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# MEROMORPHIC LÉVY-KHINTCHINE EXPONENTS WITH POLES OF ORDER TWO

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Abstract. Following the recent work of Kuznetsov, we propose the Wiener-Hopf factorization for a new class of Lévy processes. When all the poles of the Lévy exponent $\phi$ have an order equal to two and satisfy some additional conditions, we are able to locate, in the complex plane, the zeros of $q - \phi(z)$, for $q > 0$, thereby yielding the factorization. We also provide a detailed example and a set of conditions under which the factorization holds when the poles are allowed to have various orders.

## 1. Introduction

Fluctuations are one of the most interesting facets of Lévy processes (processes with stationary independent increments). They involve the running supremum $S_t = \sup_{0 \le s \le t} X_s$ and running infimum $I_t$ of the Lévy process $(X_t)_{\{t \geq 0\}}$. The most important result in the theory is the well known Wiener-Hopf factorization which describes the distribution of $S$ and $I$ at an independent exponential random time. More precisely, if $e_q$ is an independent exponential random variable with parameter $q$, then

$$\frac{q}{q-\psi(z)} = \mathbb{E}[e^{izS_{e_q}}]\mathbb{E}[e^{izI_{e_q}}] = \psi_q^+(z)\psi_q^-(z),$$

where

$$\psi(z) = \log(\mathbb{E}[e^{izX_1}]) = imz - \frac{\sigma^2 z^2}{2} + \int_{\mathbb{R}^*} (e^{izx} - 1 - izxh(x))\nu(dx) \qquad (1.1)$$

is the Lévy-Khintchine exponent of $X$ (associated to the cut-off function $h$) and $\psi_q^+, \psi_q^-$ are referred to as the Wiener-Hopf factors. For more details, see chapter VI of [3] or chapter 6 of [15].

Even though some old results exist ([4]), it is only recently that there has emerged a blossoming literature on closed forms of $\psi_q^{\pm}$: [2], [8], [9], [10],[13], [17].

The recent results rely on the fact that, in many cases, $\psi$ can be extended to a meromorphic function on the complex plane and thus can be seen as the ratio of two analytic functions. Writing $q(\psi(z) - q)^{-1} = f_q(z)/g_q(z)$, the aim is then to obtain the Hadamard/Weierstrass factorization of $f_q$ and $g_q$, which often yield formulae that are easily Laplace (or Fourier) inverted into the distributions of $S_{e_q}$ and $I_{e_q}$. Until very recently, the models in the literature either proposed

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factorizations with finitely many (possibly) complex zeros ([17]), or factorizations with an infinite number of real zeros ([8],[9],[13]). The present paper, as well as the recent preprint [14], allows for a factorization with an infinite number of non-real zeros. The Lévy measure in [14] can have any general form but its support is bounded from above while our model allows unbounded positive jumps, but for a specific type of Lévy measure.

Obtaining closed-forms of Wiener-Hopf factorizations is very useful since, given the method developed in [12], it enables to efficiently simulate the couple $(X_t, S_t)$ at fixed times. These results find applications in Quantitative Finance (pricing of barrier and lookback options) and Mathematical Insurance (ruin probabilities).

The paper is organized as follows: in section 2, we state our main result, that is, the Wiener-Hopf factorization when the Lévy density is an infinite sum of $\Gamma(2,\cdot)$ densities. Sections 3 and 4 are aimed at proving and discussing this result. Lastly, in section 5, we provide an example for which it is possible to asymptotically locate the terms in the factorization.

## 2. Notation and Main Result

Let $(X_t)_{t\geq 0}$ be a real-valued Lévy process starting at 0, with Lévy-Khintchine representation given by (1.1). The Lévy measure we are interested in is absolutely continuous and its density is given by

$$\pi(x) = \mathbb{1}_{(x>0)} \sum_{n=1}^{\infty} a_n \rho_n^2 x e^{-\rho_n x} + \mathbb{1}_{(x<0)} \sum_{n=1}^{\infty} \hat{a}_n \hat{\rho}_n^2 (-x) e^{\hat{\rho}_n x},$$

where the $(a_n, \hat{a}_n, \rho_n, \hat{\rho}_n)$ are positive, real numbers satisfying

$$\sum_{n=1}^{\infty} \frac{a_n}{\rho_n} < \infty, \qquad \sum_{n=1}^{\infty} \frac{\hat{a}_n}{\hat{\rho}_n} < \infty, \tag{2.1}$$

which ensures that $\pi$ is indeed a Lévy measure. Moreover, the sequences $\rho_n, \hat{\rho}_n$ are increasing and satisfy

$$\forall n \geq 1, \quad \min(\rho_n, \hat{\rho}_n) \geq c n^{1+\varepsilon}, \quad \max\left(\frac{\rho_{n+1}}{\rho_n}, \frac{\hat{\rho}_{n+1}}{\hat{\rho}_n}\right) \leq C, \tag{2.2}$$

for some strictly positive constants $C, c, \varepsilon$. It is easy to check that under (2.1), $\nu$ is indeed a Lévy measure since $\int_{\mathbb{R}^*} |x|\pi(x)dx < \infty$. In this case, the truncation function is not necessary and we set $h := 0$, which gives

$$\psi(z) = imz - \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} a_n \left(\frac{\rho_n^2}{(\rho_n - iz)^2} - 1\right) + \sum_{n=1}^{\infty} \hat{a}_n \left(\frac{\hat{\rho}_n^2}{(\hat{\rho}_n + iz)^2} - 1\right).$$

In fact, it will be more convenient to work with the Laplace exponent:

$$\begin{aligned} \phi(z) &= \log(\mathbb{E}[e^{zX_1}]) \\ &= mz + \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} a_n \frac{2\rho_n z - z^2}{(\rho_n - z)^2} - \sum_{n=1}^{\infty} \hat{a}_n \frac{2\hat{\rho}_n z + z^2}{(\hat{\rho}_n + z)^2}. \end{aligned} \tag{2.3}$$

This Laplace exponent is well defined on an open set containing zero because $c > 0$.

Before stating our main result, we introduce the following condition, which is discussed in the Appendix.

$$(*)\quad \begin{cases} \forall j \geq 1,\ \forall b^2 > 0, \\ \dfrac{\sigma^2}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \frac{\rho_j^2 - 4\rho_j\rho_n + 3\rho_n^2 + b^2}{((\rho_n - \rho_j)^2 + b^2)^2} + \sum_{n=1}^{\infty} \hat{a}_n \frac{\rho_j^2 + 4\rho_j\hat{\rho}_n + 3\hat{\rho}_n^2 + b^2}{((\hat{\rho}_n + \rho_j)^2 + b^2)^2} > 0, \\ \dfrac{\sigma^2}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \frac{\hat{\rho}_j^2 - 4\hat{\rho}_j\rho_n + 3\rho_n^2 + b^2}{((\rho_n - \hat{\rho}_j)^2 + b^2)^2} + \sum_{n=1}^{\infty} \hat{a}_n \frac{\hat{\rho}_j^2 + 4\hat{\rho}_j\hat{\rho}_n + 3\hat{\rho}_n^2 + b^2}{((\hat{\rho}_n + \hat{\rho}_j)^2 + b^2)^2} > 0. \end{cases}$$

We are now ready to proceed with our main result, which states that the Wiener-Hopf factorization in this setting has the same form as in [8] and the related literature. We formulate it in a probabilistic fashion using the Laplace transform of $S_{e_q}$ and $I_{e_q}$.

**Theorem 2.1.** *For $q, z > 0$, under (2.1), (2.2) and $(*)$,*

$$\mathbb{E}[e^{-zS_{e_q}}] = \frac{1}{1 + \frac{z}{\zeta_0^+}} \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n^+}} \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_n^-}}, \tag{2.4}$$

$$\mathbb{E}[e^{zI_{e_q}}] = \frac{1}{1 - \frac{z}{\zeta_0^-}} \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\hat{\zeta}_n^+}} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\hat{\zeta}_n^-}}, \tag{2.5}$$

*where $\zeta_0^+$, $\zeta_n^+$ and $\zeta_n^-$ (resp. $\zeta_0^-$, $\hat{\zeta}_n^+$ and $\hat{\zeta}_n^-$) are the zeros of $\phi(z) - q$ with positive (resp. negative) real part.*

The notation used for the zeros stems from the fact that they go by pairs since, as we shall see, each pole of order two $\rho_n$ and $\hat{\rho}_n$ will engender two roots for $\phi(z) - q$. The proof of the theorem relies heavily on the location of the zeros of $\phi(z) - q$, a topic discussed in the next section.

## 3. The Location of the Zeros

In [8], Kuznetsov addresses the problem of root localization in two steps: first he finds obvious locations and then proves that they are sufficient (there are no other zeros) using an asymptotic result. Here, we proceed differently: we show that the function has exactly two zeros in an infinite number of well-defined zones, and none outside these zones.

