

IDENTITIES AND INEQUALITIES FOR CDO TRANCHE SENSITIVITIES

CLAAS BECKER AND AMBAR N. SENGUPTA*

ABSTRACT. We examine general copula models for the valuation of CDOs and provide explicit formulas for the sensitivities with respect to spreads. In the case of Gaussian copulas with non-uniform correlations, we prove a functional relation between spread sensitivities and correlation sensitivities.

1. Introduction

The Gaussian copula model with uniform correlations has evolved as the industry standard for the valuation of collateralized debt obligations (CDOs) referencing liquidly traded credits. This model, established in the context of CDOs in the work of Li [13], can be easily calibrated to observed market prices by varying the correlation. In the context of the US subprime financial crisis and the subsequent market turmoils, it has been heavily criticized for being too simplistic. Many extensions of the Gaussian copula model are available in the literature, see [2] and [9]. A good overview which also covers the practitioner's view is given in the monograph [3].

In this paper we examine general copula models, that is, not necessarily Gaussian models, and provide explicit formulas for sensitivities with respect to spreads. In the special case of Gaussian copulas with non-uniform correlations, we provide an explicit relation between the sensitivities with respect to spreads and the correlation sensitivities.

Put more mathematically, we work with an \mathbb{R}^N -valued random variable $X = (X_1, \dots, X_N)$ and study the 'tranche loss'

$$L_{[a,b]} = \left\{ \sum_{m=1}^N l_m 1_{[X_m \leq c_m]} - a \right\}_+ - \left\{ \sum_{m=1}^N l_m 1_{[X_m \leq c_m]} - b \right\}_+,$$

where $l_1, \dots, l_N > 0$, $0 \leq a \leq b$, and $c_1, \dots, c_N \in \mathbb{R}$ are parameters; we obtain formulas for $\partial \mathbb{E}[L_{[a,b]}] / \partial c_j$ that do not depend on any specific choice of the distribution of X , and then, specializing to the case where X is Gaussian, with each X_j being normalized to the standard Gaussian, we show that $\partial \mathbb{E}[L_{[0,a]}] / \partial r_{jk}$ is ≤ 0 , where $r_{jk} = \mathbb{E}[X_j X_k]$ is the correlation for any $j \neq k$. We establish other results in this spirit and also indicate extensions to distributions other than Gaussian.

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In section 2 we work out a formula (2.21) for the ‘delta’ of a tranche for variations in the default probability of a specific constituent of the portfolio. This result does not depend on any specific choice of copula. In section 3 we further examine sensitivity to the default probability and obtain formulas using the specific copula distribution; these are contained in Proposition 7.1 and Proposition 7.2. In section 4 we show, among other things, that for the Gaussian copula the expected loss in an equity tranche decreases when correlation between the constituents of the portfolio increases.

The work of Cousin and Laurent [4] uses very general order relation theory to establish results on the sensitivity of equity tranche to correlation variation. This approach uses the general theory of stochastic orders (Müller and Stoyan [16]). The methods and strategy we use in the present paper are largely self-contained and therefore more accessible. Our approach is in the spirit of Slepian inequalities [18].

We refer to the work of Jarrow and van Deventer [10] for a thorough analysis of the dependence of the risk of the equity tranche on correlation under various scenarios and different models. In particular they observe that there are circumstances and models for which the equity tranche risk (suitably measured) does not decrease as correlation increases. The work of Ağça and Islam [1] also explores situations where an increase in correlation affects market-assessed default probabilities and thereby also impacts the equity tranche spread, potentially increasing it. Notwithstanding these cautionary findings, our work brings mathematical techniques of Gaussian and related inequalities to the analysis of default phenomena, the application to proving the relationship between equity tranche expected loss and correlation being one illustration of these techniques.

2. Tranche Loss Function and Delta

The default behavior in a CDO consisting of N names may be modeled by introducing random variables X_1, \dots, X_N , and certain threshold values $c_1, \dots, c_N \in \mathbb{R}$, with the event $[X_m \leq c_m]$ being interpreted as default of the m -th name within a given fixed time period. Thus the total loss during this period is

$$L = \sum_{m=1}^N l_m 1_{[X_m \leq c_m]}, \quad (2.1)$$

where $l_1, \dots, l_N > 0$, with l_m being the loss resulting from default of name m . Increasing any c_m potentially enlarges the corresponding set $[X_m \leq c_m]$ and hence also the value of L .

A *tranche* is described mathematically by a closed interval $[a, b]$ with $0 \leq a \leq b$, and we define the corresponding *tranche loss function* $t_{[a,b]}$ by

$$\begin{aligned} t_{[a,b]}(x) &= (x - a)_+ - (x - b)_+, \quad \text{for all } x \in \mathbb{R}, \\ &= \begin{cases} 0 & \text{if } x < a; \\ x - a & \text{if } x \in [a, b]; \\ b - a & \text{if } x > b. \end{cases} \end{aligned} \quad (2.2)$$

The first equation shows that $t_{[a,b]}(x)$ is continuous in (a, b, x) , and the second triple of formulas show that it is a non-decreasing function of x .

Applying $t_{[a,b]}$ to the loss variable L , we see that $t_{[a,b]}(L)$ represents the amount for a total loss of L that would be apportioned to the tranche specified by $[a, b]$. It will be useful to note that

$$t_{[a,b]}(x) = \int_{[a,b]} 1_{[0,x]}(s) ds \tag{2.3}$$

for all $x \geq 0$, and consequently

$$t_{[a,b]}(v) - t_{[a,b]}(u) = \int_{[a,b]} 1_{(u,v]}(s) ds \tag{2.4}$$

for all $u \leq v$.

