\mathbb{E}\left[\int_0^T e^{-r(t)}\left(2S(t)-\sigma(t,\mathbf{v}(t)),\sigma(t,\mathbf{v}(t))\right)_{\mathrm{L}_Q}\mathrm{d}t\right]$$
$$+\mathbb{E}\left[\int_0^T e^{-r(t)}\int_Z\left(2G(t)-g(\mathbf{v}(t),z),g(\mathbf{v}(t),z)\right)\lambda(dz)\mathrm{d}t\right].$$

On rearranging the terms, we obtain,

$$\mathbb{E}\left[\int_0^T e^{-r(t)}\big(2F_0(t)-2F(\mathbf{v}(t)),\mathbf{X}(t)-\mathbf{v}(t)\big)\mathrm{d}t\right]$$
$$+\mathbb{E}\left[\int_0^T e^{-r(t)}\dot{r}(t)|\mathbf{X}(t)-\mathbf{v}(t)|^2\mathrm{d}t\right]$$
$$+\mathbb{E}\left[\int_0^T e^{-r(t)}\|\mathbf{S}(t)-\sigma(t,\mathbf{v}(t))\|^2_{\mathrm{L}_Q}\mathrm{d}t\right]$$
$$+\mathbb{E}\left[\int_0^T e^{-r(t)}\int_Z\|\mathbf{G}(t)-g(\mathbf{v}(t),z)\|^2\lambda(dz)\mathrm{d}t\right]\leq 0.$$

This estimate holds for any $\mathbf{v}\in\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T;\mathbf{H}_m))$ for any $m\in\mathbb{N}$. It is clear by a density argument that the above inequality remains the same for any $\mathbf{v}\in\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T;\mathbf{H})\cap\mathrm{L}^2(0,T;\mathbf{V}))$. Indeed, for any $\mathbf{v}\in\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T;\mathbf{H})\cap\mathrm{L}^2(0,T;\mathbf{V}))$, there exits a strongly convergent sequence $\mathbf{v}_m\in\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T;\mathbf{H})\cap\mathrm{L}^2(0,T;\mathbf{V}))$ that satisfies the above inequality. Let $\mathbf{v}(t)=\mathbf{X}(t)$, then, $\mathbf{S}(t)=\sigma(t,\mathbf{X}(t))$ and $\mathbf{G}(t)=g(\mathbf{X}(t),z)$. Take $\mathbf{v}=\mathbf{X}-\mu\mathbf{Y}$ with $\mu>0$ and $\mathbf{Y}$ is an adapted process in $\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T;H)\cap\mathrm{L}^2(0,T;V)\cap\mathscr{D}(0,T;H))$. Then,

$$\mu\mathbb{E}(\int_0^T e^{-r(t)}\big(2F_0(t)-2F(\mathbf{X}-\mu\mathbf{Y})(t),\mathbf{Y}(t)\big)\mathrm{d}t+\mu\int_0^T e^{-r(t)}\dot{r}(t)|\mathbf{Y}(t)|^2\mathrm{d}t)\leq 0.$$

Dividing by μ on both sides of the above inequality and letting $\mu\to 0$, one obtains

$$\mathbb{E}\left[\int_0^T e^{-r(t)}\big(F_0(t)-F(\mathbf{X}(t)),\mathbf{Y}(t)\big)\mathrm{d}t\right]\leq 0.$$

Since $\mathbf{Y}(t)$ is arbitrary, we conclude that $F_0(t)=F(\mathbf{X}(t))$. Thus the existence of the strong solution of the stochastic MHD system (3.25) has been proved.