More specifically, we define

$$R_0 = \{z \in \mathbb{C}\,,\ -\hat{\rho}_1 < \Re(z) < \rho_1, |\Im(z)| \leq \max(\rho_1, \hat{\rho}_1)\}$$

and the two series of rectangles: $\forall n \geq 1$,

$$R_n = \{z \in \mathbb{C},\ \rho_n < \Re(z) < \rho_{n+1},\ |\Im(z)| \leq \rho_{n+1}\}$$

$$\hat{R}_n = \{z \in \mathbb{C},\ -\hat{\rho}_n > \Re(z) > -\hat{\rho}_{n+1},\ |\Im(z)| \leq |\hat{\rho}_{n+1}|\}$$

The main result of the section follows.

**Proposition 3.1.** *For any $q > 0$, under $(*)$, all the zeros of $\phi(z) - q$ are located in $R_0 \cup \bigcup_{n\geq 1}(R_n \cup \hat{R}_n)$. Moreover, for any $n \geq 1$, $R_n$ and $\hat{R}_n$ both contain exactly 2 zeros or one double zero and so does $R_0$.*

The proof of the proposition will require a few intermediate results. We first introduce two test functions, defined for $q_n, \hat{q}_n > 0$,

$$\Phi_n(z) = \frac{1}{(\rho_n - z)^2} + \frac{1}{(\rho_{n+1} - z)^2} - q_n, \quad \hat{\Phi}_n(z) = \frac{1}{(\hat{\rho}_n + z)^2} + \frac{1}{(\hat{\rho}_{n+1} + z)^2} - \hat{q}_n$$

and which have the following properties.

**Lemma 3.2.** *For all $n \geq 1$, $\Phi_n$ (resp $\hat{\Phi}_n$) has only two zeros in $R_n$ (resp $\hat{R}_n$). Furthermore, the real part of $\Phi_n$ (resp $\hat{\Phi}_n$) is strictly negative on $\partial R_n$ (resp $\partial \hat{R}_n$).*

*Proof.* We will often use the usual complex notation $z = a + ib$. We will prove the lemma for $\Phi_n$ since the transposition to $\hat{\Phi}_n$ will be straightforward. First notice that $\Phi_n$ has exactly four zeros:

$$z_q^{1,\pm} = \frac{\rho_n + \rho_{n+1}}{2} \pm \frac{\sqrt{q_n\left(4 + q_n(\rho_n - \rho_{n+1})^2 - 4\sqrt{1 + q_n(\rho_n - \rho_{n+1})^2}\right)}}{2q_n}$$

$$z_q^{2,\pm} = \frac{\rho_n + \rho_{n+1}}{2} \pm \frac{\sqrt{q_n\left(4 + q_n(\rho_n - \rho_{n+1})^2 + 4\sqrt{1 + q_n(\rho_n - \rho_{n+1})^2}\right)}}{2q_n}$$

The zeros $z_q^{2,\pm}$ are both real and outside $R_n$ and since $|4 + x - 4\sqrt{1+x}| \leq x$ for $x \geq 0$, it is obvious that

$$|4 + q_n(\rho_n - \rho_{n+1})^2 - 4\sqrt{1 + q_n(\rho_n - \rho_{n+1})^2}| \leq q_n(\rho_n - \rho_{n+1})^2,$$

hence, the $z_q^{1,\pm}$ belong to $R_n$ and are either both real or both complex with $|\Im(z_q^{1,\pm})| \leq (\rho_{n+1} - \rho_n)/2$.

Moreover,

$$\Re(\Phi_n(a + ib)) = -q_n + \frac{(\rho_n - a)^2 - b^2}{((\rho_n - a)^2 + b^2)^2} + \frac{(\rho_{n+1} - a)^2 - b^2}{((\rho_{n+1} - a)^2 + b^2)^2},$$

so that for $a = \rho_n$ and denoting $\rho = (\rho_n - \rho_{n+1})^2$,

$$\Re(\Phi_n(\rho_n + ib)) = -q_n + \frac{-1}{b^2} + \frac{\rho - b^2}{(\rho + b^2)^2} = -q_n + \frac{-\rho^2 - b^2\rho - 2b^4}{b^2(\rho + b^2)^2} < 0, \quad \forall b \in \mathbb{R}.$$

The proof is the same for $a = \rho_{n+1}$. Lastly, for $b = \pm\rho_{n+1}$,

$$\Re(\Phi_n(a \pm i\rho_{n+1})) = -q_n + \frac{(\rho_n - a)^2 - \rho_{n+1}^2}{((\rho_n - a)^2 + \rho_{n+1}^2)^2} + \frac{(\rho_{n+1} - a)^2 - \rho_{n+1}^2}{((\rho_{n+1} - a)^2 + \rho_{n+1}^2)^2}$$

and both numerators are strictly negative for $a \in (\rho_n, \rho_{n+1})$. □

The proof of Proposition 3.1 will rely on the following reinforcement, due to Estermann (see [5] p. 156), of Rouché's theorem.

**Theorem 3.3** (Estermann-Rouché's Theorem)**.** *Let $f$ and $g$ be two holomorphic functions inside and on some simple contour $\partial K$. If $|f(z)-g(z)| < |f(z)|+|g(z)|$ on $\partial K$, then $f$ and $g$ have the same number of zeros (counting multiplicities) inside $K$.*

Our objective is to apply this theorem to the functions $\phi$ and $\Phi_n$. The technical assumption $(*)$ will be required to ensure the strict inequality required by the theorem. This technique was already used in a similar context in the proof of the main theorem of [17]. However, we must choose proper contours $K_n$ to proceed. To this purpose, we introduce two series of disks: given their radiuses $\varepsilon_{n,q}, \hat{\varepsilon}_{n,q} > 0$, they are defined for all $n \geq 1$ by

$$D_{n,q} = \{z \in \mathbb{C},\ |z-\rho_n| \leq \varepsilon_{n,q}\}, \quad \hat{D}_{n,q} = \{z \in \mathbb{C},\ |z-\hat{\rho}_n| \leq \hat{\varepsilon}_{n,q}\}$$

We will need the following result related to these disks.

**Lemma 3.4.** *For any $n \geq 1$ and $q > 0$, there exist $\varepsilon_{n,q}, \hat{\varepsilon}_{n,q} > 0$ such that*

*i) $\forall z \in D_{n,q}$, $|\phi(z) - q - \Phi_n(z)| < |\phi(z) - q| + |\Phi_n(z)|$*
*i') $\forall z \in \hat{D}_{n,q}$, $|\phi(z) - q - \hat{\Phi}_n(z)| < |\phi(z) - q| + |\hat{\Phi}_n(z)|$*
*ii) there are no zeros of $\phi(z) - q$ inside $D_{n,q}$*
*ii') there are no zeros of $\phi(z) - q$ inside $\hat{D}_{n,q}$*

*Proof.* The proof of both $i$) and $ii$) relies on the fact that as $\varepsilon_{n,q}$ decreases, both $\phi$ and $\Phi_n$ behave like $t(z) = (\rho_n - z)^{-2}$ inside $D_{n,q}$. Indeed, $\phi(z) - a_n\rho_n^2 t(z)$ is bounded inside $D_{n,q}$ and so is $\Phi(z) - t(z)$.

We divide $D_{n,q}$ into eight radial subsets of equal size (and angle), as shown in Figure 1. The function $t$ has the following properties:

- in areas Im+, $\Im(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough
- in areas Im-, $-\Im(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough
- in areas Re+, $\Re(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough
- in areas Re-, $-\Re(t)$ can take arbitrarily large values for $\varepsilon_{n,q}$ small enough

First, this means that for $\varepsilon_{n,q}$ small enough, $|\phi(z) - q| > 1$, yielding $ii$). Moreover, for $\varepsilon_{n,q}$ small enough, there is either $\Im(\phi(z))\Im(\Phi_n(z)) > 0$ or $\Re(\phi(z) - q)\Re(\Phi_n(z)) > 0$ for any $z \in D_{n,q}$. This implies $i$) since $|x-y| = |x| + |y|$ if and only if 0 belongs to the segment $[x, y]$ in the complex plane (both imaginary and real parts must have opposite signs).

We prove $i'$) and $ii'$) likewise. □

Lastly, to cover the whole complex plane, we need to show the following lemma.

