

## ITÔ FORMULA AND GIRSANOV THEOREM FOR ANTICIPATING STOCHASTIC INTEGRALS

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**ABSTRACT.** We study the concept of translation of a Brownian motion by an anticipative term given by a Lebesgue integral of an instantly independent stochastic process. We introduce an equivalent probability measure that is constructed via an exponential process based on the stochastic integral of anticipative processes (in the sense of Ayed and Kuo) and show that under the new measure the translated Brownian motion is a continuous near-martingale with quadratic variation  $t$  on the interval  $[0, t]$ . Thus we obtain an anticipative version of the Girsanov theorem.

### 1. Introduction

Throughout this paper we let  $(\Omega, \mathcal{F}, P)$  be a complete probability space. We also let  $B_t$  be a Brownian motion defined on  $(\Omega, \mathcal{F}, P)$  and  $\{\mathcal{F}_t: 0 \leq t \leq T\}$ , be its natural filtration, that is  $\mathcal{F}_t = \sigma\{B_s: 0 \leq s \leq t\}$ . Since we will work only on a finite time interval, we fix it to be  $[0, T]$ . For the sake of brevity, we will write  $\{\mathcal{F}_t\}$  for  $\{\mathcal{F}_t: 0 \leq t \leq T\}$  and  $\{f_t\}$  for  $\{f_t: 0 \leq t \leq T\}$ . For a square integrable stochastic process  $\{f_t\}$  adapted to  $\{\mathcal{F}_t\}$ , we denote by  $\mathcal{E}_t(f)$  the stochastic exponential associated to  $f$  defined by

$$\mathcal{E}_t(f) = \exp \left\{ \int_0^t f_s dB_s - \frac{1}{2} \int_0^t f_s^2 ds \right\}.$$

Below we state two theorems from classical Itô calculus that have proven very important in many areas of applications of stochastic analysis, *e.g.* financial mathematics. These are the Itô formula and the Girsanov theorem. In this paper we present their extensions to the stochastic integral of adapted and instantly independent processes introduced by Ayed and Kuo in [1, 2] and briefly reviewed in Section 2 of the present paper. First of the Itô type formulas were derived by Kuo, Sae-Tang and Szozda in [10, 11, 12] and used by Khalifa *et al.* in [7] to find a unique solution to the linear stochastic differential equation with anticipating initial conditions.

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Received 2013-8-9; Communicated by the editors.

2010 *Mathematics Subject Classification.* Primary 60H05; Secondary 60H20.

*Key words and phrases.* Brownian motion, Itô integral, adapted stochastic process, anticipating integral, anticipating stochastic process, Itô formula, Girsanov theorem, instantly independent stochastic processes, Lévy characterization theorem, martingale, near-martingale.

\* Benedykt Szozda acknowledges support from the T.N. Thiele Centre for Applied Mathematics in Natural Science and from CREATES (DNRF78), funded by the Danish National Research Foundation.

**Theorem 1.1** (Itô Formula – adapted). *Suppose that  $\{X_t^{(i)} : i = 1, 2, \dots, n\}$  are continuous martingales with respect to  $\{\mathcal{F}_t : 0 \leq t \leq T\}$  and  $f(x_1, x_2, \dots, x_n)$  is a twice continuously differentiable real function on  $\mathbb{R}^n$ . Then*

$$\begin{aligned} & f(X_t^{(1)}, X_t^{(2)}, \dots, X_t^{(n)}) \\ &= f(X_0^{(1)}, X_0^{(2)}, \dots, X_0^{(n)}) \\ &+ \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i} f(X_s^{(1)}, X_s^{(2)}, \dots, X_s^{(n)}) dX_s^{(i)} \\ &+ \frac{1}{2} \int_0^t \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f(X_s^{(1)}, X_s^{(2)}, \dots, X_s^{(n)}) (dX_s^{(i)})(dX_s^{(j)}). \end{aligned}$$

**Theorem 1.2** (Girsanov, 1960, [5]). *Let  $\{f_t\}$  be a square integrable stochastic process adapted to  $\{\mathcal{F}_t\}$  such that  $\mathbb{E}_P[\mathcal{E}^t(f)] < \infty$  for all  $t \in [0, T]$ . Then*

$$\tilde{B}_t = B_t - \int_0^t f_s ds$$

*is a Brownian motion with respect to an equivalent probability measure  $Q$  given by*

$$dQ = \mathcal{E}_T(f) dP = \exp \left\{ \int_0^T f_s dB_s - \frac{1}{2} \int_0^T f_s^2 ds \right\} dP.$$

## 2. A New Stochastic Integral

In [1, 2], Ayed and Kuo proposed a new definition for a stochastic integral of a certain class of anticipating stochastic processes. In their construction, the authors exploit the independence of increments of Brownian motion. In order to do so, they decompose the integrands into sums of products of adapted and instantly independent processes (defined below). In this section, we briefly recall this construction.

We say that a stochastic process  $\{\varphi_t\}$  is *instantly independent* with respect to the filtration  $\{\mathcal{F}_t\}$  if for each  $t \in [0, T]$ , the random variable  $\varphi_t$  and the  $\sigma$ -field  $\mathcal{F}_t$  are independent. For example  $\varphi(B_1 - B_t)$  is instantly independent of  $\{\mathcal{F}_t : t \in [0, 1]\}$  for any real measurable function  $\varphi(x)$ . However,  $\varphi(B_1 - B_t)$  is adapted to  $\{\mathcal{F}_t : t \geq 1\}$ .

**Definition 2.1.** If  $\{f_t\}$  is an adapted stochastic process with respect to the filtration  $\{\mathcal{F}_t\}$  and  $\{\varphi_t\}$  is instantly independent with respect to the same filtration, we define the *stochastic integral* of  $f_t \varphi_t$  as

$$\int_0^T f_t \varphi_t dB_t = \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f_{t_{i-1}} \varphi_{t_i} \Delta B_i,$$

where  $\Delta_n = \{0 = t_0 < t_1 < \dots < t_n = T\}$  is a partition of the interval  $[0, T]$  and  $\Delta B_i = B_{t_i} - B_{t_{i-1}}$  and  $\|\Delta_n\| = \max\{t_i - t_{i-1} : i = 1, 2, \dots, n\}$ , provided the limit exists in probability.

Next, we recall some basic facts about martingales and their instantly independent counterpart, processes called near-martingales that were introduced and studied by Kuo, Sae-Tang and Szozda in [10]. It is worth mentioning that the same kind of processes are studied in [3], however they serve a different purpose and are termed *increment martingales*.

**Definition 2.2.** A stochastic process  $\{X_t\}$  is said to be a *martingale* with respect to a filtration  $\{\mathcal{F}_t\}$  if  $E|X_t| < \infty$  for all  $t \in [0, T]$  and

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \quad 0 \leq s < t \leq T.$$

Notice that the above definition immediately implies that  $\{X_t\}$  is adapted to  $\{\mathcal{F}_t\}$ , thus it is not feasible in the anticipating setting. However, Definition 2.2 is equivalent to the following three statements:

- (1)  $\mathbb{E}|X_t| < \infty$  for all  $t \in [0, T]$ ;
- (2)  $X_t$  is adapted;
- (3)  $\mathbb{E}[X_t - X_s|\mathcal{F}_s] = 0$  for all  $0 \leq s < t \leq T$ .

