

## A BOCHNER-TYPE REPRESENTATION OF POSITIVE DEFINITE MAPPINGS ON THE DUAL OF A COMPACT GROUP

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ABSTRACT. For an arbitrary compact group  $G$  with dual space  $\Sigma(G)$  a Bochner-type bijection is established between positive definite mappings on  $\Sigma(G)$  and central bounded measures on  $G$ . This bijection defined by the generalized Fourier transformation is based on the comparative study of three kinds of function algebras: the coefficient algebra, the algebra of convergent Fourier series, and the central Fourier algebra of  $G$ .

### 1. Introduction

Bochner's classical theorem on the characterization of continuous positive definite functions on the real line as Fourier transforms of bounded measures has been generalized to various algebraic-topological structures such as abelian groups, Gelfand pairs and commutative hypergroups, to name only a few examples.

If  $G$  is a locally compact abelian group and  $G^\wedge$  is its dual group, then for any continuous positive definite function  $\phi$  on  $G^\wedge$  there exists a unique bounded measure  $\tau$  on  $G$  such that  $\phi$  is represented as the Fourier transform of  $\tau$ .

This representation admits analogues provided for the underlying abstract algebraic-topological structure a dual structure can be defined on which some sort of positive definiteness makes sense.

Within the category of locally compact groups  $G$ , successful attempts have led to establishing a Bochner representation beyond the abelian case, in particular under the assumption that  $G$  is a compact Lie group. Even for this class of groups, for which the set  $\Sigma(G)$  of irreducible unitary representations serves as the dual object, a new version of positive definiteness has to be invented in order to reach the desired representation. See f.e.[6]. Moreover, there is an axiomatic approach to the Bochner property applying positive definite forms on the coefficient algebra of  $G$  in place of positive definite functions on  $\Sigma(G)$  ([4]).

In the present paper an application of the central Fourier algebra studied in the unpublished Doctoral dissertation of D.R. Beldin [2] enables us to generalize a Bochner type theorem from connected compact Lie groups to arbitrary compact groups.

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The inspiration for writing up previous thoughts on the subject going back to the author's monographs [4] and [5] is due to D. Applebaum who in his upcoming book [1] points out that the harmonic analysis of compact groups still lacks a precise analogue of Bochner's theorem.

The main result to be presented here states that for every positive definite mapping  $\phi$  on the dual  $\Sigma(G)$  of a compact group  $G$  there exists a unique central measure  $\mu$  on  $G$  such that its Fourier transform  $\hat{\mu}$  equals  $\phi$ .

The layout of our exposition is determined by the interplay among three types of function algebras.

Chapter 2 contains preliminaries from the representation theory of a compact group  $G$ . In particular the *coefficient algebra*  $F(G)$  of  $G$  and the Fourier–Stieltjes transform are introduced.

In Chapter 3, we recall the properties of central measure and functions on  $G$  and study the central Fourier mapping. Moreover, we introduce the *algebra*  $K(G)$  of *absolutely convergent Fourier series* whose spectrum coincides with  $G$ .

This result is an important tool in approaching a similar identification for the *central Fourier algebra*  $F^0(G)$  of  $G$  which is the subject of Chapter 4. This identification is achieved by extending elements of  $\text{spec}(F^0(G))$  for elements in  $\text{spec}(K(G))$ , where the extension procedure is purely Banach-algebraic, similar to but different from an idea in [3], (34.37). Finally, Chapter 5 is devoted to introducing a suitable notion of positive definite mappings on the dual  $\Sigma(G)$  of  $G$  and to proving Bochner's theorem along the lines of its classical predecessor, by applying the spectral properties of  $F^0(G)$  in place of  $L^1(G)$ . The necessary arguments are borrowed from [2]. Although it has been attempted to make the paper more easily readable by reproducing well-known facts and referring consequently to the seminal monograph [3], the reader is expected to be familiar with the Gelfand theory for commutative Banach algebras and its application to locally compact abelian groups ([8]).

## 2. Preliminaries from Representation Theory

For a multiplicatively written locally compact group  $G$  with unit element  $e$  the space of continuous functions on  $G$  will be denoted by  $C(G)$ . On  $G$  there exists a Haar measure  $\omega_G$  which is unique within a positive factor.  $\omega_G$  is bounded and hence normalizable to 1 if and only if  $G$  is compact.  $M(G)$  denotes the Banach space of bounded (Radon) measures on  $G$ , and  $L^1(G, \omega_G)$  the norm-closed subspace of  $\omega_G$ -absolutely continuous measures. In fact,  $M(G)$  is a Banach  $*$ -algebra with respect to convolution and conjugation, admitting the Dirac measure  $\varepsilon_e$  as multiplicative unit.

We denote by  $\mathfrak{B}(\mathcal{H})$  the  $*$ -algebra of bounded linear operators on a Hilbert space  $\mathcal{H}$ , with  $I$  as the identity operator.

From now on we shall assume  $G$  to be a compact group.

Let  $\Sigma(G)$  and  $\Sigma'(G)$  denote the sets of equivalence classes of irreducible and finite-dimensional (continuous, unitary) representations of  $G$  respectively. Since  $G$  is compact,  $\Sigma(G) \subset \Sigma'(G)$ . Given  $\sigma \in \Sigma'(G)$  the *character* of  $\sigma$  is defined by

$$\chi_\sigma(x) := \chi_{U(\sigma)}(x) := \text{tr} \left( U^{(\sigma)}(x) \right)$$

for all  $x \in G$ , where  $U^{(\sigma)}$  is a representative of the class  $\sigma$ .

It is a basic result of the harmonic analysis of compact groups  $G$  that every  $\sigma' \in \Sigma'(G)$  admits a decomposition

$$\sigma' = \bigoplus_{\sigma \in \Sigma(G)} M(\sigma, \sigma')\sigma,$$

where

$$M(\sigma, \sigma') := \int_G \chi_\sigma(x^{-1})\chi_{\sigma'}(x)\omega_G(dx)$$

stands for the *multiplicity* of  $\sigma$  in  $\sigma'$  ([3], (27.30)). Let  $\sigma \in \Sigma(G)$ . We choose  $U^{(\sigma)} \in \sigma$  with representing Hilbert space  $\mathcal{H}_\sigma$  having an orthonormal basis  $\{h_1^{(\sigma)}, \dots, h_{d_\sigma}^{(\sigma)}\}$ , where  $d_\sigma = \dim \sigma := \dim \mathcal{H}_\sigma$ . For  $i, j = 1, \dots, d_\sigma$  the functions

$$x \mapsto u_{ij}^{(\sigma)}(x) := \langle U^{(\sigma)}(x)h_j^{(\sigma)}, h_i^{(\sigma)} \rangle$$

are the *coefficient functions* of  $\sigma$ .

The spaces

$$F_\sigma(G) := \left\langle \left\{ u_{ij}^{(\sigma)} : i, j = 1, \dots, d_\sigma \right\} \right\rangle, \quad (\sigma \in \Sigma(G))$$

and

$$F(G) := \left\langle \bigcup_{\sigma \in \Sigma(G)} F_\sigma(G) \right\rangle$$

are independent of the choice of the representative  $U^{(\sigma)}$  of  $\sigma$ .

$F(G)$  is a subalgebra of  $C(G)$  called the *coefficient algebra* of  $G$ .

A second basic result of the harmonic analysis of compact groups  $G$  is the Peter–Weyl theorem which states that  $F(G)$  is dense in  $C(G)$  with respect to the uniform norm  $\|\cdot\|_u$ , and that the system

$$\left\{ \sqrt{d_\sigma} u_{ij}^{(\sigma)} : \sigma \in \Sigma(G), i, j = 1, \dots, d_\sigma \right\}$$

is an orthonormal basis of  $L^2(G, \omega_G)$ .