**Lemma 3.5.** *For $q > 0$, let $z^* = a + ib$ be a non-real zero of $\phi(z) - q$. Then $|a| \geq \sqrt{3}|b|$.*

*Proof.* First, we have the following identities

$$\frac{2\rho(a+bi)-(a+bi)^2}{(\rho-a-bi)^2} = \frac{-a^4+4a^3\rho-5a^2\rho^2-2a^2b^2+2a\rho^3-3b^2\rho^2+4ab^2\rho-b^4}{((\rho-a)^2+b^2)^2} + i\frac{2b\rho^2(\rho-a)}{((\rho-a)^2+b^2)^2} \tag{3.1}$$

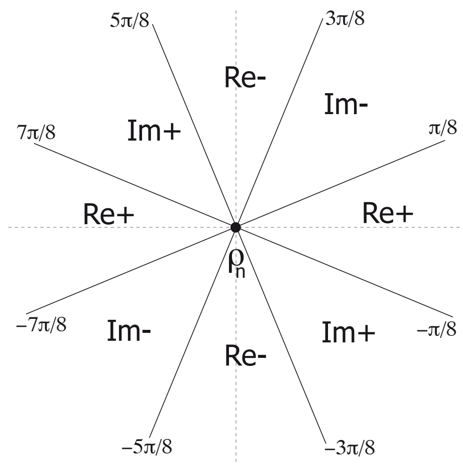


FIGURE 1. The subdivisions of $D_{n,q}$

$$\frac{2\rho(a+bi)+(a+bi)^2}{(\rho+a+bi)^2} = \frac{a^4+4a^3\rho+5a^2\rho^2+2a^2b^2+2a\rho^3+3b^2\rho^2+4ab^2\rho+b^4}{((\rho-a)^2+b^2)^2} + i\frac{2b\rho^2(\rho+a)}{((\rho+a)^2+b^2)^2} \tag{3.2}$$

which leads to

$$\begin{aligned}\Re(\phi(z^*)-q) = &-q+\frac{\sigma^2}{2}(a^2-b^2)+am \\ &+\sum_{n=1}^{\infty} a_n \frac{-a^4+4a^3\rho_n-5a^2\rho_n^2-2a^2b^2+2a\rho_n^3-3b^2\rho_n^2+4ab^2\rho_n-b^4}{((\rho_n-a)^2+b^2)^2} \\ &-\sum_{n=1}^{\infty} \hat{a}_n \frac{a^4+4a^3\hat{\rho}_n+5a^2\hat{\rho}_n^2+2a^2b^2+2a\hat{\rho}_n^3+3b^2\hat{\rho}_n^2+4ab^2\hat{\rho}_n+b^4}{((\hat{\rho}_n-a)^2+b^2)^2}\end{aligned} \tag{3.3}$$

and

$$\Im(\phi(z^*)) = \sigma^2 ab + bm + \sum_{n=1}^{\infty} a_n \frac{2b\rho_n^2(\rho_n-a)}{((\rho_n-a)^2+b^2)^2} - \sum_{n=1}^{\infty} \hat{a}_n \frac{2b\hat{\rho}_n^2(\hat{\rho}_n+a)}{((\hat{\rho}_n+a)^2+b^2)^2} \tag{3.4}$$

so that as $b \neq 0$ and $\Im(\phi(z^*)) = \Re((\phi(z^*)-q) = 0$, after simplifications,

$$\begin{aligned}0 =& \Re(\phi(z^*)-q) - \frac{a}{b}\Im(\phi(z^*)) \\ =& -q-(a^2+b^2)\frac{\sigma^2}{2} \\ &-(a^2+b^2)\left(\sum_{n=1}^{\infty} a_n \frac{a^2-4a\rho_n+3\rho_n^2+b^2}{((\rho_n-a)^2+b^2)^2} + \sum_{n=1}^{\infty} \hat{a}_n \frac{a^2+4a\hat{\rho}_n+3\hat{\rho}_n^2+b^2}{((\hat{\rho}_n+a)^2+b^2)^2}\right)\end{aligned} \tag{3.5}$$

If $a > 0$ and $|a| < \sqrt{3}|b|$, then $\forall n \geq 1$,

$$a^2 - 4a\rho_n + 3\rho_n^2 + b^2 > \frac{4}{3}(a + 3\rho_n/2)^2 \geq 0,$$

and $a^2 + 4a\hat{\rho}_n + 3\hat{\rho}_n^2 + b^2 > 0$ so that $z^*$ cannot be a zero of $\phi(z) - q$. If $a < 0$ and $|a| < \sqrt{3}|b|$, then the contradiction is the same. Hence, for $a \neq 0$, the zero $z^* = a + ib$ must satisfy $|a| \geq \sqrt{3}|b|$.

Lastly, for $a = 0$,

$$\Re(\phi(z^*) - q) = -q - \frac{\sigma^2 b^2}{2} - \sum_{n=1}^{\infty} a_n \frac{3b^2\rho_n^2 + b^4}{(\rho_n^2 + b^2)^2} - \sum_{n=1}^{\infty} \hat{a}_n \frac{3b^2\hat{\rho}_n^2 + b^4}{(\hat{\rho}_n^2 + b^2)^2},$$

which is strictly negative for any real $b$, hence there are no purely imaginary zeros of $\phi(z) - q$. □

The lemma tells us that there are no zeros of $\phi(z) - q$ in the angles $(\pi/6, 5\pi/6)$ and $(-5\pi/6, -\pi/6)$ of the complex plane. We are now ready to prove the proposition.

*Proof of Proposition 3.1.* We begin with $R_n$ and $\hat{R}_n$. The aim of the proof is to show that there are $\varepsilon_{n,q}, \varepsilon_{n+1,q} > 0$ (resp $\hat{\varepsilon}_{n,q}, \hat{\varepsilon}_{n+1,q} > 0$) such that on the boundary of $K_n = R_n \backslash \{D_{n,q} \cup D_{n+1,q}\}$ (see Figure 2) (resp $\hat{K}_n = \hat{R}_n \backslash \{\hat{D}_{n,q} \cup \hat{D}_{n+1,q}\}$), the condition of Rouché's theorem applies, that is,

$$(**) \qquad |\phi(z) - q - \Phi_n(z)| < |\phi(z) - q| + |\Phi_n(z)|$$

(resp $|\phi(z) - q - \hat{\Phi}_n(z)| < |\phi(z) - q| + |\hat{\Phi}_n(z)|$). To prove this, we will rely on the following equivalence, for $x, y \in \mathbb{C}$

$$\left.\begin{array}{c} |x - y| = |x| + |y| \\ \text{if and only if} \\ \Re(x)\Im(y) = \Im(x)\Re(y) \text{ and } \Im(x)\Im(y) \leq 0 \text{ and } \Re(x)\Re(y) \leq 0 \end{array}\right\} \tag{3.6}$$

We are first interested in the horizontal and vertical segments of $\partial K_n$, a set which we denote by $S_n$ (see Figure 2). Recall the expression of $\Re(\phi(a + ib)) - \frac{a}{b}\Im(\phi(a + ib))$ given by (3.5). Because of $(*)$ (vertical segments), and Lemma 3.5 (horizontal segments), we have, for $z = a+ib \in S_n$, $\Re(\phi(a+ib)) - \frac{a}{b}\Im(\phi(a+ib)) < 0$. Hence, near the zeros of $\Im(\phi)$, $\Re(\phi)$ is negative. More precisely, there are two cases involving $\epsilon_n := \inf\{|\Im(\phi(z))|, z \in S_n\}$:

- either $\epsilon_n > 0$ and $\Im(\phi)$ has no zero on $S_n$
- or $\epsilon_n = 0$ and there exists $\epsilon > 0$ such that $V_{n,q,\epsilon} = \{z \in S_n, |\Im(\phi(z))| < \epsilon, \Re(\phi(z) - q) < 0\} \neq \emptyset$.

In either case, on $S_n \backslash V_{n,q,\epsilon}$, $|\Im(\phi)|$ is bounded from below by, say, $\kappa_{n,q,\epsilon} > 0$.

We want to prove $(**)$ on the following three sets: $V_{n,q,\epsilon}$, $S_n \backslash V_{n,q,\epsilon}$ and $\partial K_n \backslash S_n$ (this last set consisting in the two semi-circles).

- by Lemma 3.2, $\Re(\Phi_n) < 0$ on $S_n$ thus (3.6) ensures that $(**)$ holds on $V_{n,q,\epsilon}$ if it is not empty.

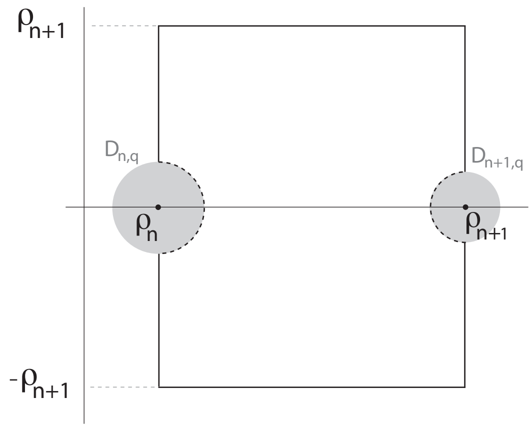


FIGURE 2. the contour $K_n$ with $S_n$ in hard line

- on $S_n \backslash V_{n,q,\epsilon}$, $|\Im(\phi)| \geq \kappa_{n,q,\epsilon} > 0$ and both $\Im(\Phi_n)$ and $\Re(\phi)$ are bounded, it is therefore possible to find a $q_n$ (in the definition of $\Phi_n$) such that

$$|\Re(\phi(z) - q)\Im(\Phi_n(z))| < |\Im(\phi(z))\Re(\Phi_n(z))|, \quad z \in S_n \backslash V_{n,q},$$

  hence, by (3.6), $(**)$ holds on $S_n \backslash V_{n,q}$.