Combining the monotonicity observations we have made before, we see that for any outcome $\omega \in \Omega$, the function

$$c_j \mapsto L_{[a,b]}(c_1, \dots, c_N)(\omega) \stackrel{\text{def}}{=} t_{[a,b]} \left(\sum_{m=1}^N l_m 1_{[X_m \leq c_m]}(\omega) \right) \tag{2.5}$$

is monotone non-decreasing, with all values lying in the interval $[0, b - a]$; we can check also that it is right-continuous. Hence the function

$$c_j \mapsto \mathbb{E} [L_{[a,b]}(c_1, \dots, c_N)] \tag{2.6}$$

is non-decreasing and right continuous.

Consequently, there is a Borel measure μ_j on \mathbb{R} such that

$$\mu_j((u, v]) = \mathbb{E} [L_{[a,b]}(c_1, \dots, v, \dots, c_N)] - \mathbb{E} [L_{[a,b]}(c_1, \dots, u, \dots, c_N)] \tag{2.7}$$

for all $u \leq v$, where u and v appear in the j -th component on the right. We note that

$$l_j 1_{[X_j \leq v]} - l_j 1_{[X_j \leq u]}$$

is 0 when X_j is $\leq u$ or is $> v$, and is equal to l_j if $X_j \in (u, v]$. Thus

$$\begin{aligned} & L_{[a,b]}(c_1, \dots, v, \dots, c_N) - L_{[a,b]}(c_1, \dots, u, \dots, c_N) \\ &= \left(t_{[a,b]}(l_j + L'_j) - t_{[a,b]}(L'_j) \right) 1_{[u < X_j \leq v]} \end{aligned} \tag{2.8}$$

where

$$L'_j = \sum_{m \in \{1, \dots, N\} \setminus \{j\}} l_m 1_{[X_m \leq c_m]}. \tag{2.9}$$

The threshold c_j governs the default probability

$$p_j = \mathbb{P}[X_j \leq c_j], \tag{2.10}$$

and so the impact of an alteration in p_j on expected tranche losses is related to the sensitivity of such tranche losses to variation in c_j .

The *delta* of the tranche $[a, b]$ for the j -th threshold c_j is a measure of the impact on

$$\mathbb{E} [L_{[a,b]}(c_1, \dots, c_N)]$$

of variation in c_j , in relation to the impact on the loss on the entire ‘index’

$$\mathbb{E} [L_{[0, L^{\max}]}(c_1, \dots, c_N)] = \sum_{m=1}^N l_m \mathbb{E}[1_{[X_m \leq c_m]}], \tag{2.11}$$

where

$$L^{\max} = l_1 + \dots + l_N. \tag{2.12}$$

Thus, since only the j -th term is varied here, the delta should be

$$\lim_{\epsilon \downarrow 0} \frac{\mathbb{E} [L_{[a,b]}(c_1, \dots, c_j + \epsilon, \dots, c_N)] - \mathbb{E} [L_{[a,b]}(c_1, \dots, c_j, \dots, c_N)]}{l_j \mathbb{E} [1_{[c_j \leq X_j \leq c_j + \epsilon]}]} \tag{2.13}$$

in some suitable sense. Here we have chosen to divide by l_j , which is not always done in practice. Notice also that the division by $\mathbb{E} [1_{[c_j \leq X_j \leq c_j + \epsilon]}]$ can also be viewed as a measurement of the impact on the expected tranche loss due to a change in the default probability p_j of the j -th name. A reasonable precise formulation of (2.13) is in terms of a Radon-Nikodym derivative. We define $\Delta_j([a, b])$ to be the Radon-Nikodym derivative

$$\Delta_j([a, b]) = \frac{d\mu_j}{l_j d\mathbb{P}_{X_j}}, \tag{2.14}$$

where μ_j is the measure defined through (2.7), and \mathbb{P}_{X_j} is the distribution measure of X_j :

$$\mathbb{P}_{X_j}(A) = \mathbb{P}[X_j \in A]. \tag{2.15}$$

Thus $\Delta_j([a, b])$ is the function on \mathbb{R} for which

$$\begin{aligned} \mu_j((u, v]) &= l_j \int_u^v \Delta_j([a, b])(c) d\mathbb{P}_{X_j}(c) \\ &= l_j \mathbb{E} [1_{[X_j \in (u, v]}] \Delta_j([a, b])(X_j)] \end{aligned} \tag{2.16}$$

for all $u \leq v$. Equivalently,

$$\mu_j(A) = l_j \mathbb{E} [1_{[X_j \in A]} \Delta_j([a, b])(X_j)] \tag{2.17}$$

for all Borel $A \subset \mathbb{R}$.