In order to introduce the Fourier mapping on  $M(G)$  we need some preparations on operators on Hilbert spaces. Let  $\{\mathcal{H}_i : i \in I\}$  be a family of Hilbert spaces  $\mathcal{H}_i$  of dimension  $d_i < \infty$  ( $i \in I$ ). The set

$$\mathcal{E}(I) := \prod_{i \in I} \mathfrak{B}(\mathcal{H}_i)$$

is a  $*$ -algebra with respect to the familiar operations. In  $\mathcal{E}(I)$  one introduces various norms:

For  $E := (E_i)_{i \in I} \in \mathcal{E}(I)$  and  $(a_i)_{i \in I}$  with  $a_i \geq 1$  for all  $i \in I$  one defines

$$\|E\|_p := \begin{cases} \left( \sum_{i \in I} a_i \|E_i\|_{\phi_p}^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \sup \{ \|E_i\|_{\phi_\infty} : i \in I \} & \text{if } p = \infty \end{cases}$$

and consider the subspaces

$$\mathcal{E}_p(I) := \{E \in \mathcal{E}(I) : \|E\|_p < \infty\} \quad (1 \leq p \leq \infty).$$

Here

$$\phi_p(u_1, \dots, u_n) := \begin{cases} \left( \sum_{j=1}^n |u_j|^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \max\{|u_1|, \dots, |u_n|\} & \text{if } p = \infty \end{cases}$$

and

$$\|X\|_p := \phi_p(x_1, \dots, x_n),$$

where  $x_1, \dots, x_n$  are the eigenvalues of the positive definite operator  $|X| := (XX^*)^{\frac{1}{2}}$ . For details see [3], (28.24).

Now we return to the given compact group  $G$  and its dual  $\Sigma(G)$ .

For each  $\sigma \in \Sigma(G)$  we choose a representative  $U^{(\sigma)}$  with corresponding Hilbert space  $\mathcal{H}_\sigma$  of dimension  $d_\sigma$ .

Then

$$\mathcal{E}(\Sigma(G)) := \prod_{\sigma \in \Sigma(G)} \mathfrak{B}(\mathcal{H}_\sigma)$$

is a  $*$ -algebra, and we obtain Banach spaces

$$\mathcal{E}_p(\Sigma(G)) := \{E \in \mathcal{E}(\Sigma(G)) : \|E\|_p < \infty\},$$

where

$$\|E\|_p := \begin{cases} \left( \sum_{\sigma \in \Sigma(G)} d_\sigma (\|E_\sigma\|_{\phi_p})^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \sup\{\|E_\sigma\|_{\phi_\infty} : \sigma \in \Sigma(G)\} & \text{if } p = \infty \end{cases}$$

In fact,  $\mathcal{E}_\infty(\Sigma(G))$  becomes a Banach  $*$ -algebra. For a measure  $\mu \in M(G)$  the Fourier(-Stieltjes) transform

$$\hat{\mu} : \Sigma(G) \rightarrow \mathcal{E}(\Sigma(G))$$

is given by

$$\langle \hat{\mu}(\sigma)h, k \rangle := \int_G \langle \bar{U}^{(\sigma)}(x)h, k \rangle \mu(dx)$$

for all  $\sigma \in \Sigma(G), h, k \in \mathcal{H}_\sigma$ . If  $\mu := f\omega_G$  with  $f \in L^1(G, \omega_G)$ , then  $\hat{\mu}$  yields the Fourier transform of  $f$ .

It is shown in [3], (28.36) that the Fourier mapping

$$\mathcal{F} : M(G) \rightarrow \mathcal{E}_\infty(\Sigma(G))$$

given by

$$\mathcal{F}(\mu) := \hat{\mu}$$

for all  $\mu \in M(G)$  is a norm-decreasing  $*$ -isomorphism of Banach  $*$ -algebras.

### 3. Central Measures and Functions

As before we keep the assumption that  $G$  is a compact group.

A measure  $\mu \in M(G)$  is called *central* if it belongs to the center  $M^z(G)$  of  $M(G)$  which means that  $\mu * \nu = \nu * \mu$  whenever  $\nu \in M(G)$ .

**Theorem 3.1.** ([3], (28.48)) *For each  $\mu \in M(G)$  the following statements are equivalent:*

- (i)  $\mu \in M^z(G)$ .

- (ii)  $\mu * u_{ij}^{(\sigma)} = u_{ij}^{(\sigma)} * \mu$  for any system  $\{u_{ij}^{(\sigma)} : i, j = 1, \dots, d_\sigma\}$  of coefficients of  $\sigma \in \Sigma(G)$ .
- (iii)  $\mu * \varepsilon_a = \varepsilon_a * \mu$  for all  $a$  from a dense subset of  $G$ .
- (iv) Given any  $\sigma \in \Sigma(G)$  there exists an  $a(\tau, \sigma) \in \mathbb{C}$  such that

$$\hat{\mu}(\sigma) = a(\mu, \sigma)I_{d_\sigma}.$$

A function  $f \in C(G)$  is said to be *central* provided  $f\omega_G \in M^z(G)$ .

There are equivalences analogous to the ones in Theorem 3.1 for the set  $C^z(G)$  of central functions in  $C(G)$  in place of  $M^z(G)$ .

Centrality of measures and functions on  $G$  can also be described as follows: For  $f \in C(G)$  let

$$f^0(x) := \int_G f(yxy^{-1})\omega_G(dy)$$

whenever  $x \in G$ . The mapping

$$P: C(G) \rightarrow C(G)$$

given by

$$P(f) := f^0$$

for all  $f \in C(G)$  is continuous.

Given  $\mu \in M(G)$  we define  $\mu^0 \in M(G)$  by

$$\mu^0(f) := \mu(f^0)$$

for all  $f \in C(G)$ . Then  $\mu \in M^z(G)$  if and only if  $\mu = \mu^0$ . From the fact that for  $f \in C(G)$ ,  $f = f^0$  if and only if  $f(x) = f(yxy^{-1})$  holds whenever  $x, y \in G$  it follows that  $P$  is surjective from  $C(G)$  onto  $C^z(G)$ .

Now we specify the Fourier transform for measures  $\mu \in M^z(G)$  and functions  $f \in C^z(G)$ .

**Definition 3.2.** For  $\mu \in M^z(G)$  the mapping  $\hat{\mu}: \Sigma(G) \rightarrow \mathbb{C}$  given by

$$\hat{\mu}(\sigma) := \int_G \chi_\sigma d\mu$$

for all  $\sigma \in \Sigma(G)$  is said to be the *central Fourier transform* of  $\mu$ .

If  $f \in C^z(G)$ , then  $\hat{\mu} = \widehat{f\omega_G}$  defines the central Fourier transform  $\hat{f}$  of  $f$ .

**Theorem 3.3.** *The central Fourier mapping*

$$\dot{F}: M^2(G) \rightarrow \mathbb{C}^{\Sigma(G)}$$

given by

$$\dot{F}(\mu) := \hat{\mu}$$

for all  $\mu \in M^z(G)$  is injective.