Part II (Uniqueness). If $\mathbf{Y}\in\mathrm{L}^2(\Omega;\mathrm{L}^\infty(0,T;H)\cap\mathrm{L}^2(0,T;V)\cap\mathscr{D}(0,T;H))$ be another solution of the equation (3.25), then $\mathbf{W}=\mathbf{X}-\mathbf{Y}$ solves the stochastic differential equation in $\mathrm{L}^2(\Omega;\mathrm{L}^2(0,T;V'))$,

$$\mathrm{d}\mathbf{W}(t)=\big(F(\mathbf{X}(t))-F(\mathbf{Y}(t))\big)\mathrm{d}t+(\sigma(t,\mathbf{X}(t))-\sigma(t,\mathbf{Y}(t)))\mathrm{d}W(t)$$
$$+\int_Z[g(\mathbf{X}(t-),z)-g(\mathbf{Y}(t-),z)]\tilde{N}(dt,dz).\qquad(3.28)$$

We denote $\sigma_d=\sigma(t,\mathbf{X}(t))-\sigma(t,\mathbf{Y}(t))$ and $g_d=g(\mathbf{X}(t-),z)-g(\mathbf{Y}(t-),z)$.

Let us apply Itô's Lemma to the function $2e^{-r(t)}|x|^2$ and to the process $\mathbf{W}(t)$ and use the local monotonicity of the sum of the linear and nonlinear operators $\mathbf{A}$

and $\mathbf{B}$, e.g. equation (3.5), one obtains,

$$\begin{aligned} \mathrm{d}\left[e^{-r(t)}|\mathbf{W}(t)|^2\right] &+ (1-\varepsilon)\|\mathbf{W}(t)\|^2 e^{-r(t)}\mathrm{d}t \\ &\leq e^{-r(t)}|\sigma_d|^2\mathrm{d}t + 2e^{-r(t)}(\sigma_d\mathrm{d}W(t), \mathbf{W}(t)) \\ &\quad + e^{-r(t)}\int_Z |g_d|^2 N(dt,dz) + 2e^{-r(t)}\int_Z \big(\mathbf{W}(t), g_d\big)\tilde{N}(dt,dz). \end{aligned}$$

Now integrating from $0 \leq t \leq T$ and taking the expectation on both sides and noting that $0 < \varepsilon < 1 - L$. Also using the fact that $2\int_0^T e^{-r(t)}(\sigma_d\mathrm{d}W(t), \mathbf{W}(t))$ and $2\int_0^T e^{-r(t)}\int_Z \big(\mathbf{W}(t), g_d\big)\tilde{N}(dt,dz)$ are martingales having zero averages, we deduce that,

$$\begin{aligned} \mathbb{E}\left[e^{-r(t)}|\mathbf{W}(t)|^2\right] &+ (1-\varepsilon)\mathbb{E}\int_0^T e^{-r(t)}\|\mathbf{W}(t)\|^2\mathrm{d}t \\ &\leq \mathbb{E}|\mathbf{W}(0)|^2 + \mathbb{E}\int_0^T e^{-r(t)}|\sigma_d|^2\mathrm{d}t + \mathbb{E}\int_0^T e^{-r(t)}\int_Z |g_d|^2\lambda(dz)\mathrm{d}t. \end{aligned}$$

Using condition (H.3), we have,

$$\mathbb{E}\left[e^{-r(t)}|\mathbf{W}(t)|^2\right] + ((1-\varepsilon)-L)\int_0^T e^{-r(t)}\|\mathbf{W}(t)\|^2\mathrm{d}t \leq \mathbb{E}|\mathbf{W}(0)|^2.$$

Sine $0 < L < (1-\varepsilon) < 1$, we obtain $\mathbb{P}$-a.s., $\mathbb{E}\left[e^{-r(t)}|\mathbf{W}(t)|^2\right] \leq \mathbb{E}|\mathbf{W}(0)|^2$, which assures the uniqueness of the strong solution. □