Looking back at the specification of μ_j in (2.7) and the observation (2.8) we have

$$\mu_j((u, v]) = \mathbb{E} [1_{[u < X_j \leq v]} \{t_{[a,b]}(l_j + L'_j) - t_{[a,b]}(L'_j)\}], \tag{2.18}$$

where

$$L'_j = \sum_{m \in \{1, \dots, N\} \setminus \{j\}} l_m 1_{[X_m \leq c_m]}.$$

We can rewrite (2.18) after conditioning with respect to $\sigma(X_j)$:

$$\mu_j((u, v]) = \mathbb{E} [1_{[u < X_j \leq v]} D], \tag{2.19}$$

where

$$D = \mathbb{E} [t_{[a,b]}(l_j + L'_j) - t_{[a,b]}(L'_j) \mid \sigma(X_j)].$$

Comparing (2.19) with the specification of the delta in (2.16) we see that

$$\Delta_j([a, b])(X_j) = \frac{1}{l_j} \mathbb{E} [t_{[a,b]}(l_j + L'_j) - t_{[a,b]}(L'_j) \mid \sigma(X_j)]. \tag{2.20}$$

This roughly conforms to the idea that the delta measures the effect on the tranche loss due to variation of the threshold c_j . Next, recalling the expression for $t_{[a,b]}(v) - t_{[a,b]}(u)$ given in (2.4), we have

$$\Delta_j([a, b])(X_j) = \int_{[a,b]} \left(\frac{1}{l_j} \mathbb{E} \left[1_{[L'_j, l_j + L'_j]}(s) \mid \sigma(X_j) \right] \right) ds, \tag{2.21}$$

where the interchange of the integral $\int \dots ds$ and the conditional expectation can be justified by Fubini's theorem. We can write (2.21) also as:

$$\Delta_j([a, b])(X_j) = \int_{[a,b]} \frac{\mathbb{P} [s - l_j \leq L'_j \leq s \mid \sigma(X_j)]}{l_j} ds. \tag{2.22}$$

If we take $a = 0$ and $b = L^{\max} = l_1 + \dots + l_N$ in (2.21) then doing the integral $\int_a^b \dots ds$ first and then $\mathbb{E}[\dots \mid X_j]$, we see that

$$\Delta_j([0, L^{\max}]) = 1.$$

Thus the tranche delta itself is a probability measure (this confirms at a very general level a result proved in [15, Theorem 3.1] for the Gaussian copula).

3. Sensitivity to Thresholds

In this section we state our result concerning the sensitivity of expected tranche losses to default probabilities p_j as expressed through the default thresholds c_j . In the next section we shall see how sensitivity to the c_j can be used to determine sensitivity to correlation when the distribution of the variables X_j is jointly Gaussian. For the sake of expository clarity we defer the proofs of the results of the present section to section 7.

Proposition 3.1. *Let $X = (X_1, \dots, X_N)$ be an \mathbb{R}^N -valued random variable, with density function p that satisfies*

$$p(y_1, y_2, y_3, \dots, y_N) \leq B(y_2, \dots, y_N) \tag{3.1}$$

for all $y \in \mathbb{R}^N$ and some $B \in L^1(\mathbb{R}^{N-1})$. Let

$$L = \sum_{m=1}^N l_m 1_{[X_m \leq c_m]},$$

where $c_1, \dots, c_N \in \mathbb{R}$, and $l_1, \dots, l_N > 0$. Then the mixed partial derivative $\frac{\partial^2 \mathbb{E}[\{L-a\}_+]}{\partial c_1 \partial c_2}$ exists and is non-negative:

$$\frac{\partial^2 \mathbb{E}[\{L-a\}_+]}{\partial c_1 \partial c_2} \geq 0 \tag{3.2}$$

for any $a \geq 0$.

4. Sensitivity to Correlations

We specialize now to the case where the variables X_j have jointly Gaussian distribution. Combining our previous results with some special properties of the Gaussian we obtain the sensitivity of tranche losses to variation in correlation. Gaussian inequalities of this nature were pioneered by Slepian [18] in the context of extrema of Gaussian processes.

Theorem 4.1. *Let (X_1, \dots, X_N) be an \mathbb{R}^N -valued Gaussian variable, with each X_m having mean 0 and variance 1, and covariance matrix $R = [r_{jk}]$ that is strictly positive definite. Let*

$$L = \sum_{m=1}^N l_m 1_{[X_m \leq c_m]},$$

where $c_1, \dots, c_N \in \mathbb{R}$, and $l_1, \dots, l_N > 0$. Then, for any $a \geq 0$ and distinct $j, k \in \{1, \dots, N\}$,

$$\partial_{r_{jk}} \mathbb{E}[(L - a)_+] = \partial_{c_j} \partial_{c_k} \mathbb{E}[(L - a)_+]. \quad (4.1)$$

Moreover,

$$\frac{\partial \mathbb{E}[(L - a)_+]}{\partial r_{jk}} \geq 0, \quad (4.2)$$

and

$$\frac{\partial \mathbb{E}[t_{[0,a]}(L)]}{\partial r_{jk}} \leq 0. \quad (4.3)$$