In order to prove this theorem we consider the  $*$ -subalgebra

$$F^z(G) := F(G) \cup C^z(G)$$

of the coefficient algebra  $F(G)$  of  $G$  and prepare the proof by showing the following

**Lemma 3.4.**  $F^z(G) = \langle \{\chi_\sigma : \sigma \in \Sigma(G)\} \rangle$

*Proof.* We first establish the inclusion

$$\langle \{\chi_\sigma : \sigma \in \Sigma(G)\} \rangle \subset F^z(G).$$

In fact, let  $\sigma \in \Sigma(G)$ . Then  $\chi_\sigma$  is continuous. Moreover,  $\chi_\sigma$  is central, since for all  $x, y \in G$

$$\begin{aligned} \chi_\sigma(y^{-1}xy) &= \operatorname{tr} \left( U^{(\sigma)}(y^{-1}xy) \right) \\ &= \operatorname{tr} \left( U^{(\sigma)}(x) \right) = \chi_\sigma(x). \end{aligned}$$

For the reverse inclusion we observe that

$$F^z(G) = P(F(G)),$$

where  $P(f) := f^0$  for all  $f \in C(G)$ , hence every element of  $F^z(G)$  is of the form  $f^0$  for some  $f \in F(G)$ . The mapping  $P$  being linear it remains to be shown that for each coefficient function  $u_{ij}^{(\sigma)}$  of  $\sigma$  the function  $(u_{ij}^{(\sigma)})^0$  is a multiple of  $\chi_\sigma$ . But this follows from the subsequent equalities valid for all  $x \in G$ :

$$\begin{aligned} (u_{ij}^{(\sigma)})^0(x) &= \int_G u_{ij}^{(\sigma)}(yxy^{-1}) \omega_G(dy) \\ &= \int_G \sum_{k,l=1}^{d_\sigma} u_{ik}^{(\sigma)}(y) u_{kl}^{(\sigma)}(x) u_{lj}^{(\sigma)}(y^{-1}) \omega_G(dy) \\ &= \sum_{k,l=1}^{d_\sigma} u_{kl}^{(\sigma)}(x) \delta_{kl} \delta_{ij} \frac{1}{d_\sigma} \\ &= \frac{\delta_{ij}}{d_\sigma} \sum_{k,l=1}^{d_\sigma} u_{kl}^{(\sigma)}(x) \delta_{kl} \\ &= \frac{\delta_{ij}}{d_\sigma} \chi_\sigma(x). \end{aligned}$$

□

*Proof of Theorem 3.3.* Clearly,  $M^z(G)$  is a linear space, and  $\dot{\mathcal{F}}$  is linear on  $M^z(G)$ . Therefore it suffices to show that  $\ker \dot{\mathcal{F}} = \{0\}$ .

Let  $\mu \in M^z(G)$  be given with  $\hat{\mu} = 0$  for all  $\sigma \in \Sigma(G)$ . Since  $\mu$  is central, we have to justify that  $\mu(f) = 0$  for all  $f \in C^z(G)$ . By assumption  $0 = \hat{\mu}(\sigma) = \mu(\chi_\sigma)$  for  $\sigma \in \Sigma(G)$ . Now we apply the Peter-Weyl theorem in order to obtain

$$\overline{F(G)}^{\|\cdot\|_u} = C(G).$$

Moreover we know from the Lemma that

$$F^z(G) = \langle \{\chi_\sigma : \sigma \in \Sigma(G)\} \rangle.$$

Since the mapping  $P: C(G) \rightarrow C^z(G)$  introduced above by  $P(f) := f^0$  for all  $f \in C(G)$  is continuous and surjective, we obtain

$$\overline{F^z(G)}^{\|\cdot\|_u} = C^z(G).$$

From this follows that  $\mu(f) = 0$  for all  $f \in C^z(G)$ . □

A further algebra to be applied in the sequel is that of absolutely convergent Fourier series. We follow the approach in [3], §34.

Given a measure  $\mu \in M(G)$  one considers the associated family  $A := \{A_\sigma \in \Sigma(G)\}$  of *coefficient operators* defined by

$$\langle A_\sigma h, k \rangle := \int_G \langle U^{(\sigma)}(x^{-1})h, k \rangle \mu(dx)$$

for all  $h, k$  in the representing Hilbert space  $\mathcal{H}_\sigma$  of  $A_\sigma$ . Clearly,  $A_\sigma = D_\sigma \hat{\mu}(\sigma)^* D_\sigma$ , where  $D_\sigma$  denotes the conjugation operator for  $\sigma \in \Sigma(G)$  in the sense of (27.25) of [3]. Moreover,  $\|A\|_p = \|\hat{\mu}\|_p$  whenever  $1 \leq p \leq \infty$ . The formal expression

$$\sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{tr}(A_\sigma U^{(\sigma)})$$

is called the *Fourier (-Stieltjes) series* of  $\mu$ , in symbols

$$\mu \sim \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{tr}(A_\sigma U^{(\sigma)}).$$

Analogously one introduces the notion of Fourier series of a function  $f \in L^1(G, \omega_G)$ . The Fourier series

$$f \sim \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{tr}(A_\sigma U^{(\sigma)})$$

of  $f \in L^1(G, \omega_G)$  is said to be *absolutely convergent* provided

$$\sum_{\sigma \in \Sigma(G)} d_\sigma \|A_\sigma\|_{\phi_1} < \infty.$$

Let  $K(G)$  denote the set of functions in  $L^1(G, \omega_G)$  which admit on absolutely convergent Fourier series. For  $f \in K(G)$  we define

$$\|f\|_{\phi_1} := \|\hat{f}\|_1,$$

where

$$\begin{aligned} \|\hat{f}\|_1 &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|\hat{f}(\sigma)\|_{\phi_1} \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|A_\sigma\|_{\phi_1} < \infty. \end{aligned}$$

$K(G)$  is a linear space,  $\mathcal{F}: K(G) \rightarrow \mathcal{E}_1(\Sigma(G))$  is a norm-preserving linear isomorphism, hence  $K(G)$  is a Banach space.

Since any  $f \in K(G)$  with

$$f \sim \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{tr}(A_\sigma U^{(\sigma)})$$

is  $\omega_G$ -almost everywhere equal to the continuous function

$$\sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{tr}(A_\sigma U^{(\sigma)}),$$

it can be regarded as an element of  $C(G)$ .

**Theorem 3.5.** (i)  $K(G)$  is a commutative Banach algebra under pointwise operations, having 1 as its unit.

(ii)  $\text{spec}(K(G)) = D(K(G))$ , i.e. every multiplicative linear functional on  $K(G)$  is an evaluation functional of the form

$$f \mapsto f(a) = \varepsilon_a(f)$$

for a fixed  $a \in G$ . Furnished with the Gelfand topology the set  $\{\varepsilon_a : a \in G\}$  is homeomorphic to  $G$ .

A proof of this theorem is contained in [3], §34. We quote the essential arguments.

(i) (34.18) Basically it has to be shown that for  $f, g \in K(G)$  with convergent Fourier series

$$f = \sum_{m \geq 1} d_m \text{tr}(A_m U^{(m)})$$

and

$$g = \sum_{n \geq 1} d_n \text{tr}(B_n U^{(n)})$$

respectively the inequalities

$$\begin{aligned} \|fg\|_{\phi_1} &\leq \sum_{m \geq 1} \sum_{n \geq 1} d_m d_n \|\text{tr}(A_m U^{(m)}) \text{tr}(B_n U^{(n)})\|_{\phi_1} \\ &\leq \|f\|_{\phi_1} \|g\|_{\phi_1} \end{aligned}$$

hold, where the crucial estimate is

$$\|\text{tr}(A_m U^{(m)}) \text{tr}(B_n U^{(n)})\|_{\phi_1} \leq \|A_m\|_{\phi_1} \|B_n\|_{\phi_1}$$

valid for all  $m, n \geq 1$ .