- for the two semi-circles of $\partial K_n$, we invoke Lemma 3.4. The radiuses must be chosen small enough for the zeros of $\Phi_n$ to be in $K_n$.

Therefore, by Rouché's theorem, for any $q > 0$, under $(*)$, the function $\phi(z) - q$ has exactly 2 zeros (or a double zero) in $R_n$.
The proof is identical for the sets $\hat{R}_n$. For $R_0$, it is easy to see that the properties of $\Phi_n$ and $\hat{\Phi}_n$ described in Lemma 3.2 also hold for $\Phi_0 = \frac{1}{(\hat{\rho}_1+z)^2} + \frac{1}{(\rho_1-z)^2} - q_0$ on $\partial R_0$ and hence the same reasoning applies (with the proper $K_0, S_0$ and $V_{0,q,\epsilon}$).
Lastly, the 3 sets were constructed so that with Lemma 3.5, the whole complex plane is covered. □

We denote by $\zeta_n^+$ and $\zeta_n^-$ the two roots in $R_n$. If they are complex then $\Im(\zeta_n^+) > \Im(\zeta_n^-)$, if not, then $\Re(\zeta_n^+) \geq \Re(\zeta_n^-)$. The equivalent notations hold for $R_0$ and $\hat{R}_n$.

## 4. Proof and Discussion of the Theorem

This section is divided into three parts. First, we prove Theorem 2.1, using Proposition 3.1 and Kuznetsov's paper [11]. Then we discuss a possible generalization when the poles of $\phi$ are allowed to have any finite order. Lastly, we introduce a simple condition which implies $(*)$.

**4.1. Proof of Theorem 2.1.** The proof will rely on the following lemma. We denote by log the principal branch of the complex logarithm defined on $\mathbb{C} \backslash \mathbb{R}_-$.

**Lemma 4.1.** *Define* $A^{\pm}(z) = \left| \frac{1}{z} \log \left( \prod_{n=1}^{\infty} \frac{\zeta_n^{\pm}(\rho_n - z)}{\rho_n(\zeta_n^{\pm} - z)} \right) \right|$ *and*

$$\hat{A}^{\pm}(z) = \left| \frac{1}{z} \log \left( \prod_{n=1}^{\infty} \frac{\hat{\zeta}_n^{\pm}(\hat{\rho}_n + z)}{\hat{\rho}_n(\hat{\zeta}_n^{\pm} - z)} \right) \right|.$$

*Then*

$$A^{\pm}(z) \ (resp.\ \hat{A}^{\pm}(z)) \to 0, \quad as\ |z| \to \infty,\ \Re(z) \leq 0\ (resp.\ \Re(z) \geq 0).$$

*Proof.* First note that by Proposition 3.1 and (2.2),

$$\left| \frac{\zeta_n^{\pm}(\rho_n - z)}{\rho_n(\zeta_n^{\pm} - z)} - 1 \right| = \left| z \frac{\rho_n - \zeta_n^{+}}{\rho_n(\zeta_n^{+} - z)} \right| = O(n^{-1-\varepsilon}),$$

so that the infinite product is well defined. Again by Proposition 3.1 and (2.2), $|\zeta_n^{+}/\rho_n| > 1$. Moreover, $\left| \frac{\rho_n - z}{\zeta_n^{+} - z} \right| \to 1$ as $|z| \to \infty$, $\Re(z) \leq 0$. The function $z \mapsto \left| \frac{\zeta_n^{+}(\rho_n - z)}{\rho_n(\zeta_n^{+} - z)} \right|$ is thus bounded from below for $\Re(z) \leq 0$ and there is $C > 0$ such that for $\Re(z) \leq 0$ and $n \geq 1$,

$$\left| \log \left( \frac{\zeta_n^{+}(\rho_n - z)}{\rho_n(\zeta_n^{+} - z)} \right) \right| \leq C \left| \frac{\zeta_n^{+}(\rho_n - z)}{\rho_n(\zeta_n^{+} - z)} - 1 \right|.$$

Hence,

$$A^{+}(z) \leq \left| \sum_{n=1}^{\infty} \frac{\rho_n - \zeta_n^{+}}{\rho_n(\zeta_n^{+} - z)} \right| = O\left( \sum_{n=1}^{\infty} \frac{1}{|\zeta_n^{+} - z|} \right).$$

Proposition 3.1 and (2.2) ensure that the infinite sum is finite for any $z$ with negative real part and by dominated convergence, $A^{+}(z) \to 0$ as $|z| \to \infty$.

The proof for $A^{-}, \hat{A}^{\pm}$ is achieved in a similar fashion. □

*Proof of Theorem 2.1.* First note that due to condition (2.2) and Proposition 3.1, $\sum_{n=1}^{\infty} \frac{1}{|\zeta_n^{\pm}|} < \infty$, so that the order of the entire function

$$(q - \phi(z)) \prod_{n=1}^{\infty} \left(1 - \frac{z}{\rho_n}\right)^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{\hat{\rho}_n}\right)^2$$

is less than one (see [16], lecture 5, for instance). Hence, the only possible Hadamard/Weierstrass representation ([16] page 26) for $\frac{q}{q-\phi(z)}$ is

$$\frac{q}{q - \phi(z)} = e^{cz} \frac{1}{1 - \frac{z}{\zeta_0^{+}}} \frac{1}{1 - \frac{z}{\zeta_0^{-}}} \prod_{n=1}^{\infty} \frac{1 - \frac{z}{\rho_n}}{1 - \frac{z}{\zeta_n^{+}}} \frac{1 - \frac{z}{\rho_n}}{1 - \frac{z}{\zeta_n^{-}}} \prod_{n=1}^{\infty} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\hat{\zeta}_n^{+}}} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 - \frac{z}{\hat{\zeta}_n^{-}}}, \tag{4.1}$$

for some $c \in \mathbb{C}$ after arrangement.

We rely on Lemma 4.1 to prove that $c = 0$ (in the same way as in the end of the proof of Theorem 5 in [8], using the fact that $\phi(iz) = O(z^2)$ for $z \to \infty$, $z \in \mathbb{R}$). Lastly, all the conditions of Theorem 1 $(f)$ from [11] are fulfilled, which completes the proof. □

**4.2. Towards a generalization.** It is natural to ask what might happen if the density of the Lévy measure had the more general form

$$\pi(x) = \mathbb{1}_{(x>0)} \sum_{n=1}^{\infty} a_n \frac{\rho_n^{m_n}}{(m_n-1)!} x^{m_n-1} e^{-\rho_n x} + \mathbb{1}_{(x<0)} \sum_{n=1}^{\infty} \hat{a}_n \frac{\hat{\rho}_n^{\hat{m}_n}}{(\hat{m}_n-1)!} (-x)^{\hat{m}_n-1} e^{\hat{\rho}_n x},$$

where $(a_n, \hat{a}_n, \rho_n, \hat{\rho}_n)$ satisfy the usual conditions and $(m_n, \hat{m}_n) \in \{1, ..., M\}$, $(M < \infty)$, which would give

$$\phi(z) = mz + \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} a_n \left( \frac{\rho_n^{m_n}}{(\rho_n - z)^{m_n}} - 1 - \frac{z m_n}{\rho_n} \right) + \sum_{n=1}^{\infty} \hat{a}_n \left( \frac{\hat{\rho}_n^{\hat{m}_n}}{(\hat{\rho}_n + z)^{\hat{m}_n}} - 1 + \frac{z \hat{m}_n}{\hat{\rho}_n} \right).$$

The case $M = 2$ can be treated using the ideas of the present paper (with a slightly different condition $(*)$ and four pairs of test functions). The central problem remains to get precise results on the location of the zeros of $\phi(z) - q$.

In the general case, finding a multiple zero is not easy and we cannot apply Theorem 2.1. However, heuristically, if we consider $q \to \infty$, we see that, asymptotically, the $(\rho_n, -\hat{\rho}_n)$ become zeros of order $m_n$ and $\hat{m}_n$. Hence, when $q$ decreases, the zeros should be located in the vicinity, in some sense, of the $\rho_n$ and $-\hat{\rho}_n$. Empirically, many computations show that, in fact, each $\rho_n$ engenders $m_n$ zeros, all of which are located inside circles with center $\rho_n$, radius $\rho_n$ and all the more close to $\rho_n$ than $q$ is large. However, this fact (which requires formal proof) does not suffice to show the convergence $A(z) \to 0$ in the proof of the theorem.