Proof. Let

$$EL(c_1, \dots, c_N) = \mathbb{E}[(L - a)_+]. \quad (4.4)$$

The expectation $\mathbb{E}[(L - a)_+]$ is given by

$$\mathbb{E}[(L - a)_+] = \int_{\mathbb{R}^N} (l_N(x) - a)_+ Q(R, x) dx, \quad (4.5)$$

where $Q(R, x)$ is the Gaussian density

$$Q(R, x) = (\det(2\pi R))^{-1/2} e^{-\frac{1}{2}(x, R^{-1}x)}, \quad (4.6)$$

and

$$l_N(x) = \sum_{m=1}^N 1_{(-\infty, c_m]}(x_m). \quad (4.7)$$

Note that l_N is constant in the neighborhood of x if and only if $x_m \neq c_m$ for each component x_m . Then

$$\begin{aligned} \partial_{r_{jk}} \mathbb{E}[(L - a)_+] &= \int_{\mathbb{R}^N} (l_N(x) - a)_+ \partial_{r_{jk}} Q(R, x) dx \\ &= \int_{\mathbb{R}^N} (l_N(x) - a)_+ \frac{\partial^2 Q(R, x)}{\partial x_j \partial x_k} dx, \end{aligned} \quad (4.8)$$

on using dominated convergence in the first step and the Gaussian identity (4.15) (proved below) in the second step. Writing the partial derivative ∂_{x_j} as the limit

of a difference quotient and using dominated convergence, and repeating this for x_k , we have

$$\partial_{r_{jk}} \mathbb{E}[(L - a)_+] = \lim_{\epsilon_j \downarrow 0, \epsilon_k \downarrow 0} \frac{1}{\epsilon_j \epsilon_k} \int_{\mathbb{R}^N} (l_N(x) - a)_+ [*] dx, \tag{4.9}$$

where

$$[*] = Q(R, x + \epsilon_j e_j + \epsilon_k e_k) - Q(R, x + \epsilon_j e_j) - Q(R, x + \epsilon_k e_k) + Q(R, x). \tag{4.10}$$

Splitting the integration on the right in (4.9) into four integrals corresponding to the terms in (4.10), and replacing x_j by $x_j - \epsilon_j$ and x_k by $x_k - \epsilon_k$ in these terms judiciously, and then using

$$1_{(-\infty, c]}(x - \epsilon) = 1_{(-\infty, c + \epsilon]}(x),$$

we obtain

$$\begin{aligned} & EL(c_1, \dots, c_j - \epsilon_j, \dots, c_k - \epsilon_k, \dots, c_N) - EL(c_1, \dots, c_k - \epsilon_k, \dots, c_N) \\ & - EL(c_1, \dots, c_j - \epsilon_j, \dots, c_N) + EL(c_1, \dots, c_N) \end{aligned} \tag{4.11}$$

where EL is as defined by (4.4).

Dividing by ϵ_j and letting $\epsilon_j \downarrow 0$ yields

$$\begin{aligned} & \frac{\partial EL(c_1, \dots, c_j, \dots, c_k - \epsilon_k, \dots, c_N)}{\partial c_j} \\ & + \frac{\partial EL(c_1, \dots, c_j, \dots, c_k - \epsilon_k, \dots, c_N)}{\partial c_j}, \end{aligned} \tag{4.12}$$

with Proposition 7.2 guaranteeing that these partial derivatives here do exist. Next, dividing by ϵ_k and letting $\epsilon_k \downarrow 0$ we obtain, again using Proposition 7.2 for the existence of the derivative,

$$\frac{\partial^2 EL}{\partial c_j \partial c_k}.$$

Looking back at (4.9) we conclude that

$$\partial_{r_{jk}} \mathbb{E}[(L - a)_+] = \partial_{c_j} \partial_{c_k} \mathbb{E}[(L - a)_+]. \tag{4.13}$$

By Proposition 7.2 the right side here is ≥ 0 . Note that Proposition 7.2 is applicable because the bound (3.1) holds for $Q(R, x)$ with B being an appropriate Gaussian function on \mathbb{R}^{N-1} .

The inequality (4.3) now follows on using

$$t_{[0,1]}(L) = L - (L - a)_+$$

and the fact that L , being the sum of functions of the individual c_j 's has all the mixed partial derivatives equal to 0. □

For the sake of completeness we include a proof of the following useful identity, possibly first discovered by Plackett [17, eqn (3)], and used by Slepian [18, eqn (34)], Joag-Dev et al. [11] and others in the context of Gaussian inequalities:

Proposition 4.2. *Let $Q(R, x)$ be the Gaussian density*

$$Q(R, x) = (\det(2\pi R))^{-1/2} e^{-\frac{1}{2}\langle x, R^{-1}x \rangle}, \quad (4.14)$$

where R is a $d \times d$ strictly positive-definite matrix $R = [r_{jk}]$ and $x \in \mathbb{R}^d$. Then

$$\frac{\partial Q(R, x)}{\partial r_{jk}} = \begin{cases} \frac{1}{2} \frac{\partial^2 Q(R, x)}{\partial x_j^2} & \text{if } j = k; \\ \frac{\partial^2 Q(R, x)}{\partial x_j \partial x_k} & \text{if } j \neq k. \end{cases} \quad (4.15)$$

Proof. We use the Gaussian integration formula

$$\mathbb{E}[e^{aX}] = e^{a\mathbb{E}[X] + \frac{a^2}{2}\text{var}(X)}, \quad (4.16)$$

for any Gaussian variable X and complex number a . Then

$$\int_{\mathbb{R}^d} e^{\langle a, x \rangle} Q(R, x) dx = \mathbb{E}[e^{a_1 Y_1 + \dots + a_d Y_d}] = e^{\frac{1}{2}\langle a, Ra \rangle}, \quad (4.17)$$

where Y_1, \dots, Y_d are centered Gaussians with covariance

$$\text{Cov}(Y_j, Y_k) = r_{jk}.$$

Taking partial derivatives of both sides of (4.17) with respect to r_{jk} we have