(ii) (34.20) Since  $\mathcal{F}$  is a Banach space isomorphism  $K(G) \rightarrow \mathcal{E}_1(\Sigma(G))$ , each bounded linear functional  $L$  on  $K(G)$  is of the form

$$f \mapsto L(f) := \sum_{\sigma \in \Sigma(G)} d_\sigma \text{tr}(\hat{f}(\sigma) F_\sigma),$$

where

$$F := \{F_\sigma : \sigma \in \Sigma(G)\} \in \mathcal{E}_\infty(\Sigma(G))$$

and

$$\sup\{\|F_\sigma\|_{\phi_\infty} : \sigma \in \Sigma(G)\} = 1.$$

From the multiplicativity of  $L$  one deduces that  $L$  is an evaluation functional on  $F(G)$ . But  $F(G)$  is dense in  $K(G)$  with respect to the norm  $\|\cdot\|_{\phi_1}$  and  $L$  is continuous. Consequently  $L$  is an evaluation functional on  $K(G)$ .

So far one knows that  $a \mapsto \phi(a) := \varepsilon_a$  is a one-to-one mapping from  $G$  into  $\text{spec}(K(G))$ . The inclusion  $K(G) \subset C(G)$  and the definition of the Gelfand topology yield the continuity of  $\phi$ . As  $\text{spec}(K(G))$  is compact (as the structure space of  $K(G)$ ),  $\phi$  is in fact a homeomorphism.



#### 4. The Central Fourier Algebra

Next to the algebras  $F(G)$  and  $K(G)$  associated with the given compact group  $G$  we shall place special emphasis on the *central Fourier algebra*  $F^0(G)$  of  $G$  consisting of all functions  $f \in C^z(G)$  that admit convergent Fourier series in the sense that

$$f = \sum_{\sigma \in \Sigma(G)} \hat{f}(\sigma^*) \chi_\sigma$$

with

$$\|f\|^0 := \sum_{\sigma \in \Sigma(G)} d_\sigma |\hat{f}(\sigma)| < \infty.$$

In analogy to the respective result for  $K(G)$  we have the following

**Theorem 4.1.**  *$F^0(G)$  is a commutative Banach \*-algebra with respect to the usual pointwise operations, complex conjugation as involution, and  $\|\cdot\|^0$  as norm.*

*Proof.* In order to show that  $F^0(G)$  is a commutative normed \*-algebra we restrict ourselves to proving the crucial inequality

$$\|fg\|^0 \leq \|f\|^0 \|g\|^0$$

valid for all  $f, g \in F^0(G)$ .

At first we note that for  $f, g \in F^0(G), \sigma \in \Sigma(G)$

$$\widehat{fg}(\sigma) = \sum_{\sigma_1, \sigma_2 \in \Sigma(G)} \hat{f}(\sigma_1^*) \hat{g}(\sigma_2^*) M(\sigma, \sigma_1^* \otimes \sigma_2^*).$$

In fact,

$$\begin{aligned} fg &= \left( \sum_{\sigma_1 \in \Sigma(G)} \hat{f}(\sigma_1^*) \chi_{\sigma_1} \right) \left( \sum_{\sigma_2 \in \Sigma(G)} \hat{g}(\sigma_2^*) \chi_{\sigma_2} \right) \\ &= \sum_{\sigma_1, \sigma_2 \in \Sigma(G)} \hat{f}(\sigma_1^*) \hat{g}(\sigma_2^*) \chi_{\sigma_1 \otimes \sigma_2}, \end{aligned}$$

hence

$$\widehat{fg}(\sigma) = \sum_{\sigma_1 \sigma_2 \in \Sigma(G)} \hat{f}(\sigma_1^*) \hat{g}(\sigma_2^*) \widehat{\chi_{\sigma_1 \otimes \sigma_2}}(\sigma),$$

where

$$\begin{aligned} \widehat{\chi_{\sigma_1 \otimes \sigma_2}}(\sigma) &= \int_G \chi_\sigma(x) \chi_{\sigma_1 \otimes \sigma_2}(x) \omega_G(dx) \\ &= \int_G \chi_{\sigma^*}(x^{-1}) \chi_{\sigma_1 \otimes \sigma_2}(x) \omega_G(dx) \\ &= M(\sigma^*, \sigma_1 \otimes \sigma_2) \\ &= M(\sigma, (\sigma_1 \otimes \sigma_2)^*) \\ &= M(\sigma, \sigma_1^* \otimes \sigma_2^*). \end{aligned}$$

Now we compute the  $\|\cdot\|^0$ -norm of the product  $fg$ :

$$\begin{aligned}
\|fg\|^0 &= \sum_{\sigma \in \Sigma(G)} \left| \widehat{fg} \right| d_\sigma \\
&= \sum_{\sigma \in \Sigma(G)} \left| \sum_{\sigma_1, \sigma_2 \in \Sigma(G)} \dot{f}(\sigma_1^*) \dot{g}(\sigma_2^*) M(\sigma, \sigma_1^* \otimes \sigma_2^*) \right| d_\sigma \\
&\leq \sum_{\sigma \in \Sigma(G)} \sum_{\sigma_1, \sigma_2 \in \Sigma(G)} |\dot{f}(\sigma_1^*)| |\dot{g}(\sigma_2^*)| M(\sigma, \sigma_1^* \otimes \sigma_2^*) d_\sigma \\
&= \sum_{\sigma_1, \sigma_2 \in \Sigma(G)} |\dot{f}(\sigma_1^*)| |\dot{g}(\sigma_2^*)| d_{\sigma_1^* \otimes \sigma_2^*} \\
&= \sum_{\sigma_1, \sigma_2 \in \Sigma(G)} |\dot{f}(\sigma_1^*)| |\dot{g}(\sigma_2^*)| d_{\sigma_1^*} d_{\sigma_2^*} \\
&= \|f\|^0 \|g\|^0.
\end{aligned}$$

Finally we have to show that  $F^0(G)$  is in fact a Banach\*-algebra. For this purpose we need to provide an embedding of  $F^0(G)$  into a Banach\*-algebra  $L^1(\Sigma(G))$  of integrable functions on the discrete dual  $\Sigma(G)$  of  $G$ . More precisely, on the power set  $\mathcal{P}(\Sigma(G))$  of  $\Sigma(G)$  one defines a mapping  $d$  into  $\bar{\mathbb{R}}_+$  by

$$d(\Sigma_1) := \begin{cases} \sum_{\sigma \in \Sigma_1} d_\sigma & \text{if } \Sigma_1 \text{ is finite} \\ \infty & \text{otherwise} \end{cases}$$

Then  $(\Sigma(G), \mathcal{P}(\Sigma(G)), d)$  is a positive measure space. Let  $L(\Sigma(G))$  denote the set of complex-valued functions on  $\Sigma(G)$  with finite support, and let  $L^1(\Sigma(G))$  be the set of  $d$ -integrable functions on  $\Sigma(G)$ . Clearly,  $L(\Sigma(G))$  is a dense subset of  $L^1(\Sigma(G))$  with respect to the norm of  $L^1(\Sigma(G))$ . In  $L^1(\Sigma(G))$  we introduce a multiplication by

$$(\phi \times \psi)(\sigma) := \sum_{\sigma_1, \sigma_2 \in \Sigma(G)} \phi(\sigma_1) \psi(\sigma_2) M(\sigma, \sigma_1 \otimes \sigma_2)$$

and an involution by

$$\phi^*(\sigma) := \overline{\phi(\sigma^*)}$$

for all  $\phi, \psi \in L^1(\Sigma(G))$ ,  $\sigma \in \Sigma(G)$ . Then  $L^1(\Sigma(G))$  becomes a Banach\*-algebra, and the mapping  $f \mapsto \hat{f}$  from  $F^0(G)$  into  $L^1(\Sigma(G))$  is an isometric \*-isomorphism between algebras. As a consequence we see that  $F^0(G)$  is a Banach\*-algebra.  $\square$

**Theorem 4.2.**  $F^0(G)$  coincides with the Banach\*-algebra  $K^z(G)$  of central functions in  $K(G)$ .

*Proof.* 1.  $F^0(G) \subset K^z(G)$ .