Theorem 3.19. *Let $\boldsymbol{X}(0)$ be $\mathscr{F}_0$ - measurable and $\mathbb{E}|\boldsymbol{X}_0|^p < \infty$. The diffusion coefficient satisfies the conditions $\sigma \in C([0,T]\times V; L_Q(H_0;H))$, $g \in \mathbb{H}^p_\lambda([0,T]\times Z;H)$, Hypothesis (H.3) and Assumption 3.14. Then there exists a unique strong solution $\boldsymbol{X}(t,x,w)$ with the regularity*

$$\boldsymbol{X} \in \mathrm{L}^p(\Omega; \mathrm{L}^\infty(0,T;H) \cap \mathrm{L}^2(0,T;V) \cap \mathscr{D}(0,T;H))$$

satisfying the stochastic MHD equation given in (3.25) and the a priori bounds in Theorem 3.15 and Theorem 3.16.

Proof. The proof of the Theorem follows the same steps as in Theorem 3.18 with minor modifications. □

4. Invariant Measures

In this section, we consider MHD system with additive Lévy noise given by

$$\mathrm{d}\mathbf{X}(t) + [\mathbf{A}\mathbf{X}(t) + \mathbf{B}(\mathbf{X}(t))]\mathrm{d}t = \sum_k \sigma_k(t)\mathrm{d}W_k(t) + \int_Z g(t,z)\tilde{N}(dt,dz), \qquad (4.1)$$

$\mathbf{X}(0) = \xi$. Here $\sigma(t) = \{\sigma_1(t), \sigma_2(t), \cdots\}$ is $\ell_2(H)$-valued for all $t \geq 0$ and $W(t)$ is given by $W(t) = \{W_1(t), W_2(t), \cdots\}$, where W_k are independent copies of the standard one-dimensional Wiener process. Hence $\sigma(t)\mathrm{d}W(t) = \sum_k \sigma_k(t)\mathrm{d}W_k(t)$ is an H-valued noise and the stochastic integral induced by the noise is given by $\mathbf{X} \mapsto \int_0^T(\sigma(t)\mathrm{d}W(t), \mathbf{X}) := \sum_k \int_0^T(\sigma_k(t), \mathbf{X})\mathrm{d}W_k(t)$. Let us assume that $\sum_k |\sigma_k(t)|^2 < \infty$ and $\int_Z |g(t,z)|^4\lambda(dz) = C_1 < \infty$. If $\lambda(\cdot)$ is of finite measure, then we have $\int_Z |g(t,z)|^2\lambda(dz) \leq [\lambda(Z)]^{1/2}C_1^{1/2} < \infty$.

Remark 4.1. If $\lambda(\cdot)$ is of σ-finite measure, then consider a measurable subset U_m of Z with $U_m \uparrow Z$ and $\lambda(U_m) < \infty$. Assume that $\int_{U_m^c} |g(t,z)|^2\lambda(dz) \to 0$ as $m \to \infty$, $t \in [0,T]$. This condition is automatically satisfied if $\lambda(Z) < \infty$. Here also we assume that $\int_{U_m} |g(t,z)|^4\lambda(dz) = C_2 < \infty$. Hence we have, $\int_{U_m} |g(t,z)|^2\lambda(dz) \leq [\lambda(U_m)]^{1/2}C_2^{1/2} < \infty$. For this case, the following results in this section will follow by replacing Z by U_m (see Fernando and Sritharan [12]).

From the definition of the linear operator (2.8) and (2.9), the eigenvalues of $\mathbf{A}$ depend on the Reynold's numbers R_e and R_m. Let the eigenvalues of $\mathbf{A}$ be denoted by $0 < \lambda_1 < \lambda_2 \leq \cdots$ and the corresponding eigenvectors by $e_1, e_2 \cdots$, which will form a complete orthonormal system in H. Also by the Poincaré inequality, we have, $\|\mathbf{X}\| \geq \lambda_1 |\mathbf{X}|^2$.