$$\int_{\mathbb{R}^d} e^{\langle a, x \rangle} \frac{\partial Q(R, x)}{\partial r_{jk}} dx = e^{\frac{1}{2}\langle a, Ra \rangle} a_j a_k \quad \text{if } j \neq k.$$

and we recognize that the right side is equal to

$$e^{\frac{1}{2}\langle a, Ra \rangle} a_j a_k = \int_{\mathbb{R}^d} \frac{\partial^2 e^{\langle a, x \rangle}}{\partial x_j \partial x_k} Q(R, x) dx$$

and using integration by parts we can move the partial derivatives over to conclude that for $j \neq k$:

$$\int_{\mathbb{R}^d} e^{\langle a, x \rangle} \frac{\partial Q(R, x)}{\partial r_{jk}} dx = e^{\frac{1}{2}\langle a, Ra \rangle} a_j a_k = \int_{\mathbb{R}^d} e^{\langle a, x \rangle} \frac{\partial^2 Q(R, x)}{\partial x_j \partial x_k} dx.$$

The second terms in both integrands are $L^2(dx)$ functions and the above identity holds for all complex $a \in \mathbb{C}^d$; taking $a \in i\mathbb{R}^d$ and using the injective nature of the Fourier transform, we conclude that

$$\frac{\partial Q(R, x)}{\partial r_{jk}} = \frac{\partial^2 Q(R, x)}{\partial x_j \partial x_k}, \quad (4.18)$$

which is the second identity in (4.15). The first identity in (4.15) follows similarly by using the integration formula:

$$\int_{\mathbb{R}^d} e^{\langle a, x \rangle} \frac{\partial Q(R, x)}{\partial r_{jj}} dx = e^{\frac{1}{2}\langle a, Ra \rangle} \frac{1}{2} a_j^2.$$

□

5. Elliptically Contoured Distributions

An *elliptically contoured* distribution is a probability measure on \mathbb{R}^N whose density function is of the form

$$q_R(x) = \frac{1}{(\det R)^{1/2}} q(\langle x, R^{-1}x \rangle), \tag{5.1}$$

where R is a strictly positive definite matrix and q is a function normalized so that $\int_{\mathbb{R}^N} q_R(x) dx = 1$. The two most widely used examples are the case of centered Gaussian distributions and multivariate- t distributions. In the latter case the function q is a constant times

$$\left[1 + \frac{1}{\nu} \langle x, R^{-1}x \rangle \right]^{-\frac{\nu+N}{2}},$$

where $\nu > 0$ is a parameter. The distribution described by q_R has mean 0 and, as may be verified by computation, covariance matrix R . Moreover,

$$\frac{\partial q_R(x)}{\partial r_{jk}} = \left(1 - \frac{\delta_{jk}}{2} \right) \frac{\partial p_R(x)}{\partial x_j \partial x_k} \quad \text{for } j, k \in \{1, \dots, N\}, \tag{5.2}$$

where p_R is related to p , given by

$$p(t) = \frac{1}{2} \int_t^\infty q(s) ds, \tag{5.3}$$

in the same way as q_R is to q in (5.1). (In the Gaussian case in N dimensions, the one-variable function q is given by $q(s) = (2\pi)^{-N/2} e^{-s/2}$, and $p(t)$ coincides with $q(t)$, so that (5.2) becomes the Gaussian differential identity (4.15) we used before.) The identity (5.2) was proved in Joag-Dev et. al [11, Prop. 2] and Gordon [7, Prop. 1]. Then the equity-tranche sensitivity relation

$$\frac{\partial \mathbb{E}[t_{[0,a]}(L)]}{\partial r_{jk}} \leq 0 \quad \text{for distinct } j, k \in \{1, \dots, N\}, \tag{5.4}$$

holds, with the proof being analogous to that of (4.3), assuming certain natural conditions of smoothness and boundedness hold for q . Slepian-type inequalities were first extended to elliptically contoured distributions in [5] and the method greatly simplified in [11].

In this context let us make some remarks on correlation measures. A natural measure of ‘rank correlation’ between two random variables X and Y is given by Kendall’s tau [12]; the idea is that if (X', Y') is another instance (independent copy) of (X, Y) then the rank correlation is the expected value of the variable that has value 1 if $(X - X')(Y - Y') > 0$, has value -1 if $(X - X')(Y - Y') < 0$. and has value 0 in the case of ties:

$$\tau = \mathbb{P}[(X - X')(Y - Y') > 0] - \mathbb{P}[(X - X')(Y - Y') < 0]. \tag{5.5}$$

If X and Y are replaced by $h(X)$ and $h(Y)$, where h is a strictly increasing function, then τ remains the same. For this reason τ is a better measure of correlation r than the usual one in terms of normalized covariance, in situations where X and Y are not directly meaningful variables but rather are proxies for some unobservable variables. It has been shown by Linskov et al. [14] that Kendall’s tau is related in

a simple way to the traditional r in the case where (X, Y) has elliptically contoured distribution. The argument goes by using a suitable transformation of (X, Y) to reduce to the case where (X, Y) is centered Gaussian, and then using the relation (attributed in [14] to Stieltjes and Sheppard):

$$\tau = \frac{2}{\pi} \arcsin r. \tag{5.6}$$

We indicate here a proof of this classical formula using the partial differential equation for the the normal density $Q(R, x)$, where now R is a 2×2 matrix, with the off-diagonal entries being r . (Note that in the case of variables X_1, \dots, X_N we focus on pairs X_j, X_k .) We assume that $r \neq 0$, in which case X and Y are nonzero and linearly independent (so that, furthermore, the probability of ties is 0); then