In [10], the authors propose to take the first and the last of the above conditions as the definition of a *near-martingale*. We recall this definition below.

**Definition 2.3.** We say that a process  $\{X_t\}$  is a *near-martingale* with respect to a filtration  $\{\mathcal{F}_t\}$  if  $E|X_t| < \infty$  for all  $0 \leq t \leq T$  and  $\mathbb{E}[X_t - X_s|\mathcal{F}_s] = 0$  for all  $0 \leq s < t \leq T$ .

It is a well-known fact that the Itô integral is a martingale, that is  $X_t = \int_0^t f_s dB_s$  is a martingale with respect to  $\{\mathcal{F}_t\}$ , for any adapted stochastic process  $\{f_t\}$  that is integrable with respect to  $B_t$  on the interval  $[0, T]$ . Similar result holds for the new stochastic integral: if  $f_t$  and  $\varphi_t$  are as in Definition 2.1 and the integral exists, then  $X_t = \int_0^t f_s \varphi_s dB_s$  is a near-martingale with respect to  $\{\mathcal{F}_t\}$  (see [10, Theorem 3.5].) Moreover, it is also a near-martingale with respect to a *natural backward filtration*  $\{\mathcal{G}^{(t)}\}$  of  $B_t$  defined by

$$\mathcal{G}^{(t)} = \sigma\{B_T - B_s : t \leq s \leq T\},$$

see [10, Theorem 3.7]. In general, a *backward filtration* is any decreasing family of  $\sigma$ -fields, *i.e.*  $\{\mathcal{G}^{(t)}\}$  satisfies  $\mathcal{G}^{(t)} \subseteq \mathcal{G}^{(s)}$  for any  $0 \leq s \leq t \leq T$ . A concept similar to that of the backward filtration is also used in [13].

In [11], Kuo, Sae-Tang and Szozda provide an Itô formula for a certain class of anticipative processes. Below, we recall special cases of two of the results of [11] that are related to the results of the present paper, namely [11, Theorem 5.1 and Corollary 6.2].

**Theorem 2.4.** *Suppose that  $f, \varphi \in C^2(\mathbb{R})$  and  $\theta(x, y) = f(x)\varphi(y - x)$ . Then for  $0 \leq t \leq T$ ,*

$$\begin{aligned} \theta(B_t, B_T) &= \theta(B_0, B_T) + \int_0^t \frac{\partial \theta}{\partial x}(B_s, B_T) dB_s \\ &\quad + \int_0^t \left[ \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(B_s, B_T) + \frac{\partial^2 \theta}{\partial x \partial y}(B_s, B_T) \right] ds. \end{aligned}$$

**Theorem 2.5.** *Suppose that  $h \in L^2[0, T]$ ,  $g \in L^1[0, T]$  and*

$$Y^{(t)} = \int_t^T h(s) dB_s + \int_t^T g(s) ds.$$

*Suppose also that  $f \in C^2(\mathbb{R} \times [0, T])$ . Then*

$$\begin{aligned} f(Y^{(t)}, t) &= f(Y^{(0)}, 0) + \int_0^t \frac{\partial f}{\partial t}(Y^{(s)}, t) ds \\ &+ \int_0^t \frac{\partial f}{\partial x}(Y^{(s)}, t) dY^{(s)} - \frac{1}{2} \int_0^t \frac{\partial^2 f}{\partial x^2}(Y^{(s)}, t) (dY^{(s)})^2. \end{aligned}$$

Finally, we recall another result from [10] that will be useful in establishing our results.

**Theorem 2.6.** *Let  $\{X_t\}$  be instantly independent with respect to the filtration  $\{\mathcal{F}_t\}$ . Then  $\{X_t\}$  is a near-martingale with respect to  $\{\mathcal{F}_t\}$  if and only if  $\mathbb{E}[X_t]$  is constant as a function of  $t$ .*

*Proof.* For the proof see [10, Theorem 3.1]. □

### 3. Itô Formula for the Anticipating Processes

In this section, we present the anticipative version of the Itô formula. We begin with several results to be used later to extend the Itô formula itself and the Girsanov type theorems. First, we present two simple but crucial observations that allow us to use some of the results from classical Itô theory in our setting.

**Theorem 3.1.** *Suppose that  $B_t$  is a Brownian motion on  $(\Omega, \mathcal{F}, P)$ ,  $\{\mathcal{F}_t\}$  and  $\{\mathcal{G}^{(t)}\}$  are its natural filtrations, forward and backward respectively. Then the probability spaces  $(\Omega, \mathcal{G}^{(0)}, P)$  and  $(\Omega, \mathcal{F}_T, P)$  coincide, that is  $\mathcal{G}^{(0)} = \mathcal{F}_T$ .*

*Proof.* Recall that the probability space  $(\Omega, \mathcal{F}_T, P)$  of  $B_t$  is a classical Wiener space with the  $\sigma$ -field  $\mathcal{F}_T$  generated by the cylinder sets. Notice that the  $\sigma$ -field  $\mathcal{G}^{(0)}$  is generated by the same cylinder sets. Thus the result follows. □

Using the above theorem, we can define a *backward Brownian motion*, that is a Brownian motion with respect to the backward filtration. Let

$$B^{(t)} = B_T - B_{T-t}.$$

By the argument above, we have the following fact.

**Proposition 3.2.** *Process  $\{B^{(t)}\}$  is a Brownian motion with respect to the filtration  $\{\overline{\mathcal{G}}^{(t)}\}$ , where  $\overline{\mathcal{G}}^{(t)} = \mathcal{G}^{(T-t)}$ .*

As we have previously mentioned, all of the classic results on Brownian motion apply to  $B^{(t)}$  and this will provide us with information on the integral for adapted and instantly independent processes.

Before we proceed with the proof of the Itô formula, we present a technical lemma. This lemma is used in the proof of the Itô formula as well as in the proof of the Girsanov theorem for the new stochastic integral.

**Lemma 3.3.** *Suppose that  $\{B_t\}$  is a Brownian motion and  $\{B^{(t)}\}$  is its backward Brownian motion, that is  $B^{(t)} = B_T - B_{T-t}$  for all  $0 \leq t \leq T$ . Suppose also that  $g(x)$  is a continuous function. Then the following two identities hold*

$$\int_t^T g(B_T - B_s) ds = \int_0^{T-t} g(B^{(s)}) ds \quad (3.1)$$

$$\int_t^T g(B_T - B_s) dB_s = \int_0^{T-t} g(B^{(s)}) dB^{(s)}. \quad (3.2)$$

*Proof.* We begin with the proof of Equation (3.1). Observe that upon a change of variables  $\bar{s} = T - s$  in the right side of Equation (3.1), we have

$$\begin{aligned} \int_0^{T-t} g(B^{(s)}) ds &= \int_0^{T-t} g(B_T - B_{T-s}) ds \\ &= - \int_T^t g(B_T - B_{\bar{s}}) d\bar{s} \\ &= \int_t^T g(B_T - B_{\bar{s}}) d\bar{s}. \end{aligned}$$

Thus Equation (3.1) holds.