Lemma 4.2. *Let $B(t)$ be an increasing, progressively measurable process with $B(0) > 0$ a. s. Let $M(t)$ be a càdlàg local martingale with $M(0) = 0$. If $\mathcal{Y}(t) = \int_0^t \frac{1}{B(s)}\mathrm{d}M(s)$ converges a. s. to a finite limit as $t \to \infty$, then $\lim t \to \infty \frac{M^*(t)}{B(t)} = 0$ a. s. on the set $\{B(\infty) = \infty\}$ where $M^*(t) = \sup_{0\leq s\leq t} |M(s)|$.*

For the proof of continuous martingale see Lemma 2.1 of Sundar [35]. A simple calculation gives the extension of the result for the càdlàg local martingale.

Lemma 4.3. *Let $\lim_{t\to\infty} \sum_k |\sigma_k(t)|^2 = M_1 < \infty$, $\lim_{t\to\infty} \int_Z |g(t,z)|^4\lambda(dz) = M_2 < \infty$ and $\lim_{t\to\infty} \int_Z |g(t,z)|^2\lambda(dz) = M_3 < \infty$. Then $\lim_{t\to\infty} \frac{M^*(t)}{t} = 0$ a. s.,*

$$\begin{aligned} M(t) &= 2\sum_k \int_0^t (\sigma_k(s), \boldsymbol{X}(s))\mathrm{d}W_k(s) \\ &\quad + \int_0^t \int_Z \left[|g(s-,z)|^2 + 2(g(s-,z), \boldsymbol{X}(s-))\right] \tilde{N}(ds,dz). \qquad (4.2) \end{aligned}$$

Proof. Let us set $B(t) = (\varepsilon + t)$ in Lemma 4.2, for any $\varepsilon > 0$ and $\mathcal{Y}(t) = \int_0^t \frac{1}{(\varepsilon+s)}\mathrm{d}M(s)$. We have (for more details about the quadratic variation of Lévy type stochastic integrals see Section 4.4.3 of Applebaum [1]),

$$\begin{aligned} [\mathcal{Y},\mathcal{Y}]_t &= 4\sum_k \int_0^t \frac{(\sigma_k(t), \mathbf{X}(s))^2}{(\varepsilon+s)^2}\mathrm{d}s \\ &\quad + \int_0^t \int_Z \frac{\left||g(s-,z)|^2 + 2(g(s-,z), \mathbf{X}(s-))\right|^2}{(\varepsilon+s)^2} N(ds,dz) \\ &\leq 4\sum_k \int_0^t \frac{|\sigma_k(s)|^2|\mathbf{X}(s)|^2}{(\varepsilon+s)^2}\mathrm{d}s + 2\int_0^t \int_Z \frac{|g(s-,z)|^4}{(\varepsilon+s)^2} N(ds,dz) \\ &\quad +4\int_0^t \int_Z \frac{|(g(s-,z), \mathbf{X}(s-))|^2}{(\varepsilon+s)^2} N(ds,dz) \\ &= 4\sum_k \int_0^t \frac{|\sigma_k(s)|^2|\mathbf{X}(s)|^2}{(\varepsilon+s)^2}\mathrm{d}s + \int_0^t \int_Z \frac{\left[2|g(s,z)|^4 + 4|g(s,z)|^2|\mathbf{X}(s)|^2\right]}{(\varepsilon+s)^2}\lambda(dz)ds \\ &\quad + \int_0^t \int_Z \frac{\left[2|g(s-,z)|^4 + 4\left|(g(s-,z), \mathbf{X}(s-))\right|^2\right]}{(\varepsilon+s)^2}\tilde{N}(ds,dz) \qquad (4.3) \end{aligned}$$

