Communications on Stochastic Analysis
Vol. 7, No. 3 (2013) 409-424

Serials Publications
www.serialspublications.com

POSITIVE HARRIS RECURRENCE OF THE CIR PROCESS AND ITS APPLICATIONS

PENG JIN, VIDYADHAR MANDREKAR, BARBARA RÜDIGER,
AND CHIRAZ TRABELSI

Abstract. In this paper, we will prove the positive Harris recurrence of the CIR process. Ergodic results on transformations of the CIR process will be given. We will also show that if $g : \mathbb{R}_+ \to \mathbb{R}$ is continuous, $f : \mathbb{R} \to \mathbb{R}_+$ is measurable and X_t is the CIR process, then $\frac{1}{N}\sum_{j=0}^{N-1} f\big(\int_j^{j+1} g(X_s)ds\big)$ converges almost surely to a constant. An application of the ergodic results in one credit migration model will be presented too.

1. Introduction

The Cox-Ingersoll-Ross model (or CIR model) was introduced in 1985, by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross in order to describe the evolution of interest rate. In this model the instantaneous interest rate X_t is assumed to be the unique solution of the following SDE:

$$dX_t = (b - aX_t)dt + \sigma\sqrt{|X_t|}dW_t, \quad X_0 \geq 0,$$

where W_t is a 1-dimensional Brownian motion and a, b, σ are positive constants. The process X_t is often called the CIR process.

This stochastic model has the following characteristics:

- The drift $b - aX_t$ ensures mean reversion of the interest rate towards the long-term value $\frac{b}{a}$.
- a is the speed of adjustment.
- When the rate X_t gets close to zero, the diffusion coefficient $\sigma\sqrt{|X_t|}$ also becomes close to zero.
- The singularity of the diffusion coefficient at the origin implies that an initially non-negative interest rate can never subsequently become negative.
- Due to the result of Yamada-Watanabe (see [12, Example 8.2]), the above SDE has a unique strong solution.
- If $b = 0$ and $X_0 = 0$, the solution of the CIR equation is $X_t \equiv 0$, and from the comparison theorem for one-dimensional diffusion processes, it follows that $X_t \geq 0$ if $X_0 \geq 0$.

As well known, the CIR process is an affine process in $\mathbb{R}_+$. General affine processes and their applications in finance have been investigated in great detail

Received 2013-4-24; Communicated by the editors.
2010 *Mathematics Subject Classification.* primary: 60H10; Secondary: 60J60.
Key words and phrases. Stochastic differential equations, CIR model, Harris recurrence, invariant measure, ergodicity, credit model.

in [7]. Among other things, it is proved in [7] that any stochastically continuous affine process is a Feller process. In particular, the CIR process is a Feller process in $\mathbb{R}_+$.

Another important issue concerning the CIR process is its long-term behavior. In the original paper [5] by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross, the steady state of the CIR model is shown to be a Gamma distribution. In other words, the corresponding Gamma distribution is an invariant measure for the CIR process. It is also well known that this invariant measure is ergodic (see [3]). [4] investigated the recurrent properties of the CIR process and proved that $\big[0, \frac{\sqrt{(2\sigma^2+4b)(3b+\sigma^2)}}{2a} + 1\big]$ is a recurrent region for the CIR process.

Different from [4], we investigate the Harris recurrent property of the CIR process in this paper.

A general continuous-time Markov process X_t with state space $(S, \mathcal{S})$ is called Harris recurrent if for some σ-finite measure μ

$$P_x\Big(\int_0^\infty \mathbf{1}_A(X_s)ds = \infty\Big) = 1,$$

for any $x \in S$ and $A \in \mathcal{S}$ with $\mu(A) > 0$. Harris recurrence guarantees the existence of a unique (up to multiplication by a constant) invariant measure for the process. If this invariant measure is finite, then the process is called positive Harris recurrent.

As the main result of this paper, we show that the CIR process, as a Feller process in $\mathbb{R}_+$, is positive Harris recurrent.

Harris recurrence was first introduced by Harris [10] for discrete Markov chains and then was extended in [1] to a general continuous time Markov process. Since then applications of Harris recurrence have been found in queueing theory and stochastic control. A recent application in interest rate models was given in [2], where Harris recurrence was used as a principal assumption to enable the authors to prove consistency of some estimators of jump-diffusion models for interest rate.

The plan of this paper is as follows. In Section 2 we prove that the CIR process is positive Harris recurrent. Ergodicity results on the transformation of the CIR process will also be given. In particular we show that if $g : \mathbb{R}_+ \to \mathbb{R}$ is continuous and $f : \mathbb{R} \to \mathbb{R}_+$ is measurable, then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f\Big(\int_j^{j+1} g\big(X_s\big)ds\Big) = E_\mu\Big[f\Big(\int_0^1 g\big(X_s\big)ds\Big)\Big]$$

almost surely, where $\{X_s\}_{s\geq 0}$ is the CIR process and μ is the unique invariant probability measure for the CIR process. An application of the ergodic results in one credit migration model will be presented in section 3.

2. Positive Harris Recurrence of the CIR Process

The CIR process X_t is given as the unique strong solution of the following stochastic differential equation

$$dX_t = (b - aX_t)dt + \sigma\sqrt{|X_t|}dW_t, \quad X_0 = x_0 \geq 0, \tag{2.1}$$

where $a, b, \sigma > 0$ are constants and W_t is a 1-dimensional Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ with $\mathcal{F}_t$ satisfying the usual conditions.

In this section we prove the main result of this paper, namely we show that the CIR process, as a Feller process on $\mathbb{R}_+$, is positive Harris recurrent.