Now, we will show Equation (3.2). Writing out the right side of Equation (3.2) using the definition of the stochastic integral, we have

$$\begin{aligned} \int_0^{T-t} g(B^{(s)}) dB^{(s)} &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g(B^{(t_{i-1})}) (B^{(t_i)} - B^{(t_{i-1})}) \\ &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g(B_T - B_{T-t_{i-1}}) (B_{T-t_{i-1}} - B_{T-t_i}), \end{aligned} \quad (3.3)$$

where  $\Delta_n$  is a partition of the interval  $[0, T-t]$  and the convergence is understood to be in probability on the space  $(\Omega, \bar{\mathcal{G}}^{(T)}, P)$  from Theorem 3.1. Applying the change of variables,

$$\bar{s} = T - s, \quad \bar{t}_i = T - t_i, \quad i = 1, 2, \dots, n,$$

we transform Equation (3.3) into

$$\begin{aligned} \int_0^{T-t} g(B^{(s)}) dB^{(s)} &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g(B_T - B_{\bar{t}_{i-1}}) (B_{\bar{t}_{i-1}} - B_{\bar{t}_i}) \\ &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n g(B_T - B_{\bar{t}_{i-1}}) (B_{\bar{t}_{i-1}} - B_{\bar{t}_i}) \end{aligned} \quad (3.4)$$

Notice that  $T = \bar{t}_0 > \bar{t}_1 > \bar{t}_2 > \dots > \bar{t}_n = t$ , and that the probability space  $(\Omega, \bar{\mathcal{G}}^{(T)}, P)$  is the same as  $(\Omega, \mathcal{F}_T, P)$ , we conclude that the last term in Equation (3.4) converges in probability to the new stochastic integral

$$\int_t^T g(B_T - B_s) dB_s.$$

Hence the Equation (3.2) holds.  $\square$

Now we are ready to prove the Itô formula for the new stochastic integral.

**Theorem 3.4** (Itô formula). *Suppose that*

$$Y_i^{(t)} = \int_t^T h_i(B_T - B_s) dB_s + \int_t^T g_i(B_T - B_s) ds, \quad i = 1, 2, \dots, n,$$

where  $h_i, g_i, i = 1, 2, \dots, n$  are continuous, square integrable functions. Then for any  $i, Y_i$  is instantly independent with respect to  $\{\mathcal{F}_t\}$ . Furthermore, let  $f(x_1, x_2, \dots, x_n)$  be a function in  $C^2(\mathbb{R}^n)$ . Then

$$\begin{aligned} df(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}) &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}) dY_i^{(t)} \\ &\quad - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}) (dY_i^{(t)})(dY_j^{(t)}). \end{aligned}$$

*Proof.* We prove the one-dimensional case only as the multi-dimensional case follows the same line of reasoning. Let

$$Y^{(t)} = \int_t^T h(B_T - B_s) dB_s + \int_t^T g(B_T - B_s) ds$$

Define

$$X_t = \int_0^t h(B^{(s)}) dB^{(s)} + \int_0^t g(B^{(s)}) ds.$$

Since  $\{B^{(s)}\}$  is a Brownian motion, we can apply the standard Itô formula to write

$$f(X_{T-t}) - f(X_0) = \int_0^{T-t} f'(X_s) dX(s) + \frac{1}{2} \int_0^{T-t} f''(X_s) (dX_s)^2 \quad (3.5)$$

Recall that  $dX_s = h(B^{(s)}) dB^{(s)} + g(B^{(s)}) ds$ , and by the Itô table we get  $(dX_s)^2 = h^2(B^{(s)}) ds$ . Hence Equation (3.5) can be written as

$$\begin{aligned} f(X_{T-t}) - f(X_0) &= \int_0^{T-t} f'(X_s) h(B^{(s)}) dB^{(s)} + \int_0^{T-t} f'(X_s) g(B^{(s)}) ds \\ &\quad + \frac{1}{2} \int_0^{T-t} f''(X_s) h^2(B^{(s)}) ds \end{aligned}$$

Writing out the integrals in the above equation as limits of the Riemann-like sums, we have

$$\begin{aligned} f(X_{T-t}) - f(X_0) &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(X_{t_{i-1}}) h(B^{(t_{i-1})}) \Delta B^{(t_i)} \\ &\quad + \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(X_{t_i}) g(B^{(t_i)}) \Delta t_i \\ &\quad + \frac{1}{2} \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f''(X_{t_i}) h^2(B^{(t_i)}) \Delta t_i, \end{aligned} \quad (3.6)$$

where  $\Delta_n = \{0 = t_0 \leq t_1 \leq \dots \leq t_n = T-t\}$  is a partition of the interval  $[0, T-t]$ .

By Lemma 3.3, we can replace  $X_{T-t}$  with  $Y^{(t)}$  in Equation (3.6) to obtain

$$\begin{aligned} & f(Y^{(t)}) - f(Y^{(T)}) \\ &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(Y^{(T-t_{i-1})}) h(B_T - B_{T-t_{i-1}}) (B_{T-t_{i-1}} - B_{T-t_i}) \\ & \quad + \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(Y^{(T-t_i)}) g(B_T - B_{T-t_i}) \Delta t_i \\ & \quad + \frac{1}{2} \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f''(Y^{(T-t_i)}) h^2(B_T - B_{T-t_i}) \Delta t_i \end{aligned}$$

Now we apply the change of variables,  $\bar{t}_i = T - t_i$  for  $i = 1, 2, \dots, n$  and notice that  $\{\bar{t}_i: i = 0, 1, 2, \dots, n\}$  is a partition of the interval  $[t, T]$ . Together with the fact that functions  $f, g$  and  $h$  are continuous this yields

$$\begin{aligned} f(Y^{(t)}) - f(Y^{(T)}) &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(Y^{(\bar{t}_{i-1})}) h(B_T - B_{\bar{t}_{i-1}}) (B_{\bar{t}_{i-1}} - B_{\bar{t}_i}) \\ & \quad + \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f'(Y^{(\bar{t}_i)}) g(B_T - B_{\bar{t}_i}) \Delta t_i \\ & \quad + \frac{1}{2} \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n f''(Y^{(\bar{t}_i)}) h^2(B_T - B_{\bar{t}_i}) \Delta t_i \\ &= \int_t^T f'(Y^{(s)}) h(B_T - B_s) dB_s + \int_t^T f'(Y^{(s)}) g(B_T - B_s) ds \\ & \quad + \frac{1}{2} \int_t^T f''(Y^{(s)}) h^2(B_T - B_s) ds. \end{aligned} \tag{3.7}$$

Writing Equation (3.7) in differential form and using the fact that  $dY^{(t)} = -h(B_T - B_t) dB_t - g(B_T - B_t) dt$  and  $(dY^{(t)})^2 = h^2(B_T - B_t) dt$ , we obtain

$$\begin{aligned} df(Y^{(t)}) &= -f'(Y^{(t)}) h(B_T - B_t) dB_t - f'(Y^{(t)}) g(B_T - B_t) dt \\ & \quad - \frac{1}{2} f''(Y^{(t)}) h^2(B_T - B_t) dt \\ &= f'(Y^{(t)}) dY^{(t)} - \frac{1}{2} f''(Y^{(t)}) (dY^{(t)})^2. \end{aligned}$$

The proof is now complete.  $\square$

*Remark 3.5.* In the above proof, we implicitly used Theorem 3.1 while taking limits. Notice that in Equation (3.6), the limit is taken in the probability space  $(\Omega, \bar{\mathcal{G}}^{(T)}, P)$ , which by definition is equal to  $(\Omega, \mathcal{G}^{(0)}, P)$ . On the other hand, in Equation and (3.7), the limit is taken in the probability space  $(\Omega, \mathcal{F}_T, P)$

Theorem 3.4 serves as the Itô formula for functions that do not explicitly depend on  $t$ , however, we can easily make this generalization using standard methods.