Taking expectation on both sides of (4.3) and noting that the last term on the right hand side is a martingale having a zero average (see Proposition 4.10 of Rüdiger [30]), we have,

$$\begin{aligned}\mathbb{E}[\mathcal{Y},\mathcal{Y}]_t &\leq 4\sum_k\int_0^t \frac{|\sigma_k(s)|^2\mathbb{E}|\mathbf{X}(s)|^2}{(\varepsilon+s)^2}\mathrm{d}s + 2\int_0^t\int_Z \frac{|g(s,z)|^4}{(\varepsilon+s)^2}\lambda(dz)ds \\ &\quad +4\int_0^t\int_Z \frac{|g(s,z)|^2\mathbb{E}|\mathbf{X}(s)|^2}{(\varepsilon+s)^2}\lambda(dz)ds. \qquad (4.4)\end{aligned}$$

Let λ_1 be the first eigenvalue of the operator $\mathbf{A}$. Then Poincaré inequality gives $\lambda_1|\mathbf{W}|^2 \leq \|\mathbf{W}\|^2$. Hence by an application of Itô's lemma and Poincaré inequality in (4.1), we have

$$\mathbb{E}|\mathbf{X}(t)|^2 \leq \mathbb{E}|\mathbf{X}(0)|^2 - 2\lambda_1\int_0^t \mathbb{E}|\mathbf{X}(s)|^2\mathrm{d}s + \int_0^t (\sum_k|\sigma_k(s)|^2 + \int_Z |g(s,z)|^2\lambda(dz))\mathrm{d}s.$$

Hence by Gronwall's inequality,

$$\mathbb{E}|\mathbf{X}(t)|^2 \leq \mathbb{E}|\mathbf{X}(0)|^2 e^{-2\lambda_1 t} + \int_0^t e^{-2\lambda_1(t-s)}\left(\sum_k|\sigma_k(s)|^2 + \int_Z |g(s,z)|^2\lambda(dz)\right)\mathrm{d}s.$$

Letting t tend to ∞ in (4.4), using the assumptions in Lemma and the above inequality, we have

$$\mathbb{E}[\mathcal{Y},\mathcal{Y}]_\infty \leq C\int_0^\infty \frac{1}{(\varepsilon+s)^2}\mathrm{d}s < \infty. \qquad (4.5)$$

Hence $[\mathcal{Y},\mathcal{Y}]_\infty < \infty$ a. s. which allows us to conclude that $\lim_{t\to\infty}\mathcal{Y}(t)$ exists almost surely. Finally we prove the Lemma by invoking Lemma 4.2. □

Theorem 4.4. *(Exponential Stability) Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two solutions of (4.1) with initial values $\boldsymbol{X}_0$ and $\boldsymbol{Y}_0$ respectively. Let us assume that $\lim_{t\to\infty}\sum_k|\sigma_k(t)|^2 = M_1 < \infty$ and $\lim_{t\to\infty}\int_Z |g(t,z)|^2\lambda(dz) = M_3 < \infty$. If $\frac{2(M_1+M_3)}{\lambda_1} < 1$, where λ_1 is the first eigenvalue of the operator $\boldsymbol{A}$, then $\lim_{t\to\infty}|\boldsymbol{X}(t)-\boldsymbol{Y}(t)| = 0$ a. s.*

Remark 4.5. A suitable choice of the Reynold's numbers R_e and R_m (i. e., viscosity of the fluid flow and the magnetic field) gives λ_1 such that $\frac{2(M_1+M_3)}{\lambda_1} < 1$.

Proof. Since $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ are two solutions of (4.1), we have

$$\mathbf{X}(t) - \mathbf{Y}(t) + \int_0^t \mathbf{A}(\mathbf{X}(s)-\mathbf{Y}(s))\mathrm{d}s + \int_0^t (\mathbf{B}(\mathbf{X}(s)) - \mathbf{B}(y(s)))\mathrm{d}s = \mathbf{X}_0 - \mathbf{Y}_0.$$