In fact, let  $f \in F^0(G)$ . Then  $f \in C(G)$ , hence  $f \in L^1(G, \omega_G)$ . Applying Theorem 3.1 we obtain

$$\begin{aligned} \|f\|_{\phi_1} &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|\hat{f}(\sigma)\|_{\phi_1} \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|a(f, \sigma) I_{d_\sigma}\|_{\phi_1} \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma^2 |a(f, \sigma)| \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma^2 \frac{|\hat{f}(\sigma)|}{d_\sigma} = \|f\|^0 < \infty. \end{aligned}$$

Together with the fact that the functions of  $F^0(G)$  are central this implies the stated inclusion.

2.  $K^z(G) \subset F^0(G)$ .

Let  $f \in K^z(G)$ . Since  $f$  is central, Theorem 3.1 yields

$$\hat{f}(\sigma) = a(f, \sigma) I_{d_\sigma}$$

for some  $a(f, \sigma) \in \mathbb{C}$ , all  $\sigma \in \Sigma(G)$ . Now consider the function

$$g := \sum_{\sigma \in \Sigma(G)} a(f, \sigma) d_\sigma \chi_\sigma$$

on  $G$ . Applying the formula

$$\hat{\chi}_\sigma(\tau) = \frac{\delta_{\sigma\tau}}{d_\tau} I_{d_\sigma}$$

valid for all  $\tau \in \Sigma(G)$ , derived from the Peter-Weyl theorem, one sees that  $\hat{g} = \hat{f}$ , hence by the injectivity of the Fourier transform that  $g = f$ . But now  $\hat{f}(\sigma^*) = a(f, \sigma) d_\sigma$  for all  $\sigma \in \Sigma(G)$ . Consequently

$$\begin{aligned} \infty > \|f\|_{\phi_1} &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|\hat{f}(\sigma)\|_{\phi_1} \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma |d_\sigma a(f, \sigma)| \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma |\hat{f}(\sigma^*)| \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma |\hat{f}(\sigma)| = \|f\|^0 \end{aligned}$$

It remains to show the continuity of  $f$ .

We know from the discussion in Chapter 3 that for  $\{A_\sigma : \sigma \in \Sigma(G)\} \in \mathcal{E}_1(\Sigma(G))$  the function

$$x \mapsto h(x) := \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{tr}(A_\sigma U^{(\sigma)}(x))$$

is continuous on  $G$ . Since  $f \in K^z(G)$ ,

$$f = \sum_{\sigma \in \Sigma(G)} a(f, \sigma) d_\sigma \chi_\sigma$$

and

$$\hat{f}(\sigma) = a(f, \sigma) I_{d_\sigma} = \hat{f}(\sigma)^t$$

or

$$A_\sigma = \hat{f}(\sigma)^t = \hat{f}(\sigma)$$

for all  $\sigma \in \Sigma(G)$ , hence  $\{\hat{f}(\sigma) : \sigma \in \Sigma(G)\} \in \mathcal{E}_1(\Sigma(G))$ . Moreover, for all  $x \in G$

$$\begin{aligned} h(x) &= \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{tr}(A_\sigma U^{(\sigma)}(x)) \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma \operatorname{tr}(a(f, \sigma) I_{d_\sigma} U^{(\sigma)}(x)) \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma a(f, \sigma) \operatorname{tr}(U^{(\sigma)}(x)) \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma a(f, \sigma) \chi_\sigma(x). \end{aligned}$$

This completes the proof.  $\square$

**Theorem 4.3.**  $G$  acts on  $K(G)$  as a compact group of continuous automorphisms of  $G$ .

*Proof.* For each  $x \in G$  consider the mapping  $i_x : G \rightarrow G$  given by

$$i_x(y) := xyx^{-1}$$

for all  $y \in G$ . Moreover, let the mapping  $i \in G \rightarrow K(G)$  be defined by

$$i(x)(f) := f \circ i_x$$

for all  $f \in K(G)$ ,  $x \in G$ . Then  $i(x)(f) \in K(G)$ .

In fact, we have the two equalities

$$i(x)(f) = f \circ i_x \in L^1(G, \omega_G)$$

and

$$\begin{aligned} \|i(x)(f)\|_{\phi_1} &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|\widehat{i(x)(f)}(\sigma)\|_{\phi_1} \\ &= \sum_{\sigma \in \Sigma(G)} d_\sigma \|\hat{f}(\sigma)\|_{\phi_1} \\ &= \|f\|_{\phi_1} < \infty. \end{aligned}$$

In order to see the second equality we compute for all  $\sigma \in \Sigma(G)$ ,  $h, k \in \mathcal{H}_\sigma$

$$\begin{aligned}
\langle \widehat{i(x)(f)}(\sigma)h, k \rangle &= \int_G \langle \bar{U}^{(\sigma)}(y)h, k \rangle f(xy x^{-1}) \omega_G(dy) \\
&= \int_G \langle \bar{U}^{(\sigma)}(x^{-1}yx)h, k \rangle f(y) \omega_G(dy) \\
&= \int_G \langle \bar{U}^{(\sigma)}(x^{-1})\bar{U}^{(\sigma)}(y)\bar{U}^{(\sigma)}(x)h, k \rangle f(y) \omega_G(dy) \\
&= \int_G \langle \bar{V}_x(y)h, k \rangle f(y) \omega_G(dy) \\
&= \langle \widehat{f}^{(U)}(\sigma)h, k \rangle,
\end{aligned}$$

where  $\widehat{f}^{(U)}$  denotes the Fourier transform of  $f$  with respect to  $V_x \in \sigma$ . This implies  $\widehat{i(x)(f)}(\sigma) = \widehat{f}^{(U)}(\sigma)$ , hence

$$\|\widehat{i(x)(f)}(\sigma)\|_{\phi_1} = \|\widehat{f}^{(U)}(\sigma)\|_{\phi_1} = \|\widehat{f}(\sigma)\|_{\phi_1}.$$

Next we observe that for every  $x \in G$  the mapping  $i(x): K(G) \rightarrow K(G)$  is an automorphism of  $K(G)$ . Obviously,  $i(x)$  is linear and multiplicative, and since  $\|i(x)(f)\|_{\phi_1} = \|f\|_{\phi_1}$  for all  $f \in K(G)$ ,  $i(x)$  is continuous. Finally, the mapping  $i: G \rightarrow \text{Aut}(K(G))$  is an anti-automorphism of groups. In fact, for  $x, y, z \in G$  and  $f \in K(G)$  we obtain

$$\begin{aligned}
(i(x) \circ i(y))(f)(z) &= (f \circ i_{xy})(z) \\
&= f(i_{xy}(z)) \\
&= f(xyz y^{-1}x^{-1}) \\
&= (f \circ i_x)(yz y^{-1}) \\
&= (i(x)(f) \circ i_y)(z) \\
&= (i(y) \circ i(x)(f))(z).
\end{aligned}$$

□

**Theorem 4.4.**  $\text{spec}(F^0(G) = D(F^0(G))$ .

*Proof.* In view of Theorem 4.3

$$F^0(G) = \{f \in K(G) : f = i(x)(f) \text{ for all } x \in G\}$$

is a closed subalgebra of  $K(G)$ . Let  $L$  be a non-vanishing multiplicative linear functional of  $F^0(G)$ . From [7], Theorem 1.3.3 we infer that there exists a multiplicative linear extension  $L'$  of  $L$  on  $K(G)$ . But by Theorem 3.3  $\text{spec}(K(G)) = D(K(G))$ , hence there exists an  $a \in G$  such that

$$L'(f) = f(a) = \varepsilon_a(f)$$

for all  $f \in F^0(G)$ . This shows the inclusion  $\text{spec}(F^0(G)) \subset D(F^0(G))$ . The other inclusion is trivial. □

### 5. Bochner's Theorem

For the given compact group  $G$  we abbreviate by  $\Sigma := \Sigma(G)$  and  $\Sigma' := \Sigma'(G)$  the sets of equivalence classes of irreducible and finite dimensional (continuous unitary) representations of  $G$  respectively.