**Corollary 3.6.** Let  $\{Y_i^{(t)} : i = 1, 2, \dots, n\}$  be defined as in Theorem 3.4 and let  $f(x_1, x_2, \dots, x_n, t)$  be a function in  $C^2(\mathbb{R}^n \times [0, T])$ . Then

$$\begin{aligned} df(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) dY_i^{(t)} \\ &\quad - \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) (dY_i^{(t)}) (dY_j^{(t)}) \\ &\quad + \frac{\partial f}{\partial t}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) dt \end{aligned}$$

We illustrate the use of the Itô formula from Theorem 3.4 with a few examples.

**Example 3.7.** Let  $Y^{(t)} = \int_t^T 1 dB_s = B_T - B_t$ . Let also  $f(x) = e^x$ ,  $g(x) = x^n$  and  $h(x, t) = \exp\{x + \frac{1}{2}t\}$ . Application of Theorem 3.4 and Corollary 3.6 yields

$$\begin{aligned} df(Y^{(t)}) &= -e^{B_T - B_t} dB_t - \frac{1}{2}e^{B_T - B_t} dt, \\ dg(Y^{(t)}) &= -n(B_T - B_t)^{n-1} dB_t - \frac{1}{2}n(n-1)(B_T - B_t)^{n-2} dt, \\ dh(Y^{(t)}, t) &= -e^{B_T - B_t + \frac{1}{2}t} dB_t. \end{aligned}$$

On the other hand, the first two of the above equalities can be derived using Theorem 2.4. The last of the above equalities can be obtained using Theorem 2.5.

The following example shows a connection between the more general Theorem 3.4 and the original Theorem 2.5.

**Example 3.8.** Let  $Z^{(t)} = \int_t^T (B_T - B_s) dB_s$  and let  $f(x)$  and  $g(x)$  be as in Example 3.7. Straightforward calculations based on Definition 2.1 yield

$$Z^{(t)} = \frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t) \quad \text{and} \quad dZ^{(t)} = -(B_T - B_t) dB_t.$$

Applying Theorem 3.4 and Corollary 3.6, for any  $0 \leq t \leq T$ , we obtain

$$\begin{aligned} df(Z^{(t)}) &= -\exp\left[\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right] (B_T - B_t) dB_t \\ &\quad - \frac{1}{2} \exp\left[\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right] (B_T - B_t)^2 dt \end{aligned}$$

and

$$\begin{aligned} dg(Z^{(t)}) &= -n\left(\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right)^{n-1} (B_T - B_t) dB_t \\ &\quad - \frac{1}{2}n(n-1)\left(\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right)^{n-2} (B_T - B_t)^2 dt. \end{aligned}$$

On the other hand, notice that  $Z^{(t)} = \frac{1}{2}(Y^{(t)})^2 - \frac{1}{2}(T - t)$ , with  $Y^{(t)}$  as in Example 3.7. Defining

$$f^*(x, t) = f\left(\frac{1}{2}x^2 - \frac{1}{2}(T - t)\right),$$

allows for an application of Theorem 2.5 to obtain the same result

$$\begin{aligned} df(Z^{(t)}) &= df\left(\frac{1}{2}(Y^{(t)})^2 - \frac{1}{2}(T - t)\right) \\ &= df^*(Y^{(t)}) \\ &= -\exp\left[\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right] (B_T - B_t) dB_t \\ &\quad - \frac{1}{2} \exp\left[\frac{1}{2}(B_T - B_t)^2 - \frac{1}{2}(T - t)\right] (B_T - B_t)^2 dt. \end{aligned}$$



The above examples demonstrate that our results are generalizations of the results presented in [11] whose special cases were cited in Theorems 2.4 and 2.5. The following example illustrates how one can define an instantly independent counterpart to the exponential process.

**Example 3.9.** Let

$$\mathcal{E}^{(t)}(\theta) = \exp\left\{-\int_t^T \theta(B_T - B_s) dB_s - \frac{1}{2} \int_t^T \theta^2(B_T - B_s) ds\right\}.$$

Then

$$d\mathcal{E}^{(t)}(\theta) = \theta(B_T - B_s)\mathcal{E}^{(t)}(\theta) dB_t.$$

We call  $\mathcal{E}^{(t)}(\theta)$  the exponential process of the instantly independent process  $\theta(B_T - B_s)$ .

*Proof.* Let  $f(x) = e^x$  and define

$$Y_t = -\int_t^T \theta(B_T - B_s) dB_s - \frac{1}{2} \int_t^T \theta^2(B_T - B_s) ds.$$

Since  $f(x) = f'(x) = f''(x)$  and  $f(Y_t) = \mathcal{E}^{(t)}(\theta)$ , application of Theorem 3.4 to  $f(Y_t)$ , yields

$$\begin{aligned} d\mathcal{E}^{(t)}(\theta) &= df(Y_t) \\ &= f'(Y_t) dY_t - \frac{1}{2} f''(Y_t) (dY_t)^2 \\ &= e^{Y_t} (\theta(B_T - B_t) dB_t + \frac{1}{2} \theta^2(B_T - B_t) dt) - \frac{1}{2} e^{Y_t} \theta^2(B_T - B_t) dt \\ &= \theta(B_T - B_s)\mathcal{E}^{(t)}(\theta) dB_t. \end{aligned}$$

Here we have used the fact that  $dY(t) = \theta(B_T - B_t) dB_t + \frac{1}{2} \theta^2(B_T - B_t) dt$ .  $\square$

In the next example, we give a solution to a simple linear stochastic differential equation with terminal condition and anticipating coefficients.

**Example 3.10.** Suppose that  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  are twice continuously differentiable functions. Then the linear SDE

$$\begin{cases} dX_t &= f(B_T - B_t)X_t dB_t + g(B_T - B_t) dt \\ X_T &= \xi_T, \end{cases}$$

where  $\xi_T$  is a real deterministic constant, has a solution given by

$$X_t = \xi_T \exp\left\{-\int_t^T f(B_T - B_s) dB_s - \int_t^T \frac{1}{2} f^2(B_T - B_s) + g(B_T - B_s) ds\right\}.$$

#### 4. Anticipative Girsanov Theorem

In this section we present several results that can be considered an instantly independent counterpart of the Girsanov theorem. In the classical case of adapted stochastic processes, Girsanov theorem states that a translated Brownian motion is again a Brownian motion in some equivalent probability measure (see Theorem 1.2.) One can prove the Girsanov theorem by using the so-called Lévy characterization theorem that we recall below.