2.1. Transition density function of the CIR process. The transition density function of the CIR process is given in [5] as

$$p(t,x,y) = ce^{-u-v}\left(\frac{v}{u}\right)^{\frac{q}{2}} I_q\big(2(uv)^{\frac{1}{2}}\big) \tag{2.2}$$

for $t > 0, x > 0$ and $y \geq 0$, where

$$c \equiv \frac{2a}{\sigma^2\Big(1-e^{-at}\Big)}, \qquad u \equiv cxe^{-at},$$

$$v \equiv cy, \qquad q \equiv \frac{2b}{\sigma^2} - 1,$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. We should remark that for $x = 0$ the formula of the density function $p(t,x,y)$ given in (2.2) is no more valid. In this case the density function is given by

$$p(t,0,y) = \frac{c}{\Gamma(q+1)} v^q e^{-v} \tag{2.3}$$

for $t > 0$ and $y \geq 0$. The transition density (2.2) is first found in [8] by Laplace transform methods. Duffie et al [7] exploited the affine structure of the CIR process to identify the Fourier transform of the law of the X_t. A more probabilistic method to get (2.2) was mentioned in Yor et al [9]. For the reader's convenience we put in the appendix a short introduction of the method used in [9].

2.2. Regularity property. So far the CIR process is defined as a solution of a stochastic differential equation on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. This setting is broadly used in the literature, especially in the area of financial mathematics. Another somewhat different setting, initiated by Duffie et al [7], is to construct the CIR process as a Markov process on the canonical path space. This approach has some advantage when we have to deal with the laws of CIR process from different starting points and it is also applicable for other affine models.

Since later we need to apply the ergodic theory of Feller processes, we adopt the approach of Duffie et al [7] in this section. To be precise, we first establish the connection between these two settings.

Let $\mathbb{R}_+ := [0,\infty)$. Consider the CIR process X_t starting from $x \in \mathbb{R}_+$, namely X_t is the unique strong solution to the following SDE

$$dX_t = (b - aX_t)dt + \sigma\sqrt{X_t}dW_t, \quad X_0 = x.$$

The semigroup (T_t) associated with the CIR process is defined as

$$T_t f(x) := \int_{\mathbb{R}_+} p(t,x,y)f(y)dy, \tag{2.4}$$

where $f : \mathbb{R}_+ \to \mathbb{R}$ is bounded and continuous. We write $C_0 = C_0(\mathbb{R}_+)$ for the class of continuous functions which vanish at infinity.

It is well known that CIR process is an affine process (see [7]). It is already shown in [7, Section 8] (see also [16]) that the semigroup of every stochastic continuous affine process is a Feller semigroup. Since CIR process is a diffusion process, it is obviously stochastic continuous. Thus we know that $(T_t)_{t\geq 0}$ defined in (2.4) is a Feller semigroup.

We denote the canonical path space by $\hat{\Omega}$, namely $\hat{\Omega} = C\big([0,\infty); \mathbb{R}_+\big)$, and let $\hat{X}_t$ be the canonical process on $\hat{\Omega}$. Let $(\hat{\mathcal{F}}_t)_{t\geq 0}$ be the filtration generated by the canonical process $\hat{X}_t$, namely $\hat{\mathcal{F}}_t := \sigma(\hat{X}_s, 0 \leq s \leq t)$ and $\hat{\mathcal{F}} := \sigma(\hat{X}_s, s \geq 0)$. The map

$$X : (\Omega, \mathcal{F}) \to (\hat{\Omega}, \hat{\mathcal{F}})$$

induces a measure $\hat{P}_x$ on $(\hat{\Omega}, \hat{\mathcal{F}})$, which is the law of the CIR process starting from x on the canonical path space. Since $(T_t)_{t\geq 0}$ is Feller, the Markov process $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ is a Feller process.

Following [14, Chapter 20] we give the definition of a regular Markov process on $\mathbb{R}_+$.

Definition 2.1. Consider a continuous-time Markov process Z with state space $\big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$ and distributions P_x. The process is said to be *regular* if there exist a locally finite measure ρ on $\mathbb{R}_+$ and a continuous function $(t,x,y) \mapsto p_t(x,y) > 0$ on $(0,\infty) \times \mathbb{R}_+^2$ such that

$$P_x\{Z_t \in B\} = \int_B p_t(x,y)\rho(dy), \quad x \in \mathbb{R}_+,\ B \in \mathcal{B}(\mathbb{R}_+),\ t > 0.$$

Proposition 2.2. *CIR process is a regular Feller process on* $\mathbb{R}_+$.

Proof. As we have mentioned it before, the Feller property is already proved in [7, Section 8]. We only need to prove the regularity property.

The modified Bessel functions of the first kind can be expanded as

$$I_q(r) = \left(\frac{r}{2}\right)^q \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}r^2\right)^k}{k!\Gamma(q+k+1)}$$

thus has the following asymptotic forms

$$I_q(r) = \frac{1}{\Gamma(q+1)}\left(\frac{r}{2}\right)^q + O(r^{q+2}) \tag{2.5}$$

for small arguments $0 < r \ll \sqrt{q+1}$. If $\frac{2b}{\sigma^2} < 1$, it follows that

$$p(t,x,0) := \lim_{y\to 0} p(t,x,y) = \infty, \quad \forall x \in \mathbb{R}_+.$$

Thus $(t,x,y) \mapsto p(t,x,y)$ is not continuous on $(0,\infty) \times \mathbb{R}_+^2$. On the other hand, if $\frac{2b}{\sigma^2} > 1$, then we have

$$p(t,x,0) := \lim_{y\to 0} p(t,x,y) = 0$$

Therefore in both cases the behavior of $p(t,x,y)$ at point $y = 0$ violates the regularity condition. To overcome this difficulty, we define a measure ρ on $\big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$

as

$$\rho(dx) := h(x)dx, \tag{2.6}$$

where

$$h(x) = \begin{cases} x^{\frac{2b}{\sigma^2}-1}, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

Then the transition density of the CIR process with respect to the new measure ρ is given by

$$\tilde{p}(t,x,y) = \frac{p(t,x,y)}{h(y)}, \quad t > 0,\ x \ge 0,\ y > 0. \tag{2.7}$$

Recall that

$$c \equiv \frac{2a}{\sigma^2\left(1 - e^{-at}\right)}, \qquad u \equiv cxe^{-at},$$

$$v \equiv cy, \qquad q \equiv \frac{2b}{\sigma^2} - 1.$$

At the point $y = 0$ we define

$$\tilde{p}(t,x,0) := \lim_{y\to 0} \tilde{p}(t,x,y) = \frac{1}{\Gamma(q+1)} c^{q+1} e^{-u} \in (0,\infty) \tag{2.8}$$

From (2.7) we get

$$0 < \tilde{p}(t,x,y) < \infty, \quad t > 0,\ x \ge 0,\ y > 0, \tag{2.9}$$

since $h(y)$ and $p(t,x,y)$ are positive and finite if $y > 0$. It follows from (2.8) and (2.9) that $0 < \tilde{p}(t,x,y) < \infty$ for all $(t,x,y) \in (0,\infty) \times \mathbb{R}_+^2$.