**Definition 5.1.** A mapping  $\phi: \Sigma \rightarrow \mathbb{C}$  is called *positive definite* (on  $\Sigma$ ) if for every  $N \geq 1$ , every sequence  $\{\sigma_1, \dots, \sigma_N\}$  in  $\Sigma$  and every sequence  $\{c_1, \dots, c_N\}$  in  $\mathbb{C}$

$$\sum_{n,m=1}^N c_n \bar{c}_m \sum_{\sigma \in \Sigma} M(\sigma, \sigma_n \otimes \sigma_m^*) \phi(\sigma) \geq 0.$$

We denote by  $P(\Sigma)$  the set of all positive definite mappings on  $\Sigma$ .

*Remark 5.2.* Any mapping  $\phi: \Sigma \rightarrow \mathbb{C}$  can be extended to a mapping  $\phi': \Sigma' \rightarrow \mathbb{C}$  by putting

$$\phi'(\sigma') := \sum_{\sigma \in \Sigma} M(\sigma, \sigma') \phi(\sigma)$$

for all  $\sigma' \in \Sigma'$ . Consequently,  $\phi: \Sigma \rightarrow \mathbb{C}$  is positive definite if and only if for all  $N \geq 1$ , sequences  $\{\sigma_1, \dots, \sigma_N\}$  in  $\Sigma$  and  $\{c_1, \dots, c_N\}$  in  $\mathbb{C}$

$$\sum_{n,m=1}^N c_n \bar{c}_m \phi'(\sigma_n \otimes \sigma_m^*) \geq 0.$$

**Special Case 5.3** If  $G$  is abelian, the above definition of positive definiteness coincides with the usual one. Indeed, in this case  $\Sigma = G^*$ , i.e. representations  $\sigma \in \Sigma$  are characters  $\chi_\sigma \in G^*$ , and  $\sigma^* = \chi_\sigma^{-1}$  as well as  $\sigma \otimes \sigma' = \chi_\sigma \chi_{\sigma'}$  whenever  $\sigma, \sigma' \in \Sigma$ . Given  $\phi \in P(\Sigma)$  we obtain with the choice  $N \geq 1$ ,  $\{\sigma_1, \dots, \sigma_N\}$  in  $\Sigma$  and  $\{c_1, \dots, c_N\}$  in  $\mathbb{C}$

$$\begin{aligned} & \sum_{n,m=1}^N c_n \bar{c}_m \sum_{\sigma \in \Sigma} \left( \int_G \overline{\chi_\sigma(x)} \chi_{\sigma_n \otimes \sigma_m^*}(x) \omega_G(dx) \right) \phi(\sigma) \\ &= \sum_{n,m=1}^N c_n \bar{c}_m \sum_{\sigma \in \Sigma} \delta_{\sigma, \sigma_n \otimes \sigma_m^*} \phi(\sigma) \\ &= \sum_{n,m=1}^N c_n \bar{c}_m \phi(\sigma_n \otimes \sigma_m^*) \\ &= \sum_{n,m=1}^N c_n \bar{c}_m \phi(\chi_n \chi_m^{-1}). \end{aligned}$$

**Properties 5.4** of  $\phi \in P(\Sigma)$ .

5.4.1  $\phi(I) \geq 0$ , where  $I$  denotes the trivial one-dimensional representation in  $\Sigma$ .

5.4.2  $\phi(\sigma^*) = \overline{\phi(\sigma)}$  for all  $\sigma \in \Sigma$ .

*Proofs.* 5.4.1 Choosing  $N = 1$  and  $c_1 = 1$  we obtain

$$\sum_{\sigma \in \Sigma} M(\sigma, \sigma_1 \otimes \sigma_1^*) \phi(\sigma) \geq 0$$

for all  $\sigma_1 \in \Sigma$ , in particular for  $\sigma_1 = I$

$$\phi(I) = \sum_{\sigma \in \Sigma} M(\sigma, I \otimes I^*) \phi(\sigma) \geq 0.$$

5.4.2 With the choices  $N = 2, c_1 = 1, c_2 = c, \sigma_1 = \sigma_0$  and  $\sigma_2 = I$  we compute

$$\begin{aligned} & \sum_{\sigma \in \Sigma} M(\sigma, \sigma_0 \otimes \sigma_0^*) \phi(\sigma) + \bar{c} \sum_{\sigma \in \Sigma} M(\sigma, \sigma_0 \otimes I^*) \phi(\sigma) \\ & + c \sum_{\sigma \in \Sigma} M(\sigma, I \otimes \sigma_0^*) \phi(\sigma) + |c|^2 \sum_{\sigma \in \Sigma} M(\sigma, I \otimes I^*) \phi(\sigma) \\ & = \sum_{\sigma \in \Sigma} M(\sigma, \sigma_0 \otimes \sigma_0^*) \phi(\sigma) + \bar{c} \phi(\sigma_0) + c \phi(\sigma_0^*) + |c|^2 \phi(I) \geq 0. \end{aligned}$$

By 5.4.1 the first and the last term of the sum is real and  $\geq 0$ , and hence the sum  $\bar{c} \phi(\sigma_0) + c \phi(\sigma_0^*)$  is real. Since this property holds for all  $c \in \mathbb{C}$ , we choose  $c = 1$  and  $c = i$  to obtain  $\phi(\sigma_0) + \phi(\sigma_0^*) \in \mathbb{R}$  as well as  $i \phi(\sigma_0^*) - i \phi(\sigma_0) \in \mathbb{R}$ . But then  $\overline{\phi(\sigma_0)} = \phi(\sigma_0^*)$  which was to be shown.  $\square$

In the following discussion we shall make use of two more symbols:  $M_+^z(G)$  for the set of non-negative measure in  $M^z(G)$  and  $P^b(\Sigma)$  for the set of mappings  $\phi \in P(\Sigma)$  that are *bounded* in the sense that there exists a constant  $c_\phi \geq 0$  such that

$$|\phi(\sigma)| \leq c_\phi d_\sigma$$

for all  $\sigma \in \Sigma$ .

**Theorem 5.5.** (Bochner) *The restriction  $\hat{\mathcal{F}}_1$  of the central Fourier mapping  $\hat{\mathcal{F}}$  to  $M_+^z(G)$  provides a bijection from  $M_+^z(G)$  onto  $P^b(\Sigma)$ .*

*Proof.* 1. By Theorem 3.3  $\hat{\mathcal{F}}_1$  is injective.

2. We show that for every  $\mu \in M^z(G)$  the central Fourier transform  $\hat{\mu} = \hat{\mathcal{F}}_1(\mu)$  of  $\mu$  belongs to  $P^b(\Sigma)$ .