**Theorem 4.1** (Lévy characterization). *A stochastic process  $\{X_t\}$  is a Brownian motion if and only if there exists a probability measure  $Q$  and a filtration  $\{\mathcal{F}_t\}$  such that*

- (1)  $\{X_t\}$  is a continuous  $Q$ -martingale
- (2)  $Q(X_0 = 0) = 1$
- (3) the  $Q$ -quadratic variation of  $\{X_t\}$  on the interval  $[0, t]$  is equal to  $t$ .

As discussed in Section 2, there is no notion of a martingale in the anticipating setting. Thus the first item above cannot be satisfied. However, as we have indicated earlier, we can relax the requirement of adaptedness and consider near-martingales instead of martingales. Thus we will show below that certain translations of Brownian motion produce continuous  $Q$ -near-martingales (condition (1) from Theorem 4.1) whose  $Q$ -quadratic variation is equal to  $t$  (condition (3) from Theorem 4.1). Here, the measure  $Q$  is a probability measure equivalent to measure  $P$  and will be introduced later. This equivalence immediately takes care of item (2) from Theorem 4.1.

Let us first present a technical result that is used in the proof of the theorems presented later in this section.

**Theorem 4.2.** *Suppose that  $\{B_t\}$  is a Brownian motion and  $\{\mathcal{F}_t\}$ ,  $\{\mathcal{G}^{(t)}\}$  are its forward and backward natural filtrations, respectively. Suppose also that stochastic processes  $\{X_t^{(i)}, i = 1, 2, \dots, n\}$  are adapted to  $\{\mathcal{F}_t\}$  and  $\{Y_t^{(i)}, i = 1, 2, \dots, n\}$  are processes that are instantly independent with respect to  $\{\mathcal{F}_t\}$  and adapted to  $\{\mathcal{G}^{(t)}\}$ . Let*

$$S_t = \int_0^t \sum_{i=1}^n X_s^{(i)} Y_s^{(i)} dB_s$$

and assume that  $E|S_t| < \infty$  for all  $0 \leq t \leq T$ . Then  $S_t$  is a near-martingale with respect to both  $\{\mathcal{F}_t\}$  and  $\{\mathcal{G}^{(t)}\}$ .

*Proof.* It is enough to show that the above holds for  $n = 1$ . The general case follows by the linearity of the conditional expectation. Thus we will show that

$$\mathbb{E}[S_t - S_s | \mathcal{F}_s] = \mathbb{E}[S_t - S_s | \mathcal{G}^{(t)}] = 0,$$

where

$$S_t = \int_0^t X_s Y_s dB_s.$$

For all  $0 \leq s \leq t \leq T$  we have

$$\begin{aligned} \mathbb{E}[S_t - S_s | \mathcal{F}_s] &= \mathbb{E} \left[ \int_s^t X_s Y_s dB_s \middle| \mathcal{F}_s \right] \\ &= \mathbb{E} \left[ \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n X_{t_{i-1}} Y_{t_i} \Delta B_i \middle| \mathcal{F}_s \right] \\ &= \lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n \mathbb{E} [X_{t_{i-1}} Y_{t_i} \Delta B_i | \mathcal{F}_s]. \end{aligned}$$

Thus we see that it is enough to show that  $\mathbb{E}[X_{t_{i-1}}Y_{t_i}\Delta B_i|\mathcal{F}_s] = 0$  for all  $i = 1, 2, \dots, n$ . In fact, using the tower property of the conditional expectation to condition on  $\mathcal{F}_{t_i}$  and the fact that  $X_{t_{i-1}}\Delta B_i$  is  $\mathcal{F}_{t_i}$  measurable and  $Y_{t_i}$  is independent of  $\mathcal{F}_{t_i}$ , we obtain

$$\begin{aligned}\mathbb{E}[X_{t_{i-1}}Y_{t_i}\Delta B_i|\mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[X_{t_{i-1}}Y_{t_i}\Delta B_i|\mathcal{F}_{t_i}]\mathcal{F}_s] \\ &= \mathbb{E}[X_{t_{i-1}}\Delta B_i\mathbb{E}[Y_{t_i}]\mathcal{F}_s]\end{aligned}$$

Conditioning on  $\mathcal{F}_{t_{i-1}}$  and using the fact that  $X_{t_{i-1}}$  is measurable with respect to  $\mathcal{F}_{t_{i-1}}$  and  $\Delta B_i$  is independent of  $\mathcal{F}_{t_{i-1}}$ , we have

$$\begin{aligned}\mathbb{E}[X_{t_{i-1}}Y_{t_i}\Delta B_i|\mathcal{F}_s] &= \mathbb{E}[Y_{t_i}]\mathbb{E}[\mathbb{E}[X_{t_{i-1}}\Delta B_i|\mathcal{F}_{t_{i-1}}]\mathcal{F}_s] \\ &= \mathbb{E}[Y_{t_i}]\mathbb{E}[X_{t_{i-1}}\mathbb{E}[\Delta B_i]\mathcal{F}_s] \\ &= 0.\end{aligned}$$

The last equality above follows from the fact that  $\mathbb{E}[\Delta B_i] = 0$ .

Now we turn our attention to the second claim of the theorem, that is we show that  $E[S_t - S_s|\mathcal{G}^{(t)}] = 0$ . For the same reasons as above, it is enough to show that  $E[X_1(t_{i-1})Y_1(t_i)\Delta B_i|\mathcal{G}^{(t)}] = 0$  for all  $i = 1, 2, \dots, n$ . We start by using the tower property, independence of  $X_{t_{i-1}}$  and  $\mathcal{G}^{(t_{i-1})}$ , and measurability of  $Y_{t_i}\Delta B_i$  with respect to  $\mathcal{G}^{(t_{i-1})}$ .

$$\begin{aligned}E[X_{t_{i-1}}Y_{t_i}\Delta B_i|\mathcal{G}^{(t)}] &= E[E[X_{t_{i-1}}Y_{t_i}\Delta B_i|\mathcal{G}^{(t_{i-1})}]\mathcal{G}^{(t)}] \\ &= E[E[X_{t_{i-1}}]Y_{t_i}\Delta B_i|\mathcal{G}^{(t)}]\end{aligned}$$

Finally, since  $\Delta B_i$  is independent of  $\mathcal{G}^{(t_i)}$  and  $Y_{t_i}$  is measurable with respect to  $\mathcal{G}^{(t_i)}$ , application of the tower property and the fact that  $\mathbb{E}[\Delta B_i] = 0$ , yields

$$\begin{aligned}E[X_{t_{i-1}}Y_{t_i}\Delta B_i|\mathcal{G}^{(t)}] &= E[X_{t_{i-1}}]E[E[Y_{t_i}\Delta B_i|\mathcal{G}^{(t_i)}]\mathcal{G}^{(t)}] \\ &= E[X_{t_{i-1}}]E[Y_{t_i}E[\Delta B_i|\mathcal{G}^{(t)}]] \\ &= 0.\end{aligned}$$

Thus the proof is complete.  $\square$

Now, we turn our attention to the study of the translated Brownian motion  $\tilde{B}_t = B_t + \int_0^t \theta(B_T - B_s) ds$ . As in the classical theory of the Itô calculus, this is the process that is described by the Girsanov theorem, with the exception that  $B_T - B_s$  is substituted by  $B_s$  in the classical case. The crucial role in the construction of the measure  $Q$ , in which  $\{\tilde{B}_t\}$  is a near-martingale, is played by the exponential process  $\mathcal{E}^{(t)}(\theta)$ . The next result gives a very useful representation of the exponential process that is applied in the proofs of our main results.