Moreover, the function $\tilde{p}(t,x,y)$ is continuous on $(0,\infty)\times(0,\infty)\times(0,\infty)$, which follows from the continuity and positivity of the functions $h(y)$ and $p(t,x,y)$ with $y > 0$. Next we prove the continuity of $\tilde{p}(t,x,y)$ at the point $(0,0,t_0)$.

Let $\delta > 0$ be sufficiently small. Then for $|t - t_0| \le \delta$ and $0 \le x, y \le \delta$ we have

$$\begin{aligned} &|\tilde{p}(t,x,y) - \tilde{p}(t_0,0,0)| \\ \le &|\tilde{p}(t,x,y) - \tilde{p}(t,0,y)| + |\tilde{p}(t,0,y) - \tilde{p}(t,0,0)| + |\tilde{p}(t,0,0) - \tilde{p}(t_0,0,0)| \\ \le &\left|\frac{p(t,x,y) - p(t,0,y)}{h(y)}\right| + \left|\frac{p(t,0,y)}{h(y)} - \tilde{p}(t,0,0)\right| + |\tilde{p}(t,0,0) - \tilde{p}(t_0,0,0)|. \end{aligned} \tag{2.10}$$

By (2.2) and (2.5) we get

$$\begin{aligned} &\left|\frac{p(t,x,y) - p(t,0,y)}{h(y)}\right| \\ =&\frac{1}{|y^q|}\left|ce^{-u-v}\left(\frac{v}{u}\right)^{\frac{q}{2}}\left(\frac{1}{\Gamma(q+1)}(uv)^{\frac{q}{2}} + O\big((uv)^{\frac{q}{2}+1}\big)\right) - \frac{c}{\Gamma(q+1)}v^q e^{-v}\right| \\ =&\frac{1}{|y^q|}\left|\frac{c}{\Gamma(q+1)}e^{-v}\big(e^{-u}-1\big)v^q + O\big(uv^{q+1}\big)\right| \\ \le&\left|\frac{c^{q+1}}{\Gamma(q+1)}e^{-v}\big(e^{-u}-1\big)\right| + O(uv). \end{aligned} \tag{2.11}$$

By (2.3) and (2.8) we have

$$\begin{aligned}&\left|\frac{p(t,0,y)}{h(y)} - \tilde{p}(t,0,0)\right| \\ &= \left|\frac{cv^q e^{-v}}{\Gamma(q+1)y^q} - \frac{1}{\Gamma(q+1)}c^{q+1}\right| = \left|\frac{c^{q+1}}{\Gamma(q+1)}\big(e^{-v}-1\big)\right|. \end{aligned} \tag{2.12}$$

Since c is a continuous function with respect to the variable t, it follows from (2.8) that

$$\lim_{t\to t_0} |\tilde{p}(t,0,0) - \tilde{p}(t_0,0,0)| = 0. \tag{2.13}$$

It follows from (2.10), (2.11), (2.12) and (2.13) that

$$\lim_{(t,x,y)\to(t_0,0,0)} |\tilde{p}(t,x,y) - \tilde{p}(t_0,0,0)| = 0.$$

The continuity at other remaining points can be proved with a similar argument. Thus $(t,x,y) \mapsto \tilde{p}(t,x,y)$ is a positive continuous function on $(0,\infty)\times\mathbb{R}_+^2$. Therefore the CIR process is regular with respect to the measure ρ. □

2.3. Positive Harris recurrence. Recall that $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ is the CIR process realized on the canonical path space.

By (2.2) we know that

$$\hat{P}_x(\hat{X}_t \in A) = \int_A p(t,x,y)dy, \quad \forall\ A \in \mathcal{B}(\mathbb{R}_+).$$

According to Proposition (2.2), we know that $(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{P}_x, x \in \mathbb{R}_+)$ is a regular Feller process with respect to the measure ρ defined in (2.6).

Definition 2.3. (i) A continuous-time Markov process Y on the state space $\big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$ is said to be *Harris recurrent* if for some σ-finite measure μ

$$P_x\Big(\int_0^\infty \mathbf{1}_A(Y_s)ds = \infty\Big) = 1,$$

for any $x \in \mathbb{R}_+$ and $A \in \mathcal{B}(\mathbb{R}_+)$ with $\mu(A) > 0$.
(ii) A continuous-time Markov process Y with state space $\big(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+)\big)$ is said to be *uniformly transient* if

$$\sup_x E_x\Big[\int_0^\infty \mathbf{1}_K(Y_s)ds\Big] < \infty \tag{2.14}$$

for every compact $K \subset \mathbb{R}_+$.

Harris recurrence guarantees the existence of a unique (up to multiplication by a constant) invariant measure for the Markov process (see e.g. [15]). If this invariant measure is finite, then the process is called positive Harris recurrent.