In this spirit, we would like expose a set of conditions under which the theorem remains valid. Namely,

**Theorem 4.2.** *If there is an ordering of the zeros and poles, repeated according to multiplicity, such that*

*i)* $\phi(z) - q$ *is meromorphic with real poles* $\rho_n, -\hat{\rho}_n$ *and zeros* $\zeta_n, \hat{\zeta}_n$ *satisfying* $\Re(\zeta_n) > 0$, $\Re(\hat{\zeta}_n) < 0$,

*ii) the series with terms* $\dfrac{1}{\rho_n}, \dfrac{1}{\hat{\rho}_n}, \dfrac{1}{\zeta_n}, \dfrac{1}{\hat{\zeta}_n}$ *are absolutely convergent,*

*iii)* $\forall n > 0$, $|\zeta_n - \rho_n| < C|\rho_n|$, $|\hat{\zeta}_n - \hat{\rho}_n| < \hat{C}|\hat{\rho}_n|$ *for some* $C, \hat{C} > 0$,

*then (2.4) and (2.5) hold.*

*Proof.* Condition *ii*) ensures that $q/(\phi(z) - q)$ is the ratio of two entire functions of order less than one and has thus a similar form as (4.1), namely

$$\frac{q}{q - \phi(z)} = e^{cz} \frac{h_+(z)}{g_+(z)} \frac{h_-(z)}{g_-(z)},$$

where $h_\pm, g_\pm$ are holomorphic in $\mathbb{C}$, with zeros in $\mathbb{C}^\pm = \{z \in \mathbb{C}, \pm\Re(z) > 0\}$ normalized so that $h_\pm(0) = g_\pm(0) = 1$. Then, as in the proof of Theorem 2.1, condition $iii$) will ensure that $z^{-1}\log\left(\frac{h_\pm(z)}{g_\pm(z)}\right) \to 0$ as $z \to \mp\infty$, so that the factorization is indeed of Wiener-Hopf type (and $c = 0$). □

*Remark* 4.3. We wish to show the connexion between the papers on meromorphic Lévy-Khintchine exponents and the ideas in [6]. For simplicity, we will consider $q \in \mathcal{Q} := \{q > 0,\ \phi(z) - q \text{ has only simple zeros}\}$. Once the zeros are located, Theorem 1 in [9] is a special case of Theorem 4.1 in [6]. It is however possible to consider complex zeros, as long as the proof of Lemma 3.1 in [6] remains valid. This can only happen if the arguments of the zeros stay away from $\pi/2$ and $-\pi/2$. Condition $iii$) in Theorem 4.2 implies this and it is verified in [8], [9] and [13], as long as the $(\rho_n, \hat\rho_n)$ in those papers have at most an exponential growth. It is thus possible to very briefly prove Theorem 4.2 using the fact that under $i$), $ii$) and $iii$),

$$\sum_{n=1}^{\infty} \frac{h_+(\zeta_n)}{g'_+(\zeta_n)} e^{-x\zeta_n}, \quad \sum_{n=1}^{\infty} \frac{h_-(\hat\zeta_n)}{g'_-(\hat\zeta_n)} e^{-x\hat\zeta_n}$$

are convergent for $x > 0$ and can then be proven to be the densities of $S_{e_q}$ and $I_{e_q}$ using dominated convergence, as in [6], Theorem 4.1 (using Lemma 3.2).

## 5. An Example

**5.1. Introducing the Laplace Exponent.** Our goal in this section is to be able to locate precisely the zeros of $\phi$ for one exponent involving known functions. We will show that even in this very simple case, this is not so easy to achieve. We set $\rho_n = \hat\rho_n = n^2$ and unit $a_n$ and $\hat a_n$, and propose the following Lévy measure,

$$\pi(x) = \mathbb{1}_{\{x>0\}} \sum_{n=1}^{\infty} x n^4 e^{-n^2 x} + \mathbb{1}_{\{x<0\}} \sum_{n=1}^{\infty} (-x) n^4 e^{n^2 x}.$$

First note that in this case, as is shown at the very end of the Appendix, $(*)$ holds. Moreover, it is easy to show (see Proposition 4 in [9]) that as $x \to 0^\pm$, $\pi(x) = O(|x|^{-3/2})$, which confirms that we can take a zero cut-off function ($h := 0$) and the following representation for $\phi$:

$$\phi(z) = mz + \frac{\sigma^2 z^2}{2} + \sum_{n=1}^{\infty} \frac{2zn^2 - z^2}{(n^2 - z)^2} - \sum_{n=1}^{\infty} \frac{2zn^2 + z^2}{(n^2 + z)^2}.$$

This particular choice of $\phi$ was made to exhibit known functions. Using the fact that

$$\frac{2zn^2 - z^2}{(n^2 - z)^2} = \frac{1}{4}\left(\frac{-6z}{z - n^2} + 2\frac{z^2 + zn^2}{(z - n^2)^2}\right)$$

and relations 1.421-3, 1.422-4 (second equality) in [7], we get

$$\sum_{n=1}^{\infty} \frac{2n^2 z - z^2}{(n^2 - z)^2} = \frac{1}{4}\left(2 - 3\pi\sqrt{z}\cot(\pi\sqrt{z}) + \pi^2 z \csc(\pi\sqrt{z})^2\right)$$

and substituting $i\sqrt{z}$ for $\sqrt{z}$,

$$-\sum_{n=1}^{\infty} \frac{2n^2 z + z^2}{(n^2 + z)^2} = \frac{1}{4}\left(2 - i3\pi\sqrt{z}\cot(i\pi\sqrt{z}) - \pi^2 z \csc(i\pi\sqrt{z})^2\right),$$

where cot and csc are the usual cotangent and cosecant functions. Recalling

$$\csc(z)^2 = \sin(z)^{-2} = \frac{\cos(z)^2 + \sin(z)^2}{\sin(z)^2} = 1 + \cot(z)^2, \tag{5.1}$$

$\phi$ can in this case be expressed solely in terms of the cotangent function

$$\begin{aligned}\phi(z) &= mz + \tfrac{\sigma^2}{2}z^2 + 1 + \frac{1}{4}\left[\pi\sqrt{z}\cot(\pi\sqrt{z})(-3+\pi\sqrt{z}\cot(\pi\sqrt{z}))\right] \\ &\quad + \frac{i}{4}\left[\pi\sqrt{z}\cot(i\pi\sqrt{z})(-3+i\pi\sqrt{z}\cot(i\pi\sqrt{z}))\right] \\ &:= mz + \tfrac{\sigma^2 z^2}{2} + 1 + \frac{1}{4}(c_+(z) + c_-(z)).\end{aligned}$$

We refer to section 4.3 of [1] for the behavior of the cotangent function in the complex plane. One of its useful properties is that if $z$ has a large imaginary part, then both the real and imaginary parts of $\cot(z)$ can be accurately estimated. More precisely, by Euler's formula and equation 4.3.58 in [1], for any $a_n > 0$ and for any $b_n > 0$ large enough,

$$\begin{aligned}\cot(a_n \pm ib_n) &= \cot(-a_n \pm ib_n) \\ &= 2\sin(2a)e^{-2b_n} + i(\mp 1 + 2\cos(2a)e^{-2b_n}) + o(e^{-2b_n})(1+i).\end{aligned} \tag{5.2}$$

Lastly, recall that the zeros $\zeta_n^\pm = a_n \pm ib_n$ of $\phi(z) - q$ are ordered so that the series $(a_n)_{\{n\geq 1\}}$ is increasing.