By using Itô's formula, we have,

$$|\mathbf{W}(t)|^2 + 2\int_0^t \|\mathbf{W}(s)\|^2\mathrm{d}s + 2\int_0^t (\mathbf{B}(\mathbf{X}(s)) - \mathbf{B}(\mathbf{Y}(s)), \mathbf{W}(s))\mathrm{d}s = |\mathbf{W}(0)|^2, \qquad (4.6)$$

where $\mathbf{W}(t) = \mathbf{X}(t) - \mathbf{Y}(t)$ and $\mathbf{W}(0) = \mathbf{X}_0 - \mathbf{Y}_0$. Using

$$|(\mathbf{B}(\mathbf{X}) - \mathbf{B}(\mathbf{Y}), \mathbf{W})| \leq 2\|\mathbf{W}\||\mathbf{W}|\|\mathbf{Y}\| \leq \frac{1}{2}\|\mathbf{W}\|^2 + 2|\mathbf{W}|^2\|\mathbf{Y}\|^2$$

and Poincaré inequality in (4.6), we have,

$$|\mathbf{W}(t)|^2 + \lambda_1 \int_0^t |\mathbf{W}(s)|^2 \mathrm{d}s \leq |\mathbf{W}(0)|^2 + 4\int_0^t |\mathbf{W}(s)|^2 \|\mathbf{Y}(s)\|^2 \, ds. \tag{4.7}$$

Hence by applying Gronwall's inequality, we get,

$$|\mathbf{W}(t)|^2 \leq |\mathbf{W}(0)|^2 \exp\left(4\int_0^t \|\mathbf{Y}(s)\|^2 \mathrm{d}s - \lambda_1 t\right). \tag{4.8}$$

By applying Itô's lemma to the function $|x|^2$ and to the process $\mathbf{Y}(t)$ and using the properties of the linear and bilinear operators, we have,

$$\begin{aligned} |\mathbf{Y}(t)|^2 + 2\int_0^t \|\mathbf{Y}(s)\|^2 \mathrm{d}s &= |\mathbf{Y}(0)|^2 + \sum_k \int_0^t |\sigma_k(s)|^2 \mathrm{d}s \\ &+ 2\sum_k \int_0^t (\sigma_k(s), \mathbf{Y}(s)) \mathrm{d}W_k(s) + \int_0^t \int_Z |g(s-,z)|^2 \lambda(dz) ds \\ &+ \int_0^t \int_Z \left[|g(s-,z)|^2 + 2(g(s-,z), \mathbf{Y}(s-)) \right] \tilde{N}(ds,dz). \end{aligned} \tag{4.9}$$

Hence from above, we get,

$$\begin{aligned} \frac{2}{t}\int_0^t \|\mathbf{Y}(s)\|^2 \mathrm{d}s &\leq \frac{|\mathbf{Y}(0)|^2}{t} + \frac{1}{t}\sum_k \int_0^t |\sigma_k(s)|^2 \mathrm{d}s \\ &+ \frac{2}{t}\sum_k \int_0^t (\sigma_k(s), \mathbf{Y}(s)) \mathrm{d}W_k(s) + \frac{1}{t}\int_0^t \int_Z |g(s-,z)|^2 \lambda(dz) ds \\ &+ \frac{1}{t}\int_0^t \int_Z \left[|g(s-,z)|^2 + 2(g(s-,z), \mathbf{Y}(s-)) \right] \tilde{N}(ds,dz). \end{aligned} \tag{4.10}$$

By Lemma 4.3, we have,

$$\begin{aligned} \lim_{t\to\infty} \frac{1}{t} \Bigg[2\sum_k \int_0^t (\sigma_k(s), \mathbf{Y}(s)) \mathrm{d}W_k(s) \\ + \int_0^t \int_Z \left[|g(s-,z)|^2 + 2(g(s-,z), \mathbf{Y}(s-)) \right] \tilde{N}(ds,dz) \Bigg] = 0. \end{aligned} \tag{4.11}$$

From the assumption of the theorem, we have,

$$\lim_{t\to\infty} \frac{1}{t}\sum_k \int_0^t |\sigma_k(s)|^2 \mathrm{d}s = M_1 \text{ and } \lim_{t\to\infty} \frac{1}{t}\int_0^t \int_Z |g(s,z)|^2 \lambda(dz) ds = M_3. \tag{4.12}$$