2.1 Let  $N \geq 1, \{\sigma_1, \dots, \sigma_N\} \subset \Sigma$  and  $\{c_1, \dots, c_N\} \subset \mathbb{C}$ .

By Remark 5.2 we may consider the extension  $\hat{\mu}'$  of  $\hat{\mu}$  to  $\Sigma'$  which is of the form

$$\hat{\mu}'(\sigma') = \int_G \chi_{\sigma'} d\mu$$

for all  $\sigma' \in \Sigma'$ . In fact, let  $\sigma'$  admit the decomposition

$$\sigma' = \bigoplus_{\sigma \in \Sigma} M(\sigma_1 \sigma) \sigma.$$

Then

$$\chi_{\sigma'} = \sum_{\sigma \in \Sigma} M(\sigma, \sigma') \chi_{\sigma},$$

hence

$$\begin{aligned} \dot{\mu}'(\sigma') &= \sum_{\sigma \in \Sigma} M(\sigma, \sigma') \dot{\mu}(\sigma) \\ &= \sum_{\sigma \in \Sigma} M(\sigma, \sigma') \int_G \chi_{\sigma} d\mu \\ &= \int_G \left( \sum_{\sigma \in \Sigma} M(\sigma, \sigma') \chi_{\sigma} \right) d\mu \\ &= \int_G \chi_{\sigma'} d\mu. \end{aligned}$$

Now,

$$\begin{aligned} &\sum_{n,m=1}^N c_n \bar{c}_m \sum_{\sigma \in \Sigma} M(\sigma, \sigma_n \otimes \sigma_m^*) \dot{\mu}(\sigma) \\ &= \sum_{n,m=1}^N c_n \bar{c}_m \dot{\mu}'(\sigma_n \otimes \sigma_m^*) \\ &= \sum_{m,n=1}^N c_n \bar{c}_m \int_G \chi_{\sigma_n \otimes \bar{\chi}_{\sigma_m}} d\mu \\ &= \sum_{m,n=1}^N c_n \bar{c}_m \int_G \chi_{\sigma_n} \bar{\chi}_{\sigma_m} d\mu \\ &= \int \left| \sum_{n=1}^N c_n \chi_{\sigma_n} \right|^2 d\mu \geq 0. \end{aligned}$$

This shows that  $\dot{\mu} \in P(\Sigma)$ .

2.2  $\dot{\mu}$  is bounded and therefore  $\dot{\mu} \in P^b(\Sigma)$ .

In fact, the bound  $c_{\dot{\mu}} := \|\mu\|$  fulfills the requirements, since  $c_{\dot{\mu}} \geq 0$  and

$$|\dot{\mu}(\sigma)| = \left| \int_G \chi_{\sigma} d\mu \right| \leq \int_G |\chi_{\sigma}| d\mu \leq \int_G d_{\sigma} d\mu = d_{\sigma} \|\mu\|$$

whenever  $\sigma \in \Sigma$ .

3. We show that given  $\phi \in P^b(\Sigma)$  there exists a measure  $\mu \in M_+^z(G)$  such that  $\phi = \dot{\mu} = \dot{\mathcal{J}}_1(\tau)$ .

3.1 Let  $\phi \in P^b(\Sigma)$  be such that  $|\phi(\sigma)| \leq c_{\phi} d_{\sigma}$  for all  $\sigma \in \Sigma$ . For every  $f \in F^0(G)$  we introduce

$$T_{\phi}(f) := \sum_{\sigma \in \Sigma} \phi(\sigma) \hat{f}(\sigma).$$



Then  $T_\phi$  is a well-defined linear functional on  $F^0(G)$  which is  $\|\cdot\|^0$ -continuous and satisfies  $\|T_\phi\| \leq c_\phi$ .

3.1.1 From the inequalities

$$\begin{aligned} |T_\phi(f)| &= \left| \sum_{\sigma \in \Sigma} \phi(\sigma) \dot{f}(\sigma) \right| \\ &\leq \sum_{\sigma \in \Sigma} |\phi(\sigma)| |\dot{f}(\sigma)| \\ &\leq \sum_{\sigma \in \Sigma} c_\phi d_\sigma |\dot{f}(\sigma)| \\ &= c_\phi \sum_{\sigma \in \Sigma} |\dot{f}(\sigma)| d_\sigma \\ &= c_\phi \|f\|^0 < \infty \end{aligned}$$

valid for all  $f \in F^0(G)$ , follows the well-definedness of  $T_\phi$ .

3.1.2 The linearity of  $T_\phi$  is clear.

3.1.3 The  $\|\cdot\|^0$ -continuity of  $T_\phi$  has already been shown in 3.1.1. It implies that  $\|T_\phi\| \leq c_\phi$ .

3.2 Now we introduce a mapping

$$[\cdot, \cdot] := F^0(G) \times F^0(G) \rightarrow \mathbb{C}$$

by

$$[f, g] := T_\phi(f\bar{g})$$

for all  $f, g \in F^0(G)$ .

Obviously

3.2.1  $f \mapsto [f, g]$  is linear for each  $g \in F^0(G)$ .

3.2.2 For all  $f, g \in F^0(G)$  we have

$$[f, g] = \overline{[g, f]},$$

since

$$\begin{aligned} [f, g] &= T_\phi(f\bar{g}) \\ &= T_\phi(\overline{g\bar{f}}) \\ &= \sum_{\sigma \in \Sigma} \overline{\phi(\sigma^*)} \widehat{g\bar{f}}(\sigma^*) \\ &= \sum_{\sigma \in \Sigma} \phi(\sigma) \widehat{g\bar{f}}(\sigma^*) \\ &= \overline{T_\phi(g\bar{f})} = \overline{[g, f]}. \end{aligned}$$

Here Property 5.4.2 of  $P(\Sigma)$  has been applied.

3.2.3 For all  $f \in F^0(G)$  we have  $[f, f] \geq 0$

Indeed,

$$\begin{aligned}
 [f, f] &= T_\phi(ff\bar{f}) \\
 &= \sum_{\sigma \in \Sigma} \phi(\sigma) \hat{f}(\sigma) \\
 &= \sum_{\sigma \in \Sigma} \phi(\sigma) \sum_{\sigma_1, \sigma_2 \in \Sigma} \dot{f}(\sigma_1^*) \dot{f}(\sigma_2^*) M(\sigma, \sigma_1^* \otimes \sigma_2^*) \\
 &= \sum_{\sigma \in \Sigma} \phi(\sigma) \sum_{\sigma_1, \sigma_2 \in \Sigma} \dot{f}(\sigma_1^*) \overline{\dot{f}(\sigma_2^*)} M(\sigma, \sigma_1^* \otimes \sigma_2^*) \\
 &= \sum_{\sigma_1, \sigma_2 \in \Sigma} \dot{f}(\sigma_1) \overline{\dot{f}(\sigma_2)} \sum_{\sigma \in \Sigma} M(\sigma, \sigma_1 \otimes \sigma_2^*) \phi(\sigma) \geq 0,
 \end{aligned}$$

the latter inequality resulting from the positive definiteness of  $\phi$ .