**Lemma 5.1.** *There are, asymptotically, no real zeros of $\phi(z) - q$, except if $\sigma = 0$ and $m \geq \pi^2/4$ (resp $m \leq -\pi^2/4$), in which case, $\hat{\zeta}_n^\pm$ (resp $\zeta_n^\pm$) are real.*

*Proof.* The proof lies in the fact that, by (5.2), for $z$ real large enough, $c_-(z) = c_+(-z) \sim (-3\pi\sqrt{z} + \pi^2 z)/4$. Moreover, both $c_+$ and $c_-$ are $U$-shaped between their poles with a negative local minimum which is close to $-0.5$. The local minima of $\phi$ thus go to $+\infty$ (yielding only complex zeros) or to $-\infty$ (yielding only real zeros). □

**5.2. Locating the zeros.** We will now focus on $\zeta_n^\pm$, as the transposition to $\hat{\zeta}_n^\pm$ is straightforward. Note that $\zeta_n^\pm$ verifies

$$\begin{aligned}&\frac{\sigma^2}{2}(\zeta_n^\pm)^2 + m\zeta_n^\pm + 1 - q + \frac{1}{4}\left[-3i\pi\sqrt{\zeta_n^\pm}\cot\left(i\pi\sqrt{\zeta_n^\pm}\right) - \pi^2\zeta_n^\pm\cot\left(i\pi\sqrt{\zeta_n^\pm}\right)^2\right] \\ &\qquad = -\frac{1}{4}\left[-3\pi\sqrt{\zeta_n^\pm}\cot\left(\pi\sqrt{\zeta_n^\pm}\right) + \pi^2\zeta_n^\pm\cot\left(\pi\sqrt{\zeta_n^\pm}\right)^2\right].\end{aligned} \tag{5.3}$$

For $\zeta_n^\pm$ away from the zeros of $\Re(\cot(\pi\sqrt{z}))$, dividing (5.3) by $\zeta_n^\pm$ yields

$$\Im\left(\frac{3\pi}{4}\frac{\cot\left(\pi\sqrt{\zeta_n^\pm}\right)}{\sqrt{\zeta_n^\pm}} - \frac{\pi^2}{4}\cot\left(\pi\sqrt{\zeta_n^\pm}\right)^2\right) = \pm\frac{\sigma^2}{2}b_n + O(\Im((\zeta_n^\pm)^{-1})), \quad n \to +\infty \tag{5.4}$$

and

$$\Re\left(-\frac{\pi^2}{4}\cot\left(\pi\sqrt{\zeta_n^{\pm}}\right)^2\right) = \frac{\sigma^2}{2}a_n + m + \frac{\pi^2}{4} + R(n) + o(R(n)), \quad n \to +\infty \quad (5.5)$$

where $R(n) = -\frac{3\pi}{4}\Re\left((\zeta_n^{\pm})^{-1/2}\left[\cot\left(\pi\sqrt{\zeta_n^{\pm}}\right) + i\cot\left(i\pi\sqrt{\zeta_n^{\pm}}\right)\right]\right)$.

We are now able to accurately locate the $\zeta_n^{\pm}$, asymptotically, in all possible cases.

**Proposition 5.2.** *As $n \to +\infty$,*

- *if $\sigma > 0$, $\zeta_n^{\pm} = n^2 \pm i\sqrt{2/\sigma^2} + O(n^{-2})(1+i)$*
- *if $\sigma = 0$ and $m > 0$, then*

$$\zeta_n^{\pm} = n^2 + c + O(n^{-1}) \pm i\frac{n}{\pi}\left(\cosh^{-1}\left(1 + \frac{\pi^2}{2m}\right) + O(n^{-1})\right)$$

- *if $\sigma = 0$ and $m \in (-\pi^2/4, 0)$, then*

$$\zeta_n^{\pm} = n^2 + n + c + O(n^{-1}) + \pm i\frac{n}{\pi}\left(\cosh^{-1}\left(-1 - \frac{\pi^2}{2m}\right) + O(n^{-1})\right)$$

- *if $\sigma = 0$ and $m = 0$,*

$$\Re(\sqrt{\zeta_n^{\pm}}) = n + 3/8 + O(\log(n)n^{-2}) \text{ and } \frac{2\pi\,\Im(\sqrt{\zeta_n^{+}})}{\log(\frac{4\pi}{3\sqrt{2}}n)} \to 1$$

- *if $\sigma = 0$ and $m \leq -\pi^2/4$, then*

$$\zeta_n^{-} = n^2 + \frac{n}{\pi}\left(\cos^{-1}\left(1 + \frac{\pi^2}{2m}\right) + cn^{-1} + o(n^{-1})\right)$$

  *and*

$$\zeta_n^{+} = (n+1)^2 - \frac{n}{\pi}\left(\cos^{-1}\left(1 + \frac{\pi^2}{2m}\right) + cn^{-1} + o(n^{-1})\right),$$

*for some irrelevant constants $c$, and where* $\cos^{-1}$ *and* $\cosh^{-1}$ *are defined on* $[-1, 1]$ *and* $[1, +\infty)$ *respectively.*

*Proof.* The proof being quite lengthy, some minor steps and details will be omitted.

Since the zeros will be located in the vicinity, in some sense, of the poles of $\phi$, we will keep the following notation throughout the proof

$$\sqrt{\zeta_n^{\pm}} = n + d_n \pm i\frac{b_n}{2(n + d_n)}.$$

The periodicity of the cotangent function in the real variable yields

$$\cot\left(\pi\sqrt{\zeta_n^{\pm}}\right) = \cot\left(\pi d_n \pm i\frac{\pi b_n}{2(n+d_n)}\right)$$
$$= \cot\left(\pi(2d_n^2 + 2nd_n \pm ib_n)\frac{1}{2(n+d_n)}\right). \tag{5.6}$$

**If** $\sigma > 0$, then by (5.5), $b_n/n \to 0$ and $d_n \to 0$ (if not the real part on the l.h.s. would be bounded).

Using the following series expansions for $z \to 0$, (4.3.70 in [1])

$$\cot(\pi(a+ib)z)^2 = (\pi z(a+ib))^{-2} - 2/3 + O(((a+ib)z)^2),$$

we have the following asymptotics as $n \to +\infty$,

$$-\frac{\pi^2}{4}\cot\left(\pi\sqrt{\zeta_n^{\pm}}\right)^2 = -\frac{(n+d_n)^2}{(2d_n^2+2nd_n\pm ib_n)^2} + \pi^2/6 + o(1)(1+i)$$
$$= -(n+d_n)^2\frac{4(d_n^2+nd_n)^2 - b_n^2}{(4(nd_n+d_n^2)^2+b_n^2)^2} + \pi^2/6$$
$$\pm i(n+d_n)^2\frac{4(nd_n+d_n^2)b_n}{(4(nd_n+d_n^2)^2+b_n^2)^2} + o(1)(1+i)$$
$$= n^2\frac{b_n^2 - 4n^2d_n^2}{(4n^2d_n^2+b_n^2)^2} + \frac{\pi^2}{6}$$
$$\pm in^2\frac{4nd_nb_n}{(4n^2d_n^2+b_n^2)^2} + o(1)(1+i). \tag{5.7}$$

From (5.5) and (5.4), it follows that

$$n^2\frac{b_n^2 - 4n^2d_n^2}{(4n^2d_n^2+b_n^2)^2} = \frac{\sigma^2}{2}n^2 + o(n^2), \quad n \to +\infty \tag{5.8}$$
$$\pm n^2\frac{4nd_nb_n}{(4(nd_n)^2+b_n^2)^2} = \pm\frac{\sigma^2}{2}b_n + o(1), \qquad n \to +\infty. \tag{5.9}$$

Because $\sigma > 0$, (5.8) imposes that both $b_n$ and $nd_n$ do not diverge. Therefore, the l.h.s. of (5.9) implies that either $b_n$ or $nd_n$ goes to 0 and because the r.h.s. in (5.8) is positive, then it must be $nd_n \to 0$, which gives $b_n \to \sqrt{2/\sigma^2}$. With (5.9), this yields $d_n = O(n^{-3})$. All these facts imply that the $o(n^2)$ in (5.8) (which stems from (5.5)) is in fact a $O(1)$, which completes the proof.

**If** $\sigma = 0$ **and** $m \neq 0$, then using the eulerian representation of the cotangent function (see 4.3.58 in [1] for instance), (5.4) and (5.5) are rewritten into

$$\frac{\sinh\left(\frac{\pi b_n}{n+d_n}\right)^2 - \sin(2\pi d_n)^2}{\left(\cosh\left(\frac{\pi b_n}{n+d_n}\right) - \cos(2\pi d_n)\right)^2} = 1 + \frac{4m}{\pi^2} + \frac{4}{\pi^2}R(n) + o(R(n)) \tag{5.10}$$

and

$$\begin{aligned} &\pm \frac{2\pi \sinh\left(\frac{\pi b_n}{n+d_n}\right)\sin(2\pi d_n)}{\left(\cosh\left(\frac{\pi b_n}{n+d_n}\right)-\cos(2\pi d_n)\right)^2} \\ &-\frac{3\left(\frac{b_n}{2(n+d_n)}\sin(2\pi d_n)\pm(n+d_n)\sinh\left(\frac{\pi b_n}{(n+d_n)}\right)\right)}{((n+d_n)^2+\frac{b_n^2}{4(n+d_n)^2})(\cosh\left(\frac{\pi b_n}{n+d_n}\right)-\cos(2\pi d_n))} \\ &= O(\Im((\zeta_n^{\pm})^{-1})), \end{aligned} \tag{5.11}$$

where both $R(n)$ and $O(\Im((\zeta_n^{\pm})^{-1}))$ converge to 0.