By using (4.11) and (4.12) in (4.10), we have, $\lim_{t\to\infty} \frac{1}{t}\int_0^t \|\mathbf{Y}(s)\|^2 \mathrm{d}s \leq \left(\frac{M_1+M_3}{2}\right)$. Using this bound in (4.8) provided $\frac{2(M_1+M_3)}{\lambda_1} < 1$, we get $\lim_{t\to\infty} |\mathbf{X}(t) - \mathbf{Y}(t)| = 0$ a. s. □

Theorem 4.6. *(Existence and Uniqueness of Invariant Measures) Let* $\sigma(t,x) = \sigma(x)$ *and* $g(t,x,z) = g(x,z)$*. Then under the hypotheses of the above theorem, there exists a unique stationary measure with support in* V*, for the solution* $\boldsymbol{X}(t,x,\omega)$ *of the stochastic MHD system with Lévy noise.*

Proof. The method of proving existence of an invariant measure is well known in the literature and interested readers may look at the works by Barbu and Da Prato [3], Chow and Khasminskii [7], Da Prato and Zabczyk [10, 11], Da Prato and Debussche [9], Flandoli [14], Odasso [26], Sundar [35, 36] for the extensive study on the subject. For the sake of completeness we are giving a very brief outline of the proof.

Since the semigroup associated with the stochastic MHD system (4.1) has the Feller property, first one proves the time averages ($\{\mu_T\}_{T\geq 0}$) of law of the solution (probability measure) is tight in H by the method developed by Chow and Khasminskii (see [7]) from the energy estimates. Hence due to Prokhorov's Theorem $\{\mu_T\}_{T\geq 0}$ in H is relatively compact. So there exists a sequence μ_{T_n} weakly convergent to a probability measure μ. Finally one proves μ is invariant by the classical method of Krylov and Bogoliubov (Theorem 3.2 (step 4) of Flandoli [14]). One can also prove the following moment estimates of the invariant measure, $\mu\left(\|\cdot\|^2\right) \leq \left(\frac{K}{2-K}\right)$ and $\mu[|\cdot|^p] \leq C(K, K_1, p)$.

For uniqueness, let us assume that μ_1 and μ_2 be two probability measures on H that are stationary for the equation (4.1). We have to show that $\mu_1 = \mu_2$. For this we have to prove for all $\phi \in C_b(H)$, $\int_H \phi(z)\mathrm{d}\mu_1(z) = \int_H \phi(z)\mathrm{d}\mu_2(z)$. Let $\mathbf{X}^\xi$ be the solution of the equation (4.1) with $\mathbf{X}(0) = \xi$. Since μ_1 and μ_2 are invariant measures, by definition we have, $\mu_1(B) = \int_H P_\xi(t, B)\mathrm{d}\mu_1(\xi)$ and $\mu_2(B) = \int_H P_\xi(t, B)\mathrm{d}\mu_2(\xi)$, where $P_\xi(t, B) = \mathbb{P}\left\{\mathbf{X}^\xi(t) \in B\right\}$ and $\mathbf{X}^\xi(0) = \xi$. Define $\mu_t^\xi(B) = \frac{1}{t}\int_0^t P_\xi(s, B)\mathrm{d}s$ for all $B \in \mathscr{B}(H)$. Also, we have, $\mathbb{E}\left(\phi(\mathbf{X}^\xi(t))\right) = \int_H \phi(x)P_\xi(t, \mathrm{d}x)$. By using Fubini's theorem, one gets,

$$\int_H \mu_T^x(\mathrm{d}z)\mathrm{d}\mu_1(x) = \frac{1}{T}\int_0^T \left(\int_H P_x(t, \mathrm{d}z)\mathrm{d}\mu_1(x)\right)\mathrm{d}t = \frac{1}{T}\int_0^T \mu_1(\mathrm{d}z)\mathrm{d}t = \mu_1(\mathrm{d}z).$$

Similarly, we have, $\int_H \mu_T^y(\mathrm{d}z)\mathrm{d}\mu_2(y) = \mu_2(\mathrm{d}z)$.