**Lemma 4.3.** *Suppose that  $\theta(x)$  is a real-valued square integrable function. Then the exponential process of  $\theta(B_T - B_t)$  given by*

$$\mathcal{E}^{(t)}(\theta) = \exp\left\{-\int_t^T \theta(B_T - B_s) dB_s - \frac{1}{2} \int_t^T \theta^2(B_T - B_s) ds\right\}$$

has the following representation

$$\mathbb{E}\left[\mathcal{E}^{(0)}(\theta)\middle|\mathcal{G}^{(t)}\right] = \mathcal{E}^{(t)}(\theta),$$

where  $\{B_t\}$  is a Brownian motion,  $\{\mathcal{G}^{(t)}\}$  is its natural backward filtration.

*Proof.* By Example 3.9, we have

$$\mathcal{E}^{(t)}(\theta) - \mathcal{E}^{(0)}(\theta) = \int_0^t \theta(B_T - B_t) \mathcal{E}^{(t)}(\theta) dB_t.$$

Note that  $\theta(B_T - B_t) \mathcal{E}^{(t)}(\theta)$  is instantly independent of  $\{\mathcal{F}_t\}$  and adapted to  $\mathcal{G}^{(t)}$ , thus by Theorem 4.2,  $\mathcal{E}^{(t)}(\theta)$  is a near-martingale relative to  $\{\mathcal{G}^{(t)}\}$ , that is

$$\mathbb{E}\left[\mathcal{E}^{(0)}(\theta) - \mathcal{E}^{(t)}(\theta)\middle|\mathcal{G}^{(t)}\right] = 0.$$

Equivalently, we have

$$\mathbb{E}\left[\mathcal{E}^{(0)}(\theta)\middle|\mathcal{G}^{(t)}\right] = \mathbb{E}\left[\mathcal{E}^{(t)}(\theta)\middle|\mathcal{G}^{(t)}\right].$$

Note that  $\mathcal{E}^{(t)}(\theta)$  is measurable with respect to  $\mathcal{G}^{(t)}$ . Hence

$$E\left[\mathcal{E}^{(0)}(\theta)\middle|\mathcal{G}^{(t)}\right] = \mathcal{E}^{(t)}(\theta),$$

and the proof is complete.  $\square$

The next theorem is an anticipative version of condition (1) of Theorem 4.1.

**Theorem 4.4.** *Suppose that  $\{B_t\}$  is a Brownian motion on  $(\Omega, \mathcal{F}_T, P)$  and  $\varphi(x)$  is a square integrable function on  $\mathbb{R}$ . Let*

$$\tilde{B}_t = B_t + \int_0^t \varphi(B_T - B_s) ds.$$

Then  $\tilde{B}_t$  is a continuous near-martingale with respect to the probability measure  $Q$  given by

$$\begin{aligned} dQ &= \mathcal{E}^{(0)}(\varphi) dP \\ &= \exp\left\{-\int_0^T \varphi(B_T - B_s) dB_s - \frac{1}{2} \int_0^T \varphi^2(B_T - B_s) ds\right\} dP. \end{aligned}$$

*Proof.* The continuity of the process  $\{\tilde{B}_t\}$  is trivial. In order to clearly present the remainder of the proof, we proceed in several steps.

**Step 1.** First, we simplify the problem at hand. We will show that it is enough to verify that the expectation of a certain process is constant as a function of  $t$ . Define

$$\widehat{B}_t = B_T - B_t + \int_t^T \varphi(B_T - B_s) ds.$$

Then for any  $0 \leq s \leq t \leq T$  we have

$$\widetilde{B}_t - \widetilde{B}_s = \widehat{B}_s - \widehat{B}_t. \quad (4.1)$$

This implies that  $\mathbb{E}[\widetilde{B}_t - \widetilde{B}_s | \mathcal{F}_s] = 0$  if and only if  $\mathbb{E}[\widehat{B}_t - \widehat{B}_s | \mathcal{F}_s] = 0$ . Thus it is enough to show that  $\widehat{B}_t$  is a  $Q$ -near-martingale. Note that  $\widehat{B}_t$  is an instantly independent process, hence by Theorem 2.6, it suffices to show that  $\mathbb{E}_Q[\widehat{B}_t]$  is constant.

**Step 2.** In this step, we show that  $\mathbb{E}_Q[\widehat{B}_t]$  is constant. First, by the property of the conditional expectation, we have

$$\begin{aligned} \mathbb{E}_Q[\widehat{B}_t] &= \mathbb{E}[\widehat{B}_t \mathcal{E}^{(0)}(\varphi)] \\ &= \mathbb{E}\left[\mathbb{E}[\widehat{B}_t \mathcal{E}^{(0)}(\varphi) | \mathcal{G}^{(t)}]\right] \end{aligned}$$

Since  $\widehat{B}_t$  is measurable with respect to  $\mathcal{G}^{(t)}$ , Lemma 4.3 yields

$$\mathbb{E}_Q[\widehat{B}_t] = \mathbb{E}[\widehat{B}_t \mathcal{E}^{(t)}(\varphi)]. \quad (4.2)$$

Next, we apply the Itô Formula (Theorem 3.4) to  $\widehat{B}_t \mathcal{E}^{(t)}(\varphi)$ . In order to do so we let  $f(x, y) = xy$ , thus

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x, \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = 1.$$

We have

$$df(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi)) = \widehat{B}_t d\mathcal{E}^{(t)}(\varphi) + \mathcal{E}^{(t)}(\varphi) d\widehat{B}_t - (d\mathcal{E}^{(t)}(\varphi))(d\widehat{B}_t). \quad (4.3)$$

Using the facts that  $d\widehat{B}_t = -dB_t - \varphi(B_T - B_t)dt$  and  $d\mathcal{E}^{(t)}(\varphi) = \varphi(B_T - B_t)\mathcal{E}^{(t)}(\varphi)dB_t$ , Equation (4.3) becomes

$$d(\widehat{B}_t \mathcal{E}^{(t)}(\varphi)) = (\mathcal{E}^{(t)}(\varphi)\varphi(B_T - B_t)\widehat{B}_t - \mathcal{E}^{(t)}(\varphi))dB_t,$$

Observe that  $\mathcal{E}^{(t)}(\varphi)\varphi(B_T - B_t)\widehat{B}_t - \mathcal{E}^{(t)}(\varphi)$  is instantly independent with respect to  $\{\mathcal{F}_t\}$ , hence by Theorem 4.2,  $\widehat{B}(t)\mathcal{E}^{(t)}(\varphi)$  is a near-martingale with respect to  $\mathcal{F}_t$ . Thus by Theorem 2.6,  $\mathbb{E}[\widehat{B}(t)\mathcal{E}^{(t)}(\varphi)]$  is constant therefore, by Equation 4.2,  $\mathbb{E}_Q[\widehat{B}(t)]$  is constant as desired.  $\square$

Next, we present a result on the process  $\widehat{B}_t$  that we introduced in the proof of Theorem 4.4. As it turns out, the properties of this process are crucial in the proof of the condition (3) as well. Moreover, in the Corollary 4.7 we will present some more properties of this process as it is interesting on its own.