Lemma 2.4. *The CIR process $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ is not uniformly transient.*

Proof. We take $K = [0, M]$ with $M > 0$. Then for any fixed $x \in (0, \infty)$

$$\begin{aligned}\hat{E}_x\Big[\int_0^\infty \mathbf{1}_{[0,M]}(\hat{X}_t)dt\Big] &= \int_0^\infty \hat{E}_x\big[\mathbf{1}_{[0,M]}(\hat{X}_t)\big]dt \\ &= \int_0^\infty \int_0^M p(t,x,y)dydt \\ &= \int_0^M dy \int_0^\infty p(t,x,y)dt.\end{aligned}$$

The modified Bessel functions of the first kind have the following asymptotic forms, for small arguments $0 < r \ll \sqrt{q+1}$, one obtains

$$I_q(r) \approx \frac{1}{\Gamma(q+1)}(\frac{r}{2})^q,$$

where Γ denotes the Gamma function. Let $\epsilon > 0$ be small enough. For any $y \in [\epsilon, M]$ and large enough t we have

$$p(t,x,y) \approx \frac{c}{\Gamma(q+1)}e^{-cy}(cy)^q$$

and thus for $y \in [\epsilon, M]$

$$\int_0^\infty p(t,x,y) = \infty.$$

It follows that

$$\hat{E}_x\Big[\int_0^\infty \mathbf{1}_{[0,M]}(\hat{X}_t)dt\Big] = \int_0^M dy \int_0^\infty p(t,x,y)dt = \infty.$$

This proves that the CIR process $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ is not uniformly transient. □

Theorem 2.5. *The CIR process* $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ *is positive Harris recurrent.*

Proof. In Proposition (2.2) we have shown that the CIR process is a regular Feller process with respect to the measure ρ defined in (2.6). It follows from Lemma 2.4 and [14, Theorem 20.17] that the CIR process is Harris recurrent and ρ can be taken as a possible reference measure in place of μ in the Definition 2.3(i). Due to [14, Theorem 20.18] (see also [18, Theorem 1.3.5]) it is possible to construct a locally finite invariant measure μ for the CIR process. Furthermore μ is equivalent to ρ and every σ-finite, invariant measure for the CIR process agrees with μ up to a normalization. It was shown in [5] that μ is a Gamma distribution and has the form

$$\mu(dy) := \frac{\omega^\nu}{\Gamma(\nu)}y^{\nu-1}e^{-\omega y}dy, \quad y \geq 0, \tag{2.15}$$

where $\omega \equiv \frac{2a}{\sigma^2}$ and $\nu \equiv \frac{2b}{\sigma^2}$. Thus the CIR process is positive Harris recurrent. □

Definition 2.6. (i) The *tail σ-field* on $\hat{\Omega}$ is defined as $\hat{\mathcal{T}} = \cap_{t\geq 0}\hat{\mathcal{T}}_t$, where $\hat{\mathcal{T}}_t = \sigma\{\hat{X}_s : s \geq t\}$.
(ii) A σ-field $\mathcal{G} \subset \hat{\mathcal{F}}$ on $\hat{\Omega}$ is said to be *$\hat{P}_\nu$-trivial* if $\hat{P}_\nu(A) = 0$ or $\hat{P}_\nu(A) = 1$ for every $A \in \mathcal{G}$, where $\hat{P}_\nu(\cdot) := \int_{\mathbb{R}_+} \hat{P}_x(\cdot)\nu(dx)$ denotes the distribution of the CIR process with initial distribution ν.

From the positive Harris recurrence of the CIR process we reproduce the following well-known fact.

Corollary 2.7. *The CIR process $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ is strongly ergodic, meaning that the tail σ-field $\hat{\mathcal{T}}$ of the CIR process is $\hat{P}_\mu$-trivial for every μ.*

Proof. According to [14, Theorem 20.12] any Harris recurrent Feller process is strongly ergodic. From Theorem 2.5 we know that the CIR process $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ is a strongly ergodic Markov process. □

Based on Corollary 2.7, we now apply Birkhoff's ergodic theorem to get an ergodicity result for a transformation of the CIR process, which will be applied the next section to calibrate parameters of a credit migration model.

Let us first recall Birkhoff's ergodic Theorem (see e.g. [14, Theorem 10.6]). Let $(S, \mathcal{S})$ be a measurable space and ξ be a random element in S with distribution μ, and let T be a μ-preserving map on S with invariant σ-field $\mathcal{I} := \{A \in \mathcal{S} : T^{-1}A = A\}$ and let $\mathcal{I}_\xi := \{\xi^{-1}A : A \in \mathcal{I}\}$. Then for any measurable function $f \geq 0$ on S,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k \xi) \to E[f(\xi)|\mathcal{I}_\xi] \quad \text{a.s.}$$

The same convergence holds in L^p for some $p \geq 1$ when $f \in L^p(\mu)$.

We now consider the canonical CIR process $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_x, x \in \mathbb{R}_+)$ which is previous constructed in this section. Recall that the measure μ defined in (2.15) is the unique invariant probability measure for the CIR process. Suppose that $g : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function. We define a random sequence

$$\xi : (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}_\mu) \to \mathbb{R}^\infty$$

by

$$\xi(\hat{\omega}) := \Big(\int_0^1 g\big(\hat{X}_s\big)(\hat{\omega}) ds, \int_1^2 g\big(\hat{X}_s\big)(\hat{\omega}) ds, \cdots \Big)$$

The shift operator $\theta : \mathbb{R}^\infty \to \mathbb{R}^\infty$ is defined as

$$\theta(x_0, x_1, x_2, x_3, \cdots) := (x_1, x_2, x_3, \cdots)$$

for $x = (x_0, x_1, x_2, \cdots)$ and the invariant σ-field $\mathcal{I}$ on $\mathbb{R}^\infty$ generated by the shift operator θ is given by

$$\mathcal{I} := \{A \in \mathcal{B}(\mathbb{R}^\infty) : \theta^{-1}A = A\}.$$

Lemma 2.8. *Suppose that $g : \mathbb{R}_+ \to \mathbb{R}$ is continuous and $f : \mathbb{R} \to \mathbb{R}_+$ is measurable. Then we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f\Big(\int_j^{j+1} g\big(\hat{X}_s\big)(\hat{\omega}) ds \Big) = \hat{E}_\mu \Big[f\Big(\int_0^1 g\big(\hat{X}_s\big) ds \Big) \Big| \mathcal{I}_\xi \Big]$$

for $\hat{P}_\mu$-almost all $\hat{\omega} \in \hat{\Omega}$, where $\mathcal{I}_\xi := \{\xi^{-1}A : A \in \mathcal{I}\}$.

