

PARTIALLY GAUSSIAN STATIONARY STOCHASTIC PROCESSES IN DISCRETE TIME

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ABSTRACT. We present here an elementary example, for every fixed positive integer k , of a strictly stationary nongaussian stochastic process in discrete time, all of whose k -marginals are gaussian.

1. Introduction

It is well known that, for every positive integer $n > 1$, there exists a probability distribution in \mathbb{R}^n which is not gaussian but has all its $(n-1)$ -dimensional marginal distributions gaussian. (See, for example, Section 10.3 in Stoyanov [2].) Using the finer theory of pathwise stochastic integrals and martingale methods, Föllmer, Wu and Yor [1] have shown that, for every positive integer k , there exists a variety of nongaussian stochastic processes with continuous trajectories in the interval $[0, \infty)$ which have the same k -dimensional marginals as the standard brownian motion process. Here we present an elementary example, for every fixed positive integer $k > 1$, a discrete time stationary stochastic process which is not gaussian but has all its $(k-1)$ -marginals gaussian. However, we do not know how to construct such processes in continuous time.

2. The Basic Construction

Let $k > 1$ be any fixed positive integer and let ν be a probability distribution in \mathbb{R}^k , which is not gaussian but has all its $(k-1)$ -marginals gaussian with mean $\mathbf{0}$ and covariance matrix identity. For example, we may choose ν to have the probability density function

$$\psi(\mathbf{x}) = (2\pi)^{-\frac{k}{2}} \left\{ 1 + x_1 x_2 \cdots x_k e^{-\frac{1}{2}|\mathbf{x}|^2} \right\} e^{-\frac{1}{2}|\mathbf{x}|^2}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_k)$. If (X_1, X_2, \dots, X_k) is an \mathbb{R}^k -valued random variable with distribution ν , then the sequence $X_1, X_2, \dots, \widehat{X}_i, \dots, X_k$ with the i -th term omitted consists of i.i.d. $N(0, 1)$ random variables, for each i .

Now consider a bilateral sequence $\{(X_{n1}, X_{n2}, \dots, X_{nk}), -\infty < n < \infty\}$ of i.i.d. \mathbb{R}^k -valued random variables with the common distribution ν as described in the preceding paragraph. Define

$$Y_n = X_{nk} + X_{n+1, k-1} + X_{n+2, k-2} + \cdots + X_{n+k-1, 1}, -\infty < n < \infty.$$

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It is to be noted that the sum of the two suffixes in each summand on the right hand side is equal to $n + k$.

Theorem 2.1. *The sequence $\{Y_n, -\infty < n < \infty\}$ is strictly stationary, $(k - 1)$ -step independent with every $(k - 1)$ -dimensional marginal being gaussian with mean $\mathbf{0}$ and covariance matrix kI , I being the identity matrix of order $k - 1$. In particular, $\{Y_n\}$ is ergodic.*

Proof. Fix an integer m and consider the two sets $\{Y_n, n \leq m\}$ and $\{Y_n, n \geq m + k\}$. Since $Y_m = X_{mk} + X_{m+1\ k-1} + \cdots + X_{m+k-1\ 1}$ and $Y_{m+k} = Y_{m+k\ k} + X_{m+k+1\ k-1} + \cdots + X_{m+2\ k-1\ 1}$ and the first suffix in the last summand in the definition of Y_m is less than the first suffix in the first summand in the definition of Y_{m+k} it follows that the two sets $\{Y_n, n \leq m\}$ and $\{Y_n, n \geq m+k\}$ are independent. In other words $\{Y_n\}$ is a $(k - 1)$ -step independent process.

We now look at the column vector-valued random variable

$$\begin{bmatrix} Y_{n+1} \\ Y_{n+2} \\ \vdots \\ Y_{n+m} \end{bmatrix} = \begin{bmatrix} X_{n+1\ k} + X_{n+2\ k-1} + \cdots + X_{n+k\ 1} \\ X_{n+2\ k} + X_{n+3\ k-1} + \cdots + X_{n+k+1\ 1} \\ \vdots \\ X_{n+m\ k} + X_{n+m+1\ k-1} + \cdots + X_{n+m+k-1\ 1} \end{bmatrix}$$

and express it as $S_1 + S_2 + S_3$ where

$$\begin{aligned} S_1 &= \begin{bmatrix} X_{n+1\ k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} X_{n+2\ k-1} \\ X_{n+2\ k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} X_{n+k-1\ 2} \\ X_{n+k-1\ 3} \\ \vdots \\ X_{n+k-1\ k} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \\ S_2 &= \begin{bmatrix} X_{n+k\ 1} \\ X_{n+k\ 2} \\ \vdots \\ X_{n+k\ k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ X_{n+k+1\ 1} \\ X_{n+k+1\ 2} \\ \vdots \\ X_{n+k+1\ k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ X_{n+m\ 1} \\ X_{n+m\ 2} \\ \vdots \\ X_{n+m\ k} \end{bmatrix}, \\ S_3 &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ X_{n+m+1\ 1} \\ X_{n+m+1\ 2} \\ \vdots \\ X_{n+m+1\ k-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ X_{n+m+2\ 1} \\ X_{n+m+2\ 2} \\ \vdots \\ X_{n+m+2\ k-2} \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ X_{n+m+k-1\ 1} \end{bmatrix}. \end{aligned}$$