**Theorem 4.5.** *Suppose that  $\{B_t\}$  is a Brownian motion on  $(\Omega, \mathcal{F}_T, P)$ . Suppose also that  $\varphi(x)$  is a square integrable function on  $\mathbb{R}$  and  $Q$  is the probability measure introduced in Theorem 4.4. Let*

$$\widehat{B}_t = B_T - B_t + \int_t^T \varphi(B_T - B_s) ds. \quad (4.4)$$

Then  $\widehat{B}_t^2 - (T - t)$  is a continuous  $Q$ -near-martingale.

*Proof.* As previously, the continuity of the process  $\widehat{B}_t^2 - (T - t)$  is obvious. Since  $\widehat{B}^2(t) - (T - t)$  is instantly independent with respect to  $\{\mathcal{F}_t\}$ , by Theorem 2.6, we only need to show that  $\mathbb{E}_Q[\widehat{B}_t^2 - (T - t)]$  is constant. In fact, using the same methods as in the proof of Theorem 4.4, we have

$$\begin{aligned} \mathbb{E}_Q \left[ \widehat{B}_t^2 - (T - t) \right] &= \mathbb{E} \left[ (\widehat{B}_t^2 - (T - t)) \mathcal{E}^{(0)}(\varphi) \right] \\ &= \mathbb{E} \left[ \mathbb{E} \left[ (\widehat{B}^2(t) - (T - t)) \mathcal{E}^{(0)}(\varphi) \middle| \mathcal{G}^{(t)} \right] \right] \\ &= \mathbb{E} \left[ (\widehat{B}_t^2 - (T - t)) \mathbb{E} \left[ \mathcal{E}^{(0)}(\varphi) \middle| \mathcal{G}^{(t)} \right] \right] \\ &= \mathbb{E} \left[ (\widehat{B}_t^2 - (T - t)) \mathcal{E}^{(t)}(\varphi) \right]. \end{aligned}$$

In the last equality above we have used Lemma 4.3. Note that now it is enough to show that  $\mathbb{E}[(\widehat{B}_t^2 - (T - t)) \mathcal{E}^{(t)}(\varphi)]$  is constant.

Next, we apply the Itô formula (see Corollary 3.6) to  $f(x, y, t) = (x^2 - (T - t))y$  with  $x = \widehat{B}_t$  and  $y = \mathcal{E}^{(t)}(\varphi)$  to obtain

$$\begin{aligned} df(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) &= \frac{\partial f}{\partial x}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) d\widehat{B}_t + \frac{\partial f}{\partial y}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) d\mathcal{E}^{(t)}(\varphi) \\ &\quad + \frac{\partial f}{\partial t}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) dt - \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) (d\widehat{B}_t)^2 \quad (4.5) \\ &\quad - \frac{\partial^2 f}{\partial x \partial y}(\widehat{B}_t, \mathcal{E}^{(t)}(\varphi), t) (d\widehat{B}_t) (d\mathcal{E}^{(t)}(\varphi)) \end{aligned}$$

Since partial derivatives of  $f(x, y, t)$  are given by

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial y} = x^2 - (T - t), \quad \frac{\partial f}{\partial t} = y, \quad \frac{\partial^2 f}{\partial x^2} = 2y, \quad \frac{\partial^2 f}{\partial x \partial y} = 2x,$$

and the stochastic differentials in Equation (4.5) are given by

$$d\widehat{B}(t) = -dB_t - \varphi(B_T - B_t) dt, \quad d\mathcal{E}^{(t)}(\varphi) = \varphi(B_T - B_t) \mathcal{E}^{(t)}(\varphi) dB_t,$$

we obtain

$$\begin{aligned} &d \left( (\widehat{B}_t^2 - (T - t)) \mathcal{E}^{(t)}(\varphi) \right) \\ &= \left( \mathcal{E}^{(t)}(\varphi) (\widehat{B}_t^2 \varphi(B_T - B_t) - 2\widehat{B}_t) - (T - t) \varphi(B_T - B_t) \right) dB_t, \end{aligned}$$

or equivalently,

$$\begin{aligned} & (\widehat{B}_t^2 - (T - t))\mathcal{E}^{(t)}(\varphi) \\ &= \int_0^t \left( \mathcal{E}^{(s)}(\varphi)(\widehat{B}_s^2 \varphi(B_T - B_s) - 2\widehat{B}_s) - (T - s)\varphi(B_T - B_s) \right) dB_s. \end{aligned}$$

It is straightforward, although tedious, to show that the term under the integral above, is instantly independent with respect to  $\{\mathcal{F}_s\}$ . Therefore, by Theorem 4.2,  $(\widehat{B}_t^2 - (T - t))\mathcal{E}^{(t)}(\varphi)$ , as an integral of an instantly independent process, is a near-martingale with respect to  $\mathcal{F}_t$ . And thus by Theorem 2.6,  $E[(\widehat{B}^2(t) - (T - t))\mathcal{E}^{(t)}(\varphi)]$  is constant, so the theorem holds.  $\square$

Now we are ready to present the proof of condition (3) for the process  $\widetilde{B}_t$ .

**Theorem 4.6.** *Suppose that  $\{B_t\}$  is a Brownian motion in the probability space  $(\Omega, \mathcal{F}_T, P)$ . Let  $Q$  be a measure introduced in Theorem 4.4. Then the  $Q$ -quadratic variation of*

$$\widetilde{B}_t = B_t + \int_0^t \varphi(B_T - B_s) ds$$

on the interval  $[0, t]$  is equal to  $t$ .