$$\begin{aligned} \frac{d\tau}{dr} &= \frac{d}{dr} \left[2 \int_{x>x', y>y'} Q(R, (x, y))Q(R, (x', y')) dx dy dx' dy' - 1 \right] \\ &= 2 \int_{x>x', y>y'} \left[\frac{\partial^2 Q(R, (x, y))}{\partial x \partial y} Q(R, (x', y')) + \frac{\partial^2 Q(R, (x', y'))}{\partial x' \partial y'} Q(R, (x, y)) \right] \\ &\quad dx dy dx' dy' \\ &= 4 \int Q(R, (x, y))^2 dx dy = 4 \int (4\pi^2 \det R)^{-1} e^{-\langle v, R^{-1}v \rangle} d\lambda(v), \end{aligned} \tag{5.7}$$

with λ being Lebesgue measure on \mathbb{R}^2 , and we have used the formula

$$Q(R, v) = \frac{1}{(\det(2\pi R))^{1/2}} e^{-\langle v, R^{-1}v \rangle/2}.$$

Setting $v = 2^{-1/2}R^{1/2}w$ in the last integral in (5.7) we then have

$$\begin{aligned} \frac{d\tau}{dr} &= \frac{1}{2\pi^2 \det R^{1/2}} \int_{\mathbb{R}^2} e^{-|w|^2/2} d\lambda(w) = \frac{1}{2\pi^2 \det R^{1/2}} (2\pi) \\ &= \frac{1}{\pi \sqrt{1-r^2}}, \end{aligned} \tag{5.8}$$

on taking, for the diagonal entries in the matrix R , the variances of X and Y to be 1. The case $r = 1$ arises from $X = Y$ and in this case it is clear from the definition of τ that $\tau = 1$. Integrating (5.8) and requiring that $\tau \rightarrow 1$ as $r \rightarrow 1$ we obtain the relation

$$\tau = \frac{2}{\pi} \arcsin r.$$

6. Correlation Between Portfolios

Consider a portfolio consisting of N_1 entities whose default behavior is governed by a random variable $X_1 = (X_{11}, \dots, X_{1N_1})$, with j -th entity defaulting when X_{1j} is below a threshold level c_{1j} . Let us assume that there is a common random factor Z_1 to which the X_{1j} are correlated through the relation:

$$X_{1j} = \rho_{1j}Z_1 + \sqrt{1 - \rho_{1j}^2}\epsilon_{1j},$$

where $Z_1, \epsilon_{11}, \dots, \epsilon_{1N_1}$ are independent variables, each with mean 0 and variance 1.

Now consider a second portfolio consisting of N_2 entities whose default behavior is governed in an analogous way by variables Z_2, X_{2j} and thresholds c_{2j} . It is then of interest to understand the sensitivity of the correlation between the equity tranche losses of the two portfolios with respect to the correlation between Z_1 and Z_2 . To compute this sensitivity we assume that $Z_1, Z_2, \epsilon_{11}, \dots, \epsilon_{1N_1}, \epsilon_{21}, \dots, \epsilon_{2N_2}$ are jointly Gaussian, each with mean 0 and unit variance. Let

$$L_m = \sum_{j=1}^{N_1} l_{mj} 1_{[X_{mj} \leq c_{mj}]}, \quad \text{for } m \in \{1, 2\}. \tag{6.1}$$

Let $a_1, a_2 \geq 0$. Then by the same argument as was used for Theorem 4.1 we have: Then

$$\partial_{r_{1j,2k}} \mathbb{E} [(L_1 - a_1)_+ (L_2 - a_2)_+] = \partial_{c_{1j}} \partial_{c_{2k}} \mathbb{E} [(L_1 - a_1)_+ (L_2 - a_2)_+] \tag{6.2}$$

where

$$r_{1j,2k} = \mathbb{E}[X_{1j} X_{2k}].$$

Again following the methods used earlier we observe first that

$$\begin{aligned} & \partial_{c_{2k}} \mathbb{E} [(L_1 - a_1)_+ (L_2 - a_2)_+] \\ &= \lim_{\epsilon_2 \downarrow 0} \frac{1}{\epsilon_2} \mathbb{E} [(L_1 - a_1)_+ (l_{2k} + L'_2 - a_2)_+ 1_{[c_{2k} < X_{2k} \leq c_{2k} + \epsilon_2]}], \end{aligned} \tag{6.3}$$

where

$$L'_2 = \sum_{m \in \{1, \dots, N_2\} \setminus \{k\}} l_m 1_{[X_{2m} \leq c_{2m}]}. \tag{6.4}$$

The existence of the limit in (6.3) follows from the boundedness properties of the Gaussian density. It is clear from the expression on the right in (6.3) that this limit is ≥ 0 . Applying the same argument now to the further derivative with respect to c_{2k} we obtain again a non-negative value. Hence,

$$\partial_{r_{1j,2k}} \mathbb{E} [(L_1 - a_1)_+ (L_2 - a_2)_+] \geq 0. \tag{6.5}$$

7. Technical Results

In this section we prove technical results and use them to prove Proposition 3.1, which was presented in section 3.