3.2.4 From 3.2.1 and 3.2.4 follows the Schwarz inequality

$$|[f, g]|^2 \leq [f, f][g, g]$$

for all  $f, g \in F^0(G)$ , which implies

$$|T_\phi(f)|^2 \leq \phi(I)T_\phi(ff\bar{f}).$$

This is seen as follows:  $1 \in F^0(G)$ , and

$$\begin{aligned}
 [1, 1] = T_\phi(1\bar{1}) &= T_\phi(1) \\
 &= \sum_{\sigma \in \Sigma} \phi(\sigma) \dot{1}(\sigma) \\
 &= \phi(I) \dot{1}(\sigma) = \phi(I).
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 |T_\phi(f)|^2 = |T_\phi(f\bar{1})|^2 &= |[f, 1]|^2 \\
 &\leq [f, f][1, 1] = \phi(I)T_\phi(ff\bar{f}).
 \end{aligned}$$

In order to obtain

3.3 the continuity of  $T_\phi$  in the sup-norm  $\|\cdot\|_u$  one applies the inequality

$$3.3.1 \quad |T_\phi(f)|^2 \leq (\phi(I))^{1+\frac{1}{2}+\dots+\frac{1}{2^n}} (T_\phi(h^{2^n}))^{\frac{1}{2^n}}$$

valid for all  $f \in F^0(G)$ , where  $h := f\bar{f}$ ,  $h^n := hh^{n-1}$  and  $h^n$  is a real-valued function for all  $n \geq 1$ .

The inequality is easily shown by induction.

3.3.2 Since  $F^0(G)$  is a  $*$ -algebra,  $h^{2^n} \in F^0(G)$  for all  $n \geq 1$ , and from the  $\|\cdot\|^0$ -continuity of  $T_\phi$  follows

$$0 \leq T_\phi(h^{2^n}) \leq c_\phi (\|h^{2^n}\|^0)^{\frac{1}{2^n}}$$

or

$$\|T_\phi(f)\|^2 \leq \phi(I)^{1+\frac{1}{2}+\dots+\frac{1}{2^n}} c_\phi^{\frac{1}{2^n}} (\|h^{2^n}\|^0)^{\frac{1}{2^n}}$$

for all  $f \in F^0(G)$ .

On the other hand, the form of the spectral radius

$$\begin{aligned} \rho(h) &= \lim_{n \rightarrow \infty} (\|h^n\|^0)^{\frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} (\|h^{2^n}\|^0)^{\frac{1}{2^n}} = \|\tilde{h}\|_{\text{sup}} \end{aligned}$$

of  $h$  in terms of the Gelfand transform  $\tilde{h}$  of  $h$ , considered as an element of  $C(\text{spec}(F^0(G)))$  with sup-norm  $\|\cdot\|_{\text{sup}}$  yields

$$\begin{aligned} |T_\phi(f)|^2 &\leq \lim_{n \rightarrow \infty} \left( (\phi(I))^{1+\frac{1}{2}+\dots+\frac{1}{2^n}} c_\phi^{\frac{1}{2^n}} (\|h^{2^n}\|^0)^{\frac{1}{2^n}} \right) \\ &= \phi(I)^2 \|\tilde{h}\|_{\text{sup}} \\ &= \phi(I)^2 \|\tilde{f}\tilde{f}\|_{\text{sup}} \\ &= \phi(I)^2 \|\tilde{f}\tilde{f}\|_{\text{sup}} \\ &\leq \phi(I)^2 \|\tilde{f}\|_{\text{sup}}^2 \end{aligned}$$

for all  $f \in F^0(G)$ .

To finish the proof of 3.3 it remains to verify that

3.3.2  $\|\tilde{f}\|_{\text{sup}} = \|f\|_u$ .

This equality follows from Theorem 4.4, since

$$\begin{aligned} \|\tilde{f}\|_{\text{sup}} &= \sup \{|f(L)| : L \in \text{spec}(F^0(G))\} \\ &= \sup \{|L(f)| : L \in \text{spec}(F^0(G))\} \\ &= \sup \{|\varepsilon_x(f)| : x \in G\} \\ &= \sup \{|f(x)| : x \in G\} = \|f\|_u. \end{aligned}$$

3.4 Since  $F^z(G) \subset F^0(G)$ , the Peter-Weyl theorem yields the  $\|\cdot\|_u$ -density of  $F^0(G)$  in  $C^z(G)$ . Consequently there exists a linear,  $\|\cdot\|_u$ -continuous extension  $T'_\phi$  of  $T_\phi$  to  $C^z(G)$ . Moreover

3.4.1  $T_\phi$  is a positive operator.

Indeed, for all  $f \in F^0(G)$  we have  $T_\phi(f\bar{f}) \geq 0$  by 3.2.3. Since  $F^0(G)$  is  $\|\cdot\|_u$ -dense in  $C^z(G)$ , for an arbitrary  $f \in C^z(G)$ , there exists a sequence  $(f_n)_{n \geq 1}$  in  $F^0(G)$  which  $\|\cdot\|_u$ -converges to  $f$ . But then  $(f_n, \bar{f}_n)_{n \geq 1}$   $\|\cdot\|_u$  converges to  $f\bar{f}$ , and the  $\|\cdot\|_u$ -continuity of  $T'_\phi$  on  $C^z(G)$  implies that

$$T'_\phi(f_n \bar{f}_n) \rightarrow T_\phi(f\bar{f})$$

as  $n \rightarrow \infty$ . From  $T'_\phi(f_n \bar{f}_n) = T_\phi(f_n \bar{f}_n) \geq 0$  follows  $T'_\phi(f \bar{f}) \geq 0$ .

Now let  $f$  belong to the cone  $C_+^z(G)$  of nonnegative functions in  $C^z(G)$  and let it be of the form  $f = gg = g\bar{g}$  with  $g \in C_+^z(G)$ . Then  $T'_\phi(f) \geq 0$ .

3.4.2 Finally, the mapping  $\mu : C(G) \rightarrow \mathbb{C}$  defined by

$$\mu(f) := T'_\phi(f)$$

for all  $f \in C(G)$  is a positive, hence  $\|\cdot\|_u$ -continuous linear functional on  $C(G)$  which by the Riesz theorem is represented by a measure  $\mu \in M_+^z(G)$ . The measure  $\bar{\mu}$  fulfills the requirements of the theorem: For any  $\sigma \in \Sigma$

$$\begin{aligned} \hat{\mu}(\sigma) &= \bar{\mu}(\chi_\sigma) = \overline{T'_\phi(\chi_\sigma)} = \overline{T_\phi(\chi_\sigma)} \\ &= \left( \sum_{\sigma' \in \Sigma} \phi(\sigma') \hat{\chi}_\sigma(\sigma') \right)^- \\ &= \left( \sum_{\sigma' \in \Sigma} \phi(\sigma') \int_G \chi_\sigma \overline{\chi_{\sigma'}} d\omega_G \right)^- \\ &= \left( \sum_{\sigma' \in \Sigma} \phi(\sigma') \delta_{\sigma', \sigma^*} \right)^- \\ &= \overline{\phi(\sigma^*)} = \phi(\sigma), \end{aligned}$$

and the proof is complete.  $\square$

*Remark 5.6.* The boundedness condition added to the positive definiteness in the statement of Theorem 5.1 can be dropped once the underlying group  $G$  is Abelian. In the case  $G := SU(2)$ , however, one shows that the condition is necessary. In fact, let  $L$  be a non-continuous  $*$ -homomorphism on  $F^0(G)$  and define a mapping  $\phi$  on  $\Sigma(G)$  by  $\phi(\sigma) := L(\chi_\sigma)$  for all  $\sigma \in \Sigma(G)$ . Then  $\phi$  is positive definite, but given  $c > 0$ , the non-continuity of  $L$  supplies a  $\sigma \in \Sigma(G)$  satisfying  $|L(\chi_\sigma)| > c d_\sigma$ . Thus  $\phi$  does not fulfill the boundedness condition.

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