*Proof.* We know that under measure  $P$ , the process  $\{B^{(t)}\}$  defined by  $B^{(t)} = B_T - B_{T-t}$  is a Brownian motion relative to filtration  $\{\widetilde{\mathcal{G}}^{(t)}\}$  (see Proposition 3.2.) By the classical Girsanov theorem (see Theorem 1.2), the process

$$\widetilde{B}^{(t)} = B^{(t)} + \int_0^t \varphi(B^{(s)}) ds \tag{4.6}$$

is a Brownian motion under measure  $\widetilde{Q}$  given by

$$\begin{aligned} d\widetilde{Q} &= \exp \left\{ - \int_0^T \varphi(B^{(s)}) dB^{(s)} - \frac{1}{2} \int_0^T \varphi^2(B^{(s)}) ds \right\} dP \\ &= \exp \left\{ - \int_0^T \varphi(B_T - B_s) dB^{(s)} - \frac{1}{2} \int_0^T \varphi^2(B_T - B_s) ds \right\} dP \\ &= dQ. \end{aligned}$$

In the second equality above, we have used Lemma 3.3 thus showing that  $\{\widetilde{B}^{(t)}\}$  is a Brownian motion under  $Q$ . Therefore the  $Q$ -quadratic variation of  $\{\widetilde{B}^{(t)}\}$  on the interval  $[T-t, T]$  is equal to  $t$ . In other words, for any partition  $\Delta_n = \{T-t = t_0 \leq t_1 \leq \dots \leq t_n = T\}$  of the interval  $[T-t, T]$ , we have that

$$\lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (\widetilde{B}^{(t_i)} - \widetilde{B}^{(t_{i-1})})^2 = t,$$

where the limit is taken in probability under measure  $Q$ . Changing the variables  $\bar{t}_i = T - t_{n-i}, i = 0, 1, \dots, n$ , yields a partition  $\bar{\Delta}_n$  of the interval  $[0, t]$  and (still under  $Q$ ),

$$\lim_{\|\bar{\Delta}_n\| \rightarrow 0} \sum_{i=1}^n (\widetilde{B}^{(T-\bar{t}_i)} - \widetilde{B}^{(T-\bar{t}_{i-1})})^2 = t. \tag{4.7}$$

Now notice that by Equation (4.6), the definition of  $B^{(s)}$  and change of variables  $v = T - s$  we have

$$\begin{aligned}\tilde{B}^{(T-\bar{t}_i)} &= B^{(T-\bar{t}_i)} + \int_0^{T-\bar{t}_i} \varphi(B^{(s)}) ds \\ &= B_T - B_{\bar{t}_i} + \int_0^{T-\bar{t}_i} \varphi(B_T - B_{T-s}) ds \\ &= B_T - B_{\bar{t}_i} + \int_{\bar{t}_i}^T \varphi(B_T - B_v) dv \\ &= \widehat{B}_{\bar{t}_i}.\end{aligned}$$

Hence Equation (4.7) becomes

$$\lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (\widehat{B}_{\bar{t}_i} - \widehat{B}_{\bar{t}_{i-1}})^2 = t, \quad (4.8)$$

where the limit is understood as a limit in probability under measure  $Q$  and  $0 = \bar{t}_0 \leq \bar{t}_1 \leq \dots \leq \bar{t}_n = t$ . Finally, Equations (4.1) and (4.8) yield

$$\lim_{\|\Delta_n\| \rightarrow 0} \sum_{i=1}^n (\tilde{B}_{\bar{t}_i} - \tilde{B}_{\bar{t}_{i-1}})^2 = t,$$

establishing the desired result. Tracing back the argument above, we can choose the points  $\bar{t}_i$  arbitrarily, hence Equation (4.8) implies that the  $Q$ -quadratic variation of  $\widehat{B}_t$  on the interval  $[0, t]$  is equal to  $t$ . Hence the proof is complete.  $\square$

**Corollary 4.7.** *Under the assumptions of Theorem 4.6, the  $Q$ -quadratic variation of the stochastic process  $\{\widehat{B}_t\}$  in Equation (4.4) on the interval  $[0, t]$  is equal to  $t$ .*

*Proof.* This follows immediately from Equation (4.8).  $\square$

Below we give two simple examples of the application of Theorems 4.5 and 4.6. The first one is the case with a simple instantly independent process driving the drift term of the translated Brownian motion.

**Example 4.8.** Let  $X_t = B_t + \int_0^t B_T - B_s ds$ . The exponential process of  $\varphi(B_T - B_t) = B_T - B_t$  is given by

$$\begin{aligned}\mathcal{E}^{(t)}(\varphi) &= \exp\left\{-\int_t^T B_T - B_s dB_s - \frac{1}{2} \int_t^T (B_T - B_s)^2 ds\right\} \\ &= \exp\left\{-\frac{1}{2}(B_T - B_t)^2 + \frac{1}{2}(T - t) - \frac{1}{2} \int_t^T (B_T - B_s)^2 ds\right\},\end{aligned}$$

where the last equality follows from the Itô formula (see Theorem 3.4.)

Thus, by Theorems 4.4 and 4.6, we conclude that  $X_t$  is a continuous  $Q$ -near-martingale with  $Q$ -quadratic variation on the interval  $[0, t]$  equal to  $t$ , where the measure  $Q$  is defined by

$$dQ = \mathcal{E}^{(0)}(\varphi) dP = \exp\left\{-\frac{1}{2}B_T^2 + \frac{1}{2}T - \frac{1}{2} \int_0^T (B_T - B_s)^2 ds\right\} dP.$$



The second example serves as a comparison of our results and the classical Girsanov theorem. The only case that we are able to compare right now is the one where the function  $\varphi(x) = a$  for some real number  $a \neq 0$ .

**Example 4.9.** Suppose that  $\varphi(x) = a$ . By Theorems 4.4 and 4.6, the stochastic process

$$X_t = B_t + \int_0^t a \, ds = B_t + at$$

is a continuous  $\overline{Q}$ -near-martingale with  $\overline{Q}$ -quadratic variation on the interval  $[0, t]$  being equal to  $t$ , where measure  $\overline{Q}$  is given by

$$d\overline{Q} = \mathcal{E}^{(0)}(\varphi) \, dP = \exp\{-aB_t - \frac{1}{2}a^2T\} \, dP.$$

Note that since the translation is deterministic, the process  $\{X_t\}$  is in fact adapted to the underlying filtration  $\{\mathcal{F}_t\}$ . Hence, as an adapted near-martingale,  $\{X_t\}$  is a martingale with respect to  $\{\mathcal{F}_t\}$ . Therefore, by the Lévy characterization theorem (see Theorem 4.1), process  $\{X_t\}$  is a Brownian motion under  $\overline{Q}$ .

On the other hand, the drift term of the process  $\{X_t\}$  is deterministic, therefore adapted, so we can use the classical Girsanov theorem (see Theorem 1.2 with  $f(s) = -a$ ) to conclude that  $X_t$  is a  $Q$ -Brownian motion, where  $Q$  is given by

$$dQ = \mathcal{E}_T(\varphi) \, dP = \exp\{-aB_t - \frac{1}{2}a^2T\} \, dP.$$

Notice that  $Q$  and  $\overline{Q}$  are actually the same measure. Thus, as expected, our results lead to the same conclusion in the case when both theorems are applicable, that is in the case when the translation is in fact deterministic. This example justifies the title of this paper as Theorems 4.4 and 4.6 actually generalize the Girsanov theorem.

As a final remark, let us note that the measure  $Q$  that was used in Theorem 4.4 is the same as the one derived in [4] where the author uses methods of Malliavin calculus to study anticipative Girsanov transformations. For more details on this approach see [4] and references therein.

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