Proof. Since the initial distribution μ is an invariant measure for the CIR process

$$\hat{X}_1 \sim \mu.$$

It follows now from homogeneous Markov property that the random sequence ξ is stationary, i.e.

$$\theta\xi \overset{d}{=} \xi.$$

According to Birkhoff's ergodic Theorem, for any measurable function $h \geq 0$ on $\big(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)\big)$,

$$\frac{1}{N}\sum_{i=0}^{N-1} h(\theta^i\xi) \to \hat{E}_\mu[h(\xi)|\mathcal{I}_\xi] \quad \text{a.s.}$$

where $\mathcal{I}_\xi := \xi^{-1}\mathcal{I}$ and $\mathcal{I}$ is the invariant σ-field on $\mathbb{R}^\infty$ generated by the shift operator θ. Especially, if we take

$$h(x) := f\big(x_0\big)$$

for $x = (x_0, x_1, x_2, \cdots) \in \mathbb{R}^\infty$, then we get

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} f\Big(\int_j^{j+1} g\big(\hat{X}_s\big)(\hat{\omega})ds\Big) = \hat{E}_\mu\Big[f\Big(\int_0^1 g\big(\hat{X}_s\big)ds\Big)\Big|\mathcal{I}_\xi\Big]$$

for $\hat{P}_\mu$-almost all $\hat{\omega} \in \hat{\Omega}$. □

Lemma 2.9. *The invariant σ-field $\mathcal{I}_\xi$ of ξ is $\hat{P}_\mu$-trivial, namely*

$$\hat{P}_\mu(A) = 0 \text{ or } \hat{P}_\mu(A) = 1 \text{ for every } A \in \mathcal{I}_\xi.$$

Proof. Since $\hat{X}_t$ is continuous and $g : \mathbb{R}_+ \to \mathbb{R}$ is continuous, thus

$$\sigma(\xi_j) = \sigma\bigg(\int_j^{j+1} g\big(\hat{X}_s\big)ds\bigg) \subset \sigma\{\hat{X}(s) : j \leq s \leq j+1\}.$$

Thus the tail σ-field of the random sequence ξ is contained in the tail σ-field $\mathcal{T}$ of the CIR process $\hat{X}_t$, where $\mathcal{T} = \cap_{t\geq 0}\mathcal{T}_t$ and $\mathcal{T}_t = \sigma\{\hat{X}_s : s \geq t\}$. Because the invariant σ-field of the random sequence ξ is contained in the tail σ-field of ξ, we get $\mathcal{I}_\xi \subset \mathcal{T}$. According to Corollary 2.7, the tail σ-field $\mathcal{T}$ of the CIR process $\hat{X}_t$ is $\hat{P}_\mu$-trivial, it follows that $\mathcal{I}_\xi$ is also $\hat{P}_\mu$-trivial. □

Theorem 2.10. *Suppose that $g : \mathbb{R}_+ \to \mathbb{R}$ is continuous and $f : \mathbb{R} \to \mathbb{R}_+$ is measurable. Then for any $x \in \mathbb{R}_+$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} f\Big(\int_j^{j+1} g\big(\hat{X}_s\big)(\hat{\omega})ds\Big) = \hat{E}_\mu\Big[f\Big(\int_0^1 g\big(\hat{X}_s\big)(\hat{\omega})ds\Big)\Big]$$

for $\hat{P}_x$-almost all $\hat{\omega} \in \hat{\Omega}$, where μ is given (2.15)and is the unique invariant probability measure for the CIR process.

Proof. Since $\mathcal{I}_\xi$ is $\hat{P}_\mu$-trivial, the conditional expectation

$$\hat{E}_\mu\Big[f\Big(\int_0^1 g\big(\hat{X}_s\big)ds\Big)\Big|\mathcal{I}_\xi\Big]$$

is a constant and equals

$$\hat{E}_\mu\Big[f\Big(\int_0^1 g\big(\hat{X}_s\big)ds\Big)\Big].$$

From Lemma 2.8 we get

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} f\Big(\int_j^{j+1} g\big(\hat{X}_s\big)(\hat{\omega})ds\Big) = \hat{E}_\mu\Big[f\Big(\int_0^1 g\big(\hat{X}_s\big)ds\Big)\Big], \tag{2.16}$$

where the convergence in (2.16) holds for $\hat{P}_\mu$-almost all $\hat{\omega}\in\hat{\Omega}$. If we set

$$N := \big\{\hat{\omega}\in\hat{\Omega}:\ \text{the convergence in (2.16) fails for } \hat{\omega}\big\},$$

then

$$\hat{P}_\mu(N) = \int_{\mathbb{R}_+}\hat{P}_x(N)\mu(dx) = 0,$$

which implies $\hat{P}_x(N) = 0$ for μ-almost all $x\in\mathbb{R}_+$. For any $x\in\mathbb{R}_+$, by the Markov property, it holds

$$\begin{aligned}\hat{P}_x(N) =&\hat{E}_x[\mathbf{1}_N] = \hat{E}_x\big[\hat{E}_x[\mathbf{1}_N|\hat{\mathcal{F}}_1]\big]\\ =&\hat{E}_x\big[\hat{P}_{\hat{X}_1}[\theta_1^{-1}(N)]\big] = \int_0^\infty \hat{P}_y(N)p(1,x,y)dy\\ =&0,\end{aligned}$$

where $p(t,x,y)$ denotes the transition density function of the CIR process. In the above calculation we have used the fact that $\theta_1^{-1}(N) = N$, namely the pre-image of N under the shift operator θ is still N. Thus we have proved that the convergence in (2.16) holds for $\hat{P}_x$-almost all $\hat{\omega}\in\hat{\Omega}$. □

3. Application in one Credit Migration Model

In this section we show a simple application of Theorem 2.10 in calibration of the parameters in one credit migration model. We should remark that the results presented in this section have been derived in [17] with a different method. Their method is very analytical and relies very much on the affine structure of the CIR process. In contrast to [17] our method is more probabilistic and can be extended to more general models.