In each column on the right hand side of S_1 or S_3 there are at most $k - 1$ nonzero entries whereas in each column on the right hand side of S_2 there are exactly k entries. All the column vectors appearing in S_1, S_2, S_3 together are mutually independent. By the choice of the measure ν , S_1 and S_3 are gaussian random vectors. Denote by $\mu([i, j])$ the $(j - i + 1)$ -dimensional standard normal distribution imbedded in \mathbb{R}^m so that the first $i - 1$ and the last $m - j$ coordinates are 0 when $1 \leq i \leq j \leq m$. Similarly, denote by $\nu([j, k + j - 1])$ the k -dimensional distribution ν imbedded in \mathbb{R}^m with the first $j - 1$ and the last $m - k - j + 1$ coordinates 0 for $1 \leq j \leq m - k + 1$, assuming $m \geq k$. Then it follows that the random variables $Y_{n+1}, Y_{n+2}, \dots, Y_{n+m}$ expressed as a single column vector has the m -dimensional distribution ν_m (in \mathbb{R}^m) given by

$$\begin{aligned} \nu_m &= \mu([1, 1]) * \mu([1, 2]) * \dots * \mu([1, k - 1]) \\ &\quad * \nu([1, k]) * \nu([2, k + 1]) * \dots * \nu([m - k + 1, m]) \\ &\quad * \mu([m - k + 2, m]) * \mu([m - k + 3, m]) * \dots * \mu([m, m]), \end{aligned}$$

for every $m \geq k$. Since ν_m is independent of n it follows that $\{Y_n\}$ is a strictly stationary process. Since $\nu([1, k])$ is nongaussian it is clear that ν_m is not gaussian for every $m \geq k$.

We now observe that Y_n , being a sum of k independent $N(0, 1)$ random variables, is an $N(0, k)$ variable with mean 0 and variance k . Now consider the pair (Y_0, Y_m) . If $m \geq k$ we have already seen that Y_0 and Y_m are independent. If $m < k$, we write

$$\begin{aligned} \begin{bmatrix} Y_0 \\ Y_m \end{bmatrix} &= \begin{bmatrix} X_{0k} + X_{1\ k-1} + \dots + X_{m-1\ k-m+1} \\ 0 \end{bmatrix} + \begin{bmatrix} X_{m\ k-m} \\ X_{m\ k} \end{bmatrix} \\ &\quad + \begin{bmatrix} X_{m+1\ k-m-1} \\ X_{m+1\ k-1} \end{bmatrix} + \dots + \begin{bmatrix} X_{k-1\ 1} \\ X_{k-1\ m+1} \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ X_{km} + X_{k+1\ m-2} + \dots + X_{k+m-1\ 1} \end{bmatrix}. \end{aligned}$$

Now the special choice of ν implies that Y_0 and Y_m are independent $N(0, k)$ random variables. Stationarity of the process $\{Y_n\}$ implies that Y_{n_1} and Y_{n_2} are independent $N(0, k)$ random variables for any n_1, n_2 .

Now consider, for any $n_1 < n_2 < \dots < n_{k-1}$ the random vector

$$\tilde{\mathbf{Y}} = \begin{bmatrix} Y_{n_1} \\ Y_{n_2} \\ \vdots \\ Y_{n_{k-1}} \end{bmatrix} = \begin{bmatrix} X_{n_1 k} + X_{n_1+1\ k-1} + \dots + X_{n_1+k-1\ 1} \\ X_{n_2 k} + X_{n_2+1\ k-1} + \dots + X_{n_2+k-1\ 1} \\ \vdots \\ X_{n_{k-1} k} + X_{n_{k-1}+1\ k-1} + \dots + X_{n_{k-1}+k-1\ 1} \end{bmatrix}.$$

The right hand side can be expressed as a sum of column vectors in which the entries in each column are either 0 or an X_{rs} where the first suffix r is fixed and the second suffix takes at most $k - 1$ values from the set $\{1, 2, \dots, k\}$. The different column vectors are independent and by the choice of ν each column has a multivariate gaussian distribution. Thus $\tilde{\mathbf{Y}}$ is gaussian. Since any two Y_i and Y_j are independent where $k > 2$, it follows that $Y_{n_1}, Y_{n_2}, \dots, Y_{n_{k-1}}$ are i.i.d $N(0, k)$ random variables. This completes the proof. \square

Remark 2.2. From the proof of Theorem 2.1 it is clear that any $k - 1$ of the random variables $\{Y_n\}$ are i.i.d $N(0, k)$. This motivates the introduction of the following notion of limited exchangeability. We say that a stationary random process $\{Z_n, -\infty < n < \infty\}$ is *k-exchangeable* if any $Z_{n_1}, Z_{n_2}, \dots, Z_{n_k}$ has the same distribution for any k -point set $\{n_1, n_2, \dots, n_k\} \subset \mathbb{Z}$. The probability measures of all such k -exchangeable stationary processes constitute a convex set. One wonders what are the extreme points of this convex set.

References

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