Notice that for $b_n/n \to +\infty$, the l.h.s. of (5.10) converges to 1, thus $b_n/n$ must be bounded (since $m \neq 0$). Hence, because 0 is a pole for $\Im(\cot(z))$, (5.11) imposes that either $\sin(2\pi d_n)$ or $b_n/n$ goes to 0. In fact, because of the positivity of (5.10), $\sin(2\pi d_n) \to 0$ while $b_n = O(n)$. Note that this gives $\Im((\zeta_n^{\pm})^{-1}) = O(n^{-3})$ and $R(n) = O(n^{-1})$. Equation (5.10) then imposes $d_n \to 0$ if $m > 0$ and $d_n \to 1/2$ if $m < 0$. For $m > 0$, it can then be rewritten into

$$\frac{\cosh\left(\frac{\pi b_n}{n+d_n}\right)^2-1}{\left(\cosh\left(\frac{\pi b_n}{n+d_n}\right)-1\right)^2} = \frac{\cosh\left(\frac{\pi b_n}{n+d_n}\right)+1}{\cosh\left(\frac{\pi b_n}{n+d_n}\right)-1} = 1+\frac{4m}{\pi^2}+O(n^{-1}),$$

thereby yielding the constant in the imaginary part. Using the Taylor expansion of the sinus function at 0, (5.11) implies $d_n = (d+\delta_n)/n$ and is simplified into

$$n^{-1}\left(\pm\frac{4\pi^2(d+\delta_n)}{\cosh\left(\frac{\pi b_n}{n+d_n}\right)-1} \mp 3 + O(n^{-2})\right) = O(n^{-3}),$$

from which $d$ and $\delta_n = O(n^{-2})$ can be inferred. Writing $b_n = (b+\beta_n)n$ for $\beta_n \to 0$, and recalling the expansion

$$\sinh(\pi(b_n+\beta_n)) = \sinh(\pi b_n) + \pi\beta_n\cosh(\pi b_n) + o(\beta_n)$$

implies that $\beta_n = O(n^{-1})$, by (5.10). Note that in this case, the constant $c$ in the proposition depends on $d$ and $b$. The case $m \in (-\pi^2/4, 0)$ is treated similarly.

**The case $\sigma = 0$ and $m = 0$** is very special as, by (5.10), it is in fact necessary that $b_n/n \to +\infty$. More precisely, (5.11) yields $\frac{\pi b_n}{n\log(\delta n)} \to 1$ for some constant $\delta > 0$ which verifies

$$\frac{4\pi\sin(2\pi d_n)}{\delta n} - 3\left(\frac{\log(\delta n)\sin(2\pi d_n)}{\delta n^3}+\frac{1}{n}\right) = O(\log(n)n^{-3}).$$

From the Taylor expansion of the sinus function at 0 and the fact that $\sin(a+b) = \sin(a)\cos(b)+\cos(a)\sin(b)$, we can deduce that $d_n = d + O(\log(n)n^{-2})$ with $4\pi\sin(2\pi d) = 3\delta$. Because $b_n/n \to +\infty$, $|\Im((\zeta_n^{\pm})^{1/2})| \to +\infty$ and hence, by (5.2), $R(n) = -3\pi/4n^{-1} + o(n^{-1})$. As $\sinh\left(\frac{\pi b_n}{n+d_n}\right)^2 \sim \cosh\left(\frac{\pi b_n}{n+d_n}\right)^2 \sim \delta n/2$, $n \to +\infty$, it can be deduced from (5.10) that

$$\frac{4\cos(2\pi d_n)}{\delta n}+O(n^{-2})=-\frac{3}{\pi n}+o(n^{-1}),$$

from which we infer $d=3/8$ and $\delta=\frac{4\pi}{3}\sin(3\pi/4)$.

Lastly, **if** $\sigma=0$ **and** $m<\pi^2/4$, writing $\sqrt{\zeta_n^-}=n+d_n$ and again using the eulerian form of the cotangent function and (5.5),

$$\begin{aligned}-\frac{\sin(2\pi d_n)^2}{(\cos(2\pi d_n)-1)^2}&=\frac{\cos(2\pi d_n)+1}{\cos(2\pi d_n)-1}\\&=1+\frac{4m}{\pi^2}+\frac{3}{\pi}\left[1+\frac{\sin(2\pi d_n)}{\cos(2\pi d_n)-1}\right]n^{-1}+o(n^{-1}),\end{aligned}$$

which gives $d_n\to d=\cos^{-1}\left(1+\frac{\pi^2}{2m}\right)/2\pi$. Recalling $\cos(a+b)=\cos(a)\cos(b)-\sin(a)\sin(b)$ yields $d_n=d+cn^{-1}+o(n^{-1})$. The same method holds for $\sqrt{\zeta_{n-1}^+}=n-d_n$. □

**5.3. Laplace transform of the first passage time.** This result enables us to compute interesting fluctuations quantities, such as the Laplace transform of the first passage time of $X$: $T_x=\inf\{t\geq 0,X_t\geq x\}$. Indeed, since $P[S_t\geq x]=P[T_x\leq t]$ it is easy to show that

$$P[S_{e_q}\geq x]=\int_0^1 P[e^{-qT_x}\geq y]dy=\mathbb{E}[e^{-qT_x}]:=h_q(x).$$

We aim at providing a graph of $h_q$ for some fixed values of $q$, $m$ and $\sigma^2$. Using residues, it is possible to perform a Laplace transform inversion on the Wiener-Hopf factors to get a series representation of the law of $S_{e_q}$ (see Theorem 1 in [9] for instance):

$$\frac{\mathrm{d}}{\mathrm{d}x}P[S_{e_q}\leq x]=\sum_{n=1}^{\infty}\left[c_n^+\zeta_n^+e^{-\zeta_n^+x}+c_n^-\zeta_n^-e^{-\zeta_n^-x}\right]+c_0\zeta_0^+e^{-\zeta_0^+x},\quad x>0,\quad(5.12)$$

$$\text{where } c_n^{\pm}=\frac{1-\frac{\zeta_n^{\pm}}{n^2}}{1-\frac{\zeta_n^{\pm}}{\zeta_0^+}}\prod_{\substack{k\geq 1\\k\neq n}}\frac{1-\frac{\zeta_n^{\pm}}{k^2}}{1-\frac{\zeta_n^{\pm}}{\zeta_k^{\pm}}}\prod_{k\geq 1}\frac{1-\frac{\zeta_n^{\pm}}{k^2}}{1-\frac{\zeta_n^{\pm}}{\zeta_k^{\mp}}},\quad c_0=\prod_{k\geq 1}\frac{1-\frac{\zeta_0^+}{k^2}}{1-\frac{\zeta_0^+}{\zeta_k^+}}\frac{1-\frac{\zeta_0^+}{k^2}}{1-\frac{\zeta_0^+}{\zeta_k^-}},$$

$$\text{and}\quad P[S_{e_q}=0]=\zeta_0^+\prod_{k\geq 1}\frac{\zeta_k^+}{k^2}\frac{\zeta_k^-}{(k+1)^2}=\mathbb{E}[e^{-qT_{0^+}}],$$

with $T_{0^+}=\inf\{t\geq 0,X_t>0\}$. The Laplace transform of this latter random variable is not equal to 1 whenever 0 is not regular for $(0,\infty)$ (see chapter 6 in [15] for further details).

We start by providing a sample of the the locations of the roots with positive real and imaginary parts for the following parametrization: $\sigma^2/2\in\{0,1\}$, $m=0$ and $q=1$:

| | Case $\sigma^2/2 = 1$ | Case $\sigma^2 = 0$ |
|---|---|---|
| n | $\zeta_n^+$ | |
| 0 | 0.4431 | 0.4596 |
| 1 | 1.5284+0.4173$i$ | 1.800+0.273$i$ |
| 2 | 4.2785+0.9257$i$ | 5.4705+1.2401$i$ |
| 3 | 9.1176+0.9813$i$ | 11.1627+2.2600$i$ |
| 4 | 16.0638+0.9884$i$ | 18.866+3.3642$i$ |
| 5 | 25.0403+0.9914$i$ | 28.5772+4.5378$i$ |
| 6 | 36.0278+0.9933$i$ | 40.2934+5.7690$i$ |
| 7 | 49.0204+0.9947$i$ | 54.0134+7.0492$i$ |
| 8 | 64.0156+0.9957$i$ | 69.7364+8.3720$i$ |
| 9 | 81.0123+0.9964$i$ | 87.4620+9.7321$i$ |
| 10 | 100.0100+0.9971$i$ | 107.1894+11.1263$i$ |
| ⋮ | ⋮ | ⋮ |
| 50 | 2500.0004+0.9998$i$ | 2536.771+80.085$i$ |

The last line of the table is coherent with the asymptotic results of Proposition 5.2. Integrating (5.12) and taking $x = 0$ yields that

$$\sum_{n\geq 0} c_n = 1 - P[S_{e_q}],$$

from which we infer that the $c_n$ are bounded (in fact , their moduli decrease very rapidly). Hence, it is possible to compute $h_q$ using a finite number of terms, even for $x$ close to zero. We provide below the graphs for the cases $\sigma^2/2 \in \{0, 1\}$ and $q \in \{1, 2, 3\}$ in figures 3 and 4.