We consider a simpler version of the credit migration model of Hurd and Kuznetsov [11]. Consider the finite state space $\{1,2,\cdots,8\}$, which can be identified with Moody's rating classes via the mapping:

$$\{1,2,\cdots,8\}\leftrightarrow\{\text{AAA, AA, A},\cdots,\text{default}\}.$$

The credit migration matrix $P(s,t), 0\le s\le t$, is a stochastic 8×8 matrix and describes all possible transition probabilities between rating classes from time s to time t, namely

$$P(s,t) = \Big(p_{ij}(s,t)\Big)_{1\le i,j\le 8},$$

where $p_{ij}(s,t)$ represents the transition probability from state i to state j from time s to time t. The last column of the migration matrix $P(s,t)$ represents probabilities of Default. It was assumed in [11] that the migration matrix $P(s,t)$ is given by

$$P(s,t) = \exp\left(\left(\int_s^t X_r dr\right) \cdot \hat{P}\right), \tag{3.1}$$

where $\hat{P}$ is a 8×8 constant matrix and X_t is a CIR process with long-term average 1, namely X_t satisfies

$$X_t = X_0 + \int_0^t a(1 - X_s)ds + \int_0^t \sigma\sqrt{X_s}dW_s, \quad t \geq 0.$$

The matrix $\hat{P}$ is called the generator matrix. Thus the dynamics of the migration matrix is determined by two factors: the generator $\hat{P}$ and the CIR process X_t.

A natural question is how to calibrate the parameters of the above credit migration model. More precisely, how can one determine the generator matrix $\hat{P}$ and the parameters a, b, σ of the CIR process?

Among many other things, this problem was considered by [17] and they presented the following way to calibrate the parameters of the above model, with extra assumptions on the generator matrix $\hat{P}$. The starting point is the Moody-matrix P_{Moody}, which is derived by Moody as the historical average of one year migration matrix, based on the historical data from 1920 to 1996. The logarithm matrix of P_{Moody} is given by

$$\hat{P}_{Moody} := \log(P_{Moody}) = \log(id - (id - P_{Moody})) = -\sum_{j=1}^{\infty} \frac{1}{j}(id - P_{Moody})^j. \tag{3.2}$$

According to [13, Theorem 2.2], the right-hand side in (3.2) converges and thus $\hat{P}_{Moody}$ is well-defined and

$$\exp(\hat{P}_{Moody}) = P_{Moody}.$$

It was indicated by [17] that $\hat{P}_{Moody}$ is diagonalizable and thus can be written as

$$\hat{P}_{Moody} = G \cdot (-\hat{D}_{Moody,ii}) \cdot G^{-1}, \tag{3.3}$$

where $\hat{D}_{Moody,ii}$ is a diagonal matrix and $G = (g_{ik})_{1 \leq j,k \leq 8}$ is a matrix whose columns are the corresponding eigenvectors of $\hat{P}_{Moody}$. Thus we get

$$P_{Moody} = G \cdot (e^{-\hat{D}_{Moody,ii}}) \cdot G^{-1}.$$

Instead of finding a full 8×8 generator matrix, it was proposed in [17] to seek the generator matrix in the form

$$\hat{P} = G \cdot (-\hat{D}) \cdot G^{-1},$$

where $\hat{D} = (\hat{D}_{ii})$ is a diagonal matrix with diagonal elements $\hat{D}_{ii} \geq 0$ and G is given in (3.3).

According to (3.1) the migration matrix from time j to $j+1$, $j = 0, 1, 2, \cdots$ is given by

$$\begin{aligned} P(j, j+1) &= \exp\left(\left(\int_j^{j+1} X_s ds\right) \cdot \hat{P}\right) \\ &= G \cdot \left(e^{-\hat{D}_{ii} \int_j^{j+1} X_s ds}\right) \cdot G^{-1} \\ &= G \cdot \begin{pmatrix} e^{-\hat{D}_{11} \int_j^{j+1} X_s ds} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & \cdots & e^{-\hat{D}_{88} \int_j^{j+1} X_s ds} \end{pmatrix} \cdot G^{-1}. \end{aligned}$$

Thus the Cesàro average $\frac{1}{N}\sum_{j=0}^{N-1} P(j, j+1)$ equals

$$G \cdot \begin{pmatrix} \frac{1}{N}\sum_{j=0}^{N-1} e^{-\hat{D}_{11} \int_j^{j+1} X_s ds} & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & \cdots & \frac{1}{N}\sum_{j=0}^{N-1} e^{-\hat{D}_{88} \int_j^{j+1} X_s ds} \end{pmatrix} \cdot G^{-1}. \quad (3.4)$$

For each $1 \leq i \leq 8$ by taking $g(x) = \hat{D}_{ii} x$ and $f(x) = e^{-x}$ in Theorem 2.10 of last section we get

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} e^{-\hat{D}_{ii} \int_j^{j+1} X_s ds} = \mathrm{E}_\mu \left[e^{-\hat{D}_{ii} \int_0^1 X_s ds} \right] \quad (3.5)$$

and the convergence in (3.5) holds almost surely. Since each term under the limit sign on the left hand side of (3.5) is bounded, by dominated convergence theorem, the convergence in (3.5) holds also in L^p for any $p \geq 1$. We should remark that the L^2 convergence in (3.5) was obtained in [17] with a different method. Our method used here is more probabilistic and provides us with a stronger convergence in (3.5).