Proposition 7.1. *Let X be a random variable and Y an \mathbb{R}^M -valued random variable, both defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let f_1 and f_0 be bounded measurable functions on \mathbb{R}^M , and let us denote $f_j(y)$ by $f(j, y)$. Suppose (X, Y) has a continuous density function p on $\mathbb{R} \times \mathbb{R}^M$ such that p is bounded by an integrable function B on \mathbb{R}^M in the sense that $p(x, y) \leq B(y)$ for all $(x, y) \in \mathbb{R}^{M+1}$. Then the function*

$$\phi : \mathbb{R} \rightarrow \mathbb{R} : c \mapsto \mathbb{E}[f(1_{[X \leq c]}, Y)]$$

is differentiable everywhere, the derivative being

$$\phi'(c) = \int_{\mathbb{R}^M} \{f(1, y) - f(0, y)\} p(c, y) dy. \tag{7.1}$$

Suppose now that the partial derivative $\partial_1 p$ exists, and there is an integrable function B on \mathbb{R}^M in the sense that $p(x, y) + |\partial_1 p(x, y)| \leq B(y)$ for all $(x, y) \in \mathbb{R}^{M+1}$. Then $\phi''(c)$ exists and

$$\frac{d^2\phi(c)}{dc^2} = \int_{\mathbb{R}^M} \{f(1, y) - f(0, y)\} \partial_1 p(c, y) dy. \tag{7.2}$$

Proof. For any $\epsilon > 0$ we have

$$f(1_{[X \leq c+\epsilon]}, Y) - f(1_{[X \leq c]}, Y) = \{f(1, Y) - f(0, Y)\} 1_{[c < X \leq c+\epsilon]}, \tag{7.3}$$

and so

$$\begin{aligned} \mathbb{E}[f(1_{[X \leq c+\epsilon]}, Y)] - \mathbb{E}[f(1_{[X \leq c]}, Y)] &= \mathbb{E} [\{f(1, Y) - f(0, Y)\} 1_{[c < X \leq c+\epsilon]}] \\ &= \int_{[c, c+\epsilon]} \left[\int_{\mathbb{R}^M} \{f(1, y) - f(0, y)\} p(x, y) dy \right] dx. \end{aligned} \tag{7.4}$$

Because of the boundedness condition we have imposed on p it follows by dominated convergence that the term $[\dots]$ on the right in (7.4) is continuous as a function of x . Consequently,

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \mathbb{E}[f(1_{[X \leq c+\epsilon]}, Y)] - \mathbb{E}[f(1_{[X \leq c]}, Y)] \\ = \int_{\mathbb{R}^M} \{f(1, y) - f(0, y)\} p(c, y) dy. \end{aligned} \tag{7.5}$$

Working similarly with c and $c - \epsilon$, we conclude that

$$\frac{d\mathbb{E}[f(1_{[X \leq c]}, Y)]}{dc} = \int_{\mathbb{R}^M} \{f(1, y) - f(0, y)\} p(c, y) dy. \tag{7.6}$$

Finally, the derivative formula (7.2) follows then using the boundedness assumption on $\partial_1 p$. \square

Now we apply Proposition 7.1 with X being X_1 ,

$$Y = (X_2, \dots, X_N),$$

and

$$\begin{aligned} f_1(y) &= t_{[a,b]}(l_1 + l') \\ f_0(y) &= t_{[a,b]}(0 + l'). \end{aligned} \tag{7.7}$$

where $y = (y_2, \dots, y_m)$ and

$$l' = \sum_{m=2}^N l_m 1_{(-\infty, c_m]}(y_m). \tag{7.8}$$

Then

$$\begin{aligned} f(1_{[X \leq c+\epsilon]}, Y) - f(1_{[X \leq c]}, Y) \\ = L_{[a,b]}(c_1, \dots, c_j + \epsilon, \dots, c_N) - L_{[a,b]}(c_1, \dots, c_j, \dots, c_N) \\ = \mu_j((c_j, c_j + \epsilon]) \quad (\text{using (2.7)}). \end{aligned} \tag{7.9}$$

Applying Proposition 7.1 we obtain:

$$\frac{d\mathbb{E}[L_{[a,b]}(c_1, \dots, c_N)]}{dc_1} = \int_{\mathbb{R}^{N-1}} \{f(1, y) - f(0, y)\} p(c_1, y) dy, \quad (7.10)$$

where p is the density function (assumed to exist and satisfy the conditions of Proposition 7.1) of (X_1, \dots, X_N) . Further, we can take the delta to be given by the function $\Delta_1([a, b])$:

$$\Delta_1([a, b])(c_1) = \frac{\int_{\mathbb{R}^{N-1}} \{f(1, y) - f(0, y)\} p(c_1, y) dy}{l_1 p_1(c_1)} \quad (7.11)$$

As a check we can take $[a, b] = [0, L^{\max}]$, in which case $f(a, y) = l_1 a + y$, and the value on the right in (7.11) reduces to 1.