## Appendix A. Some Remarks on the Condition of the Theorem

It is easy to find examples for which $(*)$ fails. However, for many cases when $a_n, \hat{a}_n$ and $\rho_n, \hat{\rho}_n$ are expressed using basic functions (exponential, power), $(*)$ will in fact hold. We provide an example below.

We split $(*)$ into its two inequalities: the upper $(*_1)$ and the lower $(*_2)$, and we will only comment on $(*_1)$ because the transposition to $(*_2)$ will be immediate. We

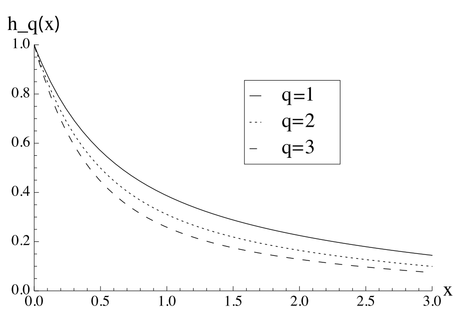


FIGURE 3. Plot of the function $h_q$ for $\sigma^2 = 2$ and $m = 0$

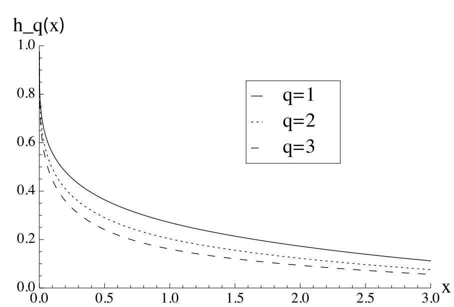


FIGURE 4. Plot of the function $h_q$ for $\sigma^2 = 0$ and $m = 0$

are in fact only interested in the term $T(\rho_j, b) = \sum_{n=1}^{\infty} a_n \frac{(\rho_j - 3\rho_n)(\rho_j - \rho_n) + b^2}{((\rho_n - \rho_j)^2 + b^2)^2}$, since it is the only one which can be negative. Furthermore, if we consider $l(j) = \sup\{k \geq 1,\, 3\rho_{j-k} \geq \rho_j\}$ (with $l(j) = 0$ if $3\rho_{j-1} \leq \rho_j$), we have

$$\begin{aligned} T(\rho_j, b) > \sum_{k=1}^{l(j)} \Bigg[ \frac{a_j}{b^2 l(j)} &+ a_{j+k} \frac{(\rho_{j+k} - \rho_j)(3\rho_{j+k} - \rho_j) + b^2}{((\rho_{j+k} - \rho_j)^2 + b^2)^2} \\ &+ a_{j-k} \frac{(\rho_{j-k} - \rho_j)(3\rho_{j-k} - \rho_j) + b^2}{((\rho_{j-k} - \rho_j)^2 + b^2)^2} \Bigg] \qquad \text{(A.1)} \\ &:= \sum_{k=1}^{l(j)} t_k(\rho_j, b). \end{aligned}$$

Now, $t_k(\rho_j, b)$ can only be negative if $3\rho_{j-k} - \rho_j > 0$. This seldom happens if $\rho_n$ increases quicky. For instance, if $\rho_n = c^n$ for $c > 1$, then $3\rho_{j-k} - \rho_j > 0 \Longleftrightarrow k < \frac{ln(3)}{ln(c)}$. In this case, if $(a_n)_{n\geq 1}$ is smooth enough (increasing, for instance), it is not hard to show that $(*_1)$ holds since for any $j \geq 1$, there are only a fixed number of negative terms. However, if $\rho_n$ increases at a slower rate (typically of power type), then the number of negative terms increases with $j$. To show that $(*_1)$ holds then requires some additional conditions, a set of which is detailed below.

**Proposition A.1.** *If the following conditions are fulfilled,*

*i)* $(a_n)_{n\geq 1}$ *is increasing*

*ii)* $\forall n \geq 2,\ \ \rho_{n+1} - \rho_n \geq (\rho_n - \rho_{n-1})$

*iii)* $\forall j > k \geq 1,\ 3\rho_{j-k} - \rho_j \geq 0 \implies 2(\rho_j - \rho_{j-k})^3 \geq (3\rho_{j-k} - \rho_j)(\rho_{j+k} - \rho_j)(\rho_{j+k} - 2\rho_j + \rho_{j-k})$,

*then* $(*_1)$ *holds.*

*Proof.* We want to study $t_k(\rho_j, b)$ as a function of $b$. The idea is to show that any possible negative term indexed by $j-k$ in (A.1) is absolutely smaller than its $j+k$ counterpart. It is obvious that, by *i)*, it is sufficient to prove this for a constant sequence $(a_n)_{n\geq 1}$; hence we set $a_j := 1$ for all $j \geq 1$. For notational convenience, we denote

$$A_1 = \rho_{j+k} - \rho_j, \quad A_2 = 3\rho_{j+k} - \rho_j, \quad B_1 = \rho_j - \rho_{j-k}, \quad B_2 = 3\rho_{j-k} - \rho_j,$$

which are all positive (the case $B_2 < 0$ is irrelevant) and satisfy $A_2 = 3A_1 + 3B_1 + B_2$. Omitting the constant term in (A.1),

$$\begin{aligned} t_k(\rho_j, b) &\geq \frac{A_1 A_2 + b^2}{(A_1^2 + b^2)^2} - \frac{B_1 B_2 - b^2}{(B_1^2 + b^2)^2} \\ &= \frac{A_1(3A_1 + 3B_1 + B_2) + b^2}{(A_1^2 + b^2)^2} - \frac{B_1 B_2 - b^2}{(B_1^2 + b^2)^2} \\ &\geq \frac{c_0 + c_2 b^2 + c_4 b^4 + 2b^6}{(A_1^1 + b^2)^2 (B_1^2 + b^2)^2}, \end{aligned}$$

where

$$\begin{aligned} c_0 &= -A_1^4B_1B_2 + A_1B_1^4B_2 + 3A_1^2B_1^4 + 3A_1B_1^5 \\ c_2 &= -2A_1^2B_1B_2 + 2A_1B_1^2B_2 + 6A_1^2B_1^2 + 6A_1B_1^3 + A_1^4 + B_1^4 \\ c_4 &= -B_1B_2 + A_1B_2 + 5A_1^2 + 3A_1B_1 + 2B_1^2 \end{aligned}$$

Note that by $ii)$, $A_1 \geq B_1$, hence $c_4 > 0$. Condition $iii) : 2B_1^3 \geq A_1B_2(A_1 - B_1)$ implies

$$c_0 \geq -2B_1^6 + A_1^2B_1^4 + A_1B_1^5 \quad c_2 \geq -4B_1^4 + 6A_1^2B_1^2 + 6A_1B_1^3 + A_1^4 + B_1^4,$$

which are both positive, by $ii)$. This leads to $t_k(\rho_j, b) \geq 0$; the sum in (A.1) is therefore positive and $(*_1)$ holds. □

This result calls for a few comments. First of all, many positive terms have been left out in the proof, thus $(*_1)$ holds under much weaker conditions. Furthermore, for any explicit formulation of $a_n$ and $\rho_n$, $i)$ is usually easily verified, and so is the convexity condition $ii)$ which is in fact not too restrictive, given (2.2). However, $iii)$ is much harder to prove. If we consider $\rho_n = n^\alpha$ for $\alpha > 1$, then we are interested in

$$2(j^\alpha - (j-k)^\alpha)^3 - (3(j-k)^\alpha - j^\alpha)((j+k)^\alpha - j^\alpha)((j+k)^\alpha - 2j^\alpha + (j-k)^\alpha)$$

and denoting $k$ as a proportion of $j$: $k = cj$, this becomes

$$[2(1-(1-c)^\alpha)^3 - (3(1-c)^\alpha - 1)((1+c)^\alpha - 1)((1+c)^\alpha - 2 + (1-c)^\alpha)]j^{3\alpha}.$$

Using numerical softwares, it is possible to show that this function of the variable $c$ is increasing and positive on $(0,1)$ for $1 < \alpha < 15$ (and in fact positive for $1 < \alpha \leq 15.87$). Note that $3\rho_{j-k} - \rho_j \geq 0 \iff k \leq \frac{3^{1/\alpha}-1}{3^{1/\alpha}}j$, hence the positivity criterion should only be checked for $c \in \left(0, \frac{3^{1/\alpha}-1}{3^{1/\alpha}}\right)$.

Other techniques can be used to show that $(*_1)$ also holds for $\alpha > 15.87$ when $a_n$ is increasing. They rely on the fact that, as in the exponential case, there are, proportionally, very few negative terms in $T(\rho_j, b)$.

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