Since the Laplace transform of $\int_0^1 X_s ds$ is well-known (for example, see [3, Lemma 2]), we get

$$E_x \left[e^{-\hat{D}_{ii} \int_0^1 X_s ds} \right] = e^{aA(0, \hat{D}_{ii}, 1) + x \cdot B(0, \hat{D}_{ii}, 1)}$$

with deterministic functions

$$B(\lambda, u, t) := -\frac{\lambda(h - a + (h+a)e^{-ht} + 2u(1 - e^{-ht})}{\sigma^2 \lambda (1 - e^{-ht}) + h + a + (h-a)e^{-ht}}$$

$$A(\lambda, u, t) := \frac{2}{\sigma^2} \log\left(\frac{2he^{(a-h)t}}{\sigma^2\lambda(1-e^{-ht}) + h + a + (h-a)e^{-ht}}\right)$$

and $h := \sqrt{a + 2u\sigma^2}$. Therefore

$$\begin{aligned} E_\mu\left[e^{-\hat{D}_{ii}\int_0^1 X_s ds}\right] &= \int_0^\infty E_x\left[e^{-\hat{D}_{ii}\int_0^1 X_s ds}\right]\mu(dx) \\ &= \int_0^\infty e^{aA(0,\hat{D}_{ii},1)+x\cdot B(0,\hat{D}_{ii},1)} \frac{\omega^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\omega x} dx \\ &= e^{a\hat{D}_{ii}A(0,\hat{D}_{ii},1)}\Big(\frac{2a}{2a - \sigma^2\hat{D}_{ii}B(0,\hat{D}_{ii},1)}\Big)^{\frac{2a}{\sigma^2}}. \end{aligned} \tag{3.6}$$

On the other hand P_{Moody} is the historical average of one year migration matrix and thus it is reasonable to assume that

$$P_{Moody} = \lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} P(j, j+1), \tag{3.7}$$

if the limit on the right-hand side exists.

It now follows from (3.4), (3.5), (3.6) and (3.7) that

$$-\hat{D}_{Moody,ii} = a\hat{D}_{ii}A(0,\hat{D}_{ii},1) + \frac{2a}{\sigma^2}\log\left(\frac{2a}{2a - \sigma^2\hat{D}_{ii}B(0,\hat{D}_{ii},1)}\right) \tag{3.8}$$

for each $1 \le i \le 8$.

To get more equations for the unknown parameters, we consider the probability of rating downgrade from level A to the level BBB within time j und $j+1$, which is given by:

$$p_{3,4}(j, j+1) := \sum_{k=1}^{8} g_{3k} \cdot e^{-\hat{D}_{kk}\int_j^{j+1} X_s ds} \cdot g^{k4}, \tag{3.9}$$

where $(g^{ik})_{1\le i,k\le 8}$ is the inverse of the matrix G. Similar to (3.5) we know that as $N \to \infty$

$$\frac{1}{N}\sum_{j=0}^{N-1} p_{3,4}(j, j+1)$$

converges almost surely to a constant, denoted by $E_{3,4,\infty}$, and we have

$$E_{3,4,\infty} = \sum_{k=1}^{8} g_{3k} e^{a\hat{D}_{kk}A(0,\hat{D}_{kk},1)} \left(\frac{2a}{2a - \sigma^2\hat{D}_{kk}B(0,\hat{D}_{kk},1)}\right)^{\frac{2a}{\sigma^2}} g^{k4}.$$

Define the ergodic variation

$$\begin{aligned} V_{3,4,\infty} &:= \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1}\left(p_{3,4}(j,j+1) - E_{3,4,\infty}\right)^2 \\ &= \lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} p_{3,4}^2(j,j+1) - E_{3,4,\infty}^2. \end{aligned} \tag{3.10}$$

Since

$$p_{3,4}^2(j,j+1) = \sum_{i,k=1}^{8} g_{3i}g_{3k}e^{-(\hat{D}_{ii}+\hat{D}_{kk})\int_j^{j+1} X_s ds} \cdot g^{i4}g^{k4},$$

it follows again from Theorem 2.10 that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=0}^{N-1} p_{3,4}^2(j,j+1) = \sum_{i,k=1}^{8} g_{3i}g_{3k}e^{(\hat{D}_{ii}+\hat{D}_{kk})aA(0,\hat{D}_{ii}+\hat{D}_{kk},1)} \cdot \left(\frac{2a}{2a-\sigma^2(\hat{D}_{ii}+\hat{D}_{kk})B(0,\hat{D}_{ii}+\hat{D}_{kk},1)}\right)^{\frac{2a}{\sigma^2}} g^{i4}g^{k4}.$$

Thus we get

$$\begin{aligned} V_{P,3,4,\infty} = \sum_{i,k=1}^{8} & g_{3i}g_{3k}e^{(\hat{D}_{ii}+\hat{D}_{kk})bA(0,\hat{D}_{ii}+\hat{D}_{kk},1)} \cdot \\ & \left(\frac{2a}{2a-\sigma^2(\hat{D}_{ii}+\hat{D}_{kk})B(0,\hat{D}_{ii}+\hat{D}_{kk},1)}\right)^{\frac{2b}{\sigma^2}} g^{i4}g^{k4} - \\ & \sum_{i,k=1}^{8} g_{3i}g_{3k}e^{\hat{D}_{ii}bA(0,\hat{D}_{ii},1)+\hat{D}_{kk}bA(0,\hat{D}_{kk},1)} \cdot \\ & \left(\frac{2a}{2a-\sigma^2\hat{D}_{ii}B(0,\hat{D}_{ii},1)}\right)^{\frac{2b}{\sigma^2}} \left(\frac{2a}{2a-\sigma^2\hat{D}_{kk}B(0,\hat{D}_{kk},1)}\right)^{\frac{2b}{\sigma^2}} g^{i4}g^{k4}. \end{aligned} \tag{3.11}$$

Based on the historical data from 1920 to 1996 Moody also gave the standard variation of one year transition probabilities between different rating classes. For example the standard variation of the one year transition probability from rating class A to BBB is 0.053. We could approximately assume that this value coincides with the square of the ergodic variation defined in (3.10), namely

$$V_{3,4,\infty} = (0.053)^2. \tag{3.12}$$

Summarizing (3.8), (3.11) and (3.12) we get 9 equations for 10 unknown parameters. By fitting Moody's standard variation of one year transition probability of another rating transition we will get an extra equation. Thus all parameters of this migration matrix model can be uniquely determined by solving the 10 equations we have derived.

Appendix A. Computing the Transition Density Function of the CIR Process via Squared Bessel Processes

For the reader's convenience we briefly explain how to compute the transition density function of the CIR process via squared Bessel processes. For full details the readers are referred to [9, 19].