We can determine other second order sensitivities by using the following extension of Proposition 7.1:

Proposition 7.2. *Let X_1 and X_2 be random variables and Y an \mathbb{R}^M -valued random variable, all defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $f_{j,k}$, for $j, k \in \{0, 1\}$, be bounded measurable functions on \mathbb{R}^M , and let us denote $f_{j,k}(y)$ by $f(j, k, y)$. Suppose (X_1, X_2, Y) has a continuous density function p on $\mathbb{R}^2 \times \mathbb{R}^M$ such that p is bounded by an integrable function B on \mathbb{R}^{1+M} in the sense that $p(c_1, c_2, y) \leq B(c_2, y)$ for all $(c_1, c_2, y) \in \mathbb{R}^{2+M}$. Then for the function*

$$\phi : \mathbb{R}^2 \rightarrow \mathbb{R} : (c_1, c_2) \mapsto \mathbb{E}[f(1_{[X_1 \leq c_1]}, 1_{[X_2 \leq c_2]}, Y)]$$

the mixed partial derivative $\partial^2 \phi / \partial c_1 \partial c_2$ exists and

$$\begin{aligned} & \frac{\partial^2 \phi(c_1, c_2)}{\partial c_1 \partial c_2} \\ &= \int_{\mathbb{R}^M} \{f(1, 1, y) - f(1, 0, y) - f(0, 1, y) + f(0, 0, y)\} p(c_1, c_2, y) dy. \end{aligned} \quad (7.12)$$

Proof. By Proposition 7.1 we have

$$\begin{aligned} & \partial_{c_1} \phi(c_1, c_2, y) \\ &= \int_{\mathbb{R}^{M+1}} \{f(1, 1_{(-\infty, c_2]}(x_2), y) - f(0, 1_{(-\infty, c_2]}(x_2), y)\} p(c_1, x_2, y) dx_2 dy. \end{aligned} \quad (7.13)$$

Then for any $\epsilon > 0$ we have

$$\begin{aligned} & \partial_{c_1} \phi(c_1, c_2 + \epsilon, y) - \partial_{c_1} \phi(c_1, c_2, y) \\ &= \int_{\mathbb{R}^{M+1}} \{f(1, 1_{(-\infty, c_2 + \epsilon]}(x_2), y) - f(0, 1_{(-\infty, c_2 + \epsilon]}(x_2), y)\} \cdot \\ & \quad \cdot p(c_1, x_2, y) dx_2 dy \\ & \quad - \int_{\mathbb{R}^{M+1}} \{f(1, 1_{(-\infty, c_2]}(x_2), y) - f(0, 1_{(-\infty, c_2]}(x_2), y)\} p(c_1, x_2, y) dx_2 dy \\ &= \int_{\mathbb{R}^{1+M}} D 1_{(c_2, c_2 + \epsilon]}(x_2) p(c_1, x_2, y) dx_2 dy, \end{aligned} \quad (7.14)$$

where

$$D = f(1, 1, y) - f(1, 0, y) - f(0, 1, y) + f(0, 0, y).$$

Note that $\int_{\mathbb{R}}[\cdots]1_{(c_2, c_2+\epsilon]}(x_2) dx_2$ is just $\int_{c_2}^{c_2+\epsilon}[\cdots] dx_2$. Dividing by ϵ and letting $\epsilon \downarrow 0$, and arguing entirely similarly with $(c_2 - \epsilon, c_2]$ instead of $(c_2, c_2 + \epsilon]$, we obtain the desired formula (7.12). \square

We apply the preceding result to the special case where f is given as follows

$$f^e(j, k, y) = \{l_1 j + l_2 k + y - a\}_+, \quad (7.15)$$

for some fixed $a \in \mathbb{R}$, and $Y = (X_3, \dots, X_N)$. We assume that $X = (X_1, \dots, X_N)$ has a density function p satisfying the hypotheses in Proposition 7.2.

Lemma 7.3. *For any $l_1, l_2 \geq 0$ and any $w \in \mathbb{R}$:*

$$\{l_1 + l_2 + w\}_+ - \{l_1 + w\}_+ - \{l_2 + w\}_+ + \{w\}_+ \geq 0. \quad (7.16)$$

Proof. The function $x \mapsto x_+$ has increasing slopes in the sense that

$$(b + h)_+ - (a + h)_+ \geq b_+ - a_+ \quad \text{if } b \geq a \text{ and } h \geq 0, \quad (7.17)$$

which we can check separately in three cases: (i) if $a < -h$ then the left side is $(b + h)_+$ which is $\geq b_+$, the value of the right side; (ii) if $a \in [-h, 0)$ then the left side is $b - a$ and the right side is b_+ (which is $\leq b - a$ here since $a < 0$); (iii) if $a > 0$ both sides equal $b - a$. We now take $a = w$, $b = l_1 + w$, $h = l_2$, and rearrange the terms in (7.17) to obtain the desired inequality (7.16). \square

Restating Lemma 7.3 in terms of the function f^e given in (7.15) we see that

$$f^e(1, 1, y) - f^e(1, 0, y) - f^e(0, 1, y) + f^e(0, 0, y) \geq 0 \quad (7.18)$$

for all $y \in \mathbb{R}$.

We can finally turn to the proof of the main result of section 3.

Proof. (Proof of Proposition 3.1.) We apply Proposition 7.2 applied with $Y = (X_3, \dots, X_N)$, and taking for f the function f^e given by

$$f^e(j, k, y) = \{jl_1 + kl_2 + y - a\}_+,$$

for all $y \in \mathbb{R}$ and $j, k \in \{0, 1\}$. Proposition 7.2 can be applied because the inequality (7.18) holds. \square

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C. BECKER: KÖNIGSBERGER STR. 33, 65830 KRIFTEL, GERMANY
E-mail address: `Claas.Becker@web.de`

A. N. SENGUPTA: DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LA 70810, USA
E-mail address: `ambarnsg@gmail.com`