Now for proving the uniqueness of stationary measures, for $\phi \in C_b(H)$, we use Fubini's theorem, Jensen's inequality and stationarity of invariant measures to get,

$$\begin{aligned}
&\left|\int_H \phi(z)\mathrm{d}\mu_1(z) - \int_H \phi(z)\mathrm{d}\mu_2(z)\right| \\
&= \left|\int_H\int_H \phi(z)\mu_T^x(\mathrm{d}z)\mathrm{d}\mu_1(x) - \int_H\int_H \phi(z)\mu_T^y(\mathrm{d}z)\mathrm{d}\mu_2(y)\right| \\
&= \left|\frac{1}{T}\int_0^T \left[\int_H\int_H \phi(z)P_x(t, \mathrm{d}z)\mathrm{d}\mu_1(x) - \int_H\int_H \phi(z)P_y(t, \mathrm{d}z)\mathrm{d}\mu_2(y)\right]\mathrm{d}t\right| \\
&= \left|\frac{1}{T}\int_0^T\int_H \mathbb{E}\left(\phi(\mathbf{X}^x(t))\right)\mathrm{d}\mu_1(x)\mathrm{d}t - \frac{1}{T}\int_0^T\int_H \mathbb{E}\left(\phi(\mathbf{X}^y(t))\right)\mathrm{d}\mu_2(y)\mathrm{d}t\right| \\
&= \left|\frac{1}{T}\int_0^T \left[\int_H\int_H \mathbb{E}\left(\phi(\mathbf{X}^x(t))\right)\mathrm{d}\mu_1\mathrm{d}\mu_2 - \int_H\int_H \mathbb{E}\left(\phi(\mathbf{X}^y(t))\right)\mathrm{d}\mu_2\mathrm{d}\mu_1\right]\mathrm{d}t\right| \\
&\leq \frac{1}{T}\int_H\int_H\int_0^T \mathbb{E}\left|\phi(\mathbf{X}^x(t)) - \phi(\mathbf{X}^y(t))\right|\mathrm{d}t\mathrm{d}\mu_1(x)\mathrm{d}\mu_2(y).
\end{aligned} \tag{4.13}$$

By the exponential stability (Theorem 4.4) and the continuity of ϕ, we have

$$|\phi(\mathbf{X}^x(t)) - \phi(\mathbf{X}^y(t))| \to 0 \quad \text{a. s.}$$

as $t \to \infty$. Therefore, we get,

$$\frac{1}{T}\int_0^T |\phi(\mathbf{X}^x(t)) - \phi(\mathbf{X}^y(t))|\mathrm{d}t \to 0$$

as $T \to \infty$. Hence, by dominated convergence theorem, the last term in the inequality (4.13) tends to 0 as $T \to \infty$. □

Remark 4.7. Since the abstract functional setting for a class of nonlinear stochastic hydrodynamic models perturbed by Lévy noise, namely 2*D Navier-Stokes equations*, 2*D Boussinesq model for the Bénard convection*, 2*D magnetic Bénard problem*, 3*D Leray* α*-model for Navier-Stokes equations*, *Shell models of turbulence* are same as that of 2*D magneto-hydrodynamic equations*, the main results discussed in this paper will hold for these models also.

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Utpal Manna: School of Mathematics, Indian Institute of Science Education and Research, Thiruvananthapuram, India
E-mail address: manna.utpal@iisertvm.ac.in

Manil T. Mohan: School of Mathematics, Indian Institute of Science Education and Research, Thiruvananthapuram, India
E-mail address: manil@iisertvm.ac.in