Definition A.1. For every $\delta \geq 0$ and $x_0 \geq 0$ the unique strong solution to the equation

$$X_t = x_0 + \delta t + 2\int_0^t \sqrt{X_s} dB_s \tag{A.1}$$

is called the *square* of a δ-dimensional Bessel process started at x_0 and is denoted by $\mathrm{BESQ}^\delta_{x_0}$.

Remark A.2. The number δ is called the *dimension* of BESQ^δ, since a BESQ^δ process X_t can be represented by the square of the Euclidean norm of δ-dimensional Brownian motion B_t: $X_t = |B_t|^2$.

Definition A.3. The square root of BESQ^δ, $\delta \geq 0, y \geq 0$ is called the *Bessel process* of dimension δ started at y and is denoted by BES^δ.

We recall that a CIR process is the unique solution to the following SDE

$$\begin{cases} dX_t = (b - aX_t)dt + \sigma\sqrt{X_t}dW_t, \quad t \geq 0, \\ X_0 = x_0 \geq 0, \end{cases} \tag{A.2}$$

where a,b, σ are positive constants and W_t is a 1-dimensional Brownian motion. The CIR process (A.2) can be represented as

$$X_t = e^{-at} Y\left(\frac{\sigma^2}{4a}\left(e^{at} - 1\right)\right),$$

where Y is a squared Bessel process with dimension $\delta = \frac{4b}{\sigma^2}$ started at x_0. This relation is used by Delbaen and Shirakawa [6] and Szatzschneider [20].

For $\delta > 0$, the transition density for BESQ^δ is equal to

$$q_t^\delta(x, y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \exp\left\{-\frac{x + y}{2t}\right\} I_\nu\left(\frac{\sqrt{xy}}{t}\right),$$

where $t > 0, x > 0, \nu \equiv \frac{\delta}{2} - 1$ and I_ν is the modified Bessel function of the first kind of index ν, see e.g. [19]. Thus it is easy to see that the transition density of CIR process is given by (2.2).

Acknowledgements. The problem in section 3 was posed by the insurance company Debeka in Koblenz, who prior to this paper solved the problem with different methods [17] and supported our research not only by contributing with many useful comments, but also by financing the stay of V. Mandrekar and C. Trabelsi in Wuppertal.

References

1. Azéma, J., Kaplan-Duflo, M. and Revuz, D.: Mesure invariante sur les classes récurrentes des processus de Markov, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **8** (1967), 157–181.
2. Bandi, F.M. and Nguyen, T.H.: On the functional estimation of jump-diffusion models, *J. Econometrics* **116** (2003), 293–328.
3. Ben Alaya, M. and Kebaier, A.: Parameter Estimation for the Square-root Diffusions: Ergodic and Nonergodic Cases, *Preprint* (2011).
4. Bhan, C. and Mandrekar, V.: Recurrence properties of term structure models, *Int. J. Contemp. Math. Sci.* **5** (2010), no. 33, 1645–1652.

5. Cox, J.C., Ingersoll, J.J.E., and Ross, S.A.: A theory of the term structure of interest rates, *Econometrica* **53** (1985), no. 2, 385–407.
6. Delbaen, F. and Shirakawa, H: A Note of Option pricing for constant elasticity of variance model, *Preprint* (1996).
7. Duffie, D. and Filipović, D. and Schachermayer, W.: Affine processes and applications in finance, *Ann. Appl. Probab.* **13** (2003), no. 3, 984–1053.
8. Feller, W.: Two singular diffusion problems, *Ann. of Math.* **54** (1951), no. 1, 173–182.
9. Göing-Jaeschke, A. and Yor, M.: A survey and some generalizations of Bessel processes, *Bernoulli* **9** (2003), no. 2, 313–349.
10. Harris, T.E.: The existence of stationary measures for certain Markov processes, in: *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability* **II** (1956), 113–124, University of California Press, Berkeley and Los Angeles.
11. Hurd, T. R. and Kuznetsov, A.: Affine Markov chain models of multifirm credit migration, *J. of credit risk* **3** (2007), 3–29.
12. Ikeda, N. and Watanabe, S.: *Stochastic differential equations and diffusion processes*, North-Holland Publishing Co., Amsterdam, 1989.
13. Israel, R.B., Rosenthal, J.S. and Wei, J.Z.: Finding generators for Markov chains via empirical transition matrices, with applications to credit ratings, *Math. Finance* **11** (2001), no. 2, 245–265.
14. Kallenberg, O.: *Foundations of modern probability*, Springer-Verlag., New York, 2002.
15. Kaspi, H. and Mandelbaum, A.: On Harris recurrence in continuous time, *Math. Oper. Res.* **19** (1994), no. 1, 211–222.
16. Keller-Ressel, M., Schachermayer, W. and Teichmann, J.: Affine processes are regular, *Probab. Theory Related Fields* **151** (2011), 591–611.
17. Koehler, B. and Kafft,V.: Ein Kreditmodell basierend auf Migrationsmatrizen, *Preprint* (2011).
18. Kunita, Hiroshi: *Stochastic flows and stochastic differential equations*, Cambridge University Press, Cambridge, 1997.
19. Revuz, D. and Yor, M.: *Continuous martingales and Brownian motion*, Springer-Verlag, Berlin, 1999.
20. Szatzschneider, W: Extended Cox, Ingersoll & Ross Model, *Revista Mexicana de Economía y Finanzas* **1** (2012), no. 4.

Peng Jin: Fachebereich C, Bergische Universität Wuppertal, 42119 Wuppertal, Germany

E-mail address: jin@uni-wuppertal.de

Vidyadhar Mandrekar: Department of Statistics and Probability Michigan State University, A436 Wells Hall, East Lansing, Michigan, USA

E-mail address: atmah@stt.msu.edu

Barbara Rüdiger: Fachebereich C, Bergische Universität Wuppertal, 42119 Wuppertal, Germany

E-mail address: ruediger@uni-wuppertal.de

Chiraz Trabelsi: Department of Mathematics, University of Tunis El-Manar, 1060 Tunis, Tunisia

E-mail address: trabelsichiraz@hotmail.fr