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ON OPTIMAL PROPORTIONAL REINSURANCE AND INVESTMENT IN A PARTIAL MARKOVIAN REGIME-SWITCHING ECONOMY

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ABSTRACT. In this paper, we consider the problem of optimal reinsurance and investment in a multiple risky assets market with appreciation rate driven by a hidden Markov chain. The surplus of the insurance company is modeled by a Brownian motion with drift and the objective function is the expected exponential utility. By using the filtering theory, we establish the separation principle and reduce the problem to the complete observed case. Through the dynamic programming approach and the Girsanov change of measure, we characterize the value function as the unique viscosity solution of a linear parabolic partial differential equation and obtain the Feynman-Kac representation of the value function.

1. Introduction

The problem of maximizing the expected utility from terminal wealth has been largely studied in both the insurance and finance literature. Optimal investment problems of maximizing utility of terminal wealth in complete or incomplete finance market are studied by Merton [19, 20], Cox and Huang [6], Karatzas [11], Karatzas et al. [12], He and Pearson [10] and so on. Browne [4] considered the optimal investment problem for an insurance company whose surplus is described by the Brownian motion with drift to maximize the exponential utility from terminal wealth. Besides, the problem of minimizing the ruin probability was also studied in that paper. Since then, many insurance theory researchers turned into this field (maximizing the utility from terminal wealth or minimizing the ruin probability), see Schmidli [25, 26], Liu and Yang [17], Promislow and Young [22], Bayraktar and Young [3], Zhang et al. [32], Bai and Guo [1] and so on.

A salient feature in the above papers is the assumption of complete observations: investors have a complete knowledge of all parameters involved in the stochastic differential equation for the asset price. However, in practice situations one may not observe these quantities directly. For example, one may estimate the diffusion coefficient of the stock price process precisely from the sample path, but neither appreciate rate of stock price process nor the underlying Brownian motion can be

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observed. It seems reasonable to consider the market model that only the stock price can be observed, since the prices are published and available to public. Such a model is also called a Hidden Markov model. For a general treatment of such model see, e.g. Elliott et al. [7].

Since the reasonability in practice situations, maximizing utility from terminal wealth with partial observation attracted more attention in the finance literature. Lakner [14] considered the optimal investment and consumption problem when appreciate rate are unobservable variable through martingale approach and obtained explicit results for log and power utility. The same methodology was applied to the case when the appreciate rate process is a linear diffusion in Lakner [15]. Pham and Quenez [21] addressed the maximization problem of expected utility from terminal wealth in a financial market where price process of risky assets follows a stochastic volatility model and only the vector of stock prices are observed by the investors. By using stochastic filtering techniques and adapting martingale duality methods in this partially observed incomplete model, they characterize the value function and the optimal portfolio policies. Sass and Haussmann [24] studied a generalized utility maximization problem in a financial market with multiple asset whose appreciate rate processes are modulated by unknown Markov chain. Through martingale method and Malliavin calculus, explicit representation of the optimal trading strategy in terms of the unnormalized filter of the drift process were obtained. Rieder and Bäuerle [23] studied portfolio selection problem when appreciate rate process is modulated by unknown Markov chain through dynamic programming approach and explicit results for log and power utility were obtained. Recently, Bäuerle and Rieder [2] and Callegaro et al. [5] began to considering the optimization problem under partial information in a financial market where the asset prices are driven by jump process instead of the usual Brownian motion. Xiong and Zhou [27] studied mean-variance portfolio selection problem in a (possibly incomplete) market with multiple stocks and a bond under partial information.

In the literature, there are a lot of works on the optimal control problems in the Markovian regime-switching market, see for example, Elliott et al. [9], Elliott and Siu [8], Zhang et al. [29, 30], Zhang and Siu [31], Liu et al. [18] and so on. The mean-variance portfolio selection problem under a hidden Markovian regime-switching model was studied by Elliott et al. [9]. Elliott and Siu [8] introduced a model to discuss an optimal investment problem of an insurance company in a Markovian regime-switching market using a game theoretic approach. Zhang et al. [30] first constructed the Markov jump assets and added these asset to the market in order to complete the Markovian regime-switching market, then the authors studied the option pricing problems in the enlarged complete market. Liu et al. [18] investigated an optimal investment problem of an insurance company in the presence of risk constraint and regime-switching using a game theoretic approach. In the paper of Zhang and Siu [31], the optimal proportional reinsurance and investment in a Markovian regime-switching market are studied. In that paper, the appreciate rates of the stock prices and the underlying Brownian motion appeared in the stock prices are observable, while in our paper only the prices of stocks are observable for us. Therefore, we can not use the method of Zhang and Siu [31] to solve our problem. In this paper, we need to establish the separation principle by using the filtering theory in order to reduce the partial observed problem to the complete observable problem. Due to the separate principle, the reduced complete observable optimal control problem is based on two stochastic differential equations (3.4) and (3.5), while in the paper of Zhang and Siu [31], the optimal control problem is only based on the stochastic differential equation of the wealth process. This makes our problem much more complicated than that of Zhang and Siu [31].

Our paper is organized as follows. In section 2, we introduce the insurance and Hidden regime-switching market and formulate the problem we need to solve. In section 3, we establish the separate principle and reduce the problem to the complete observed case. Through the dynamic programming approach, we characterize the value function as the unique viscosity solution of a linear parabolic partial differential equation.

2. Insurance Risk Model and Hidden Regime Switching Market

We introduce the insurance risk model in the frame work of Promislow and Young [22]. The claim process $C := \{C(t)\}$ is governed by the following stochastic differential equation:

$$dC(t) = \alpha_0(t)dt - \sigma_0(t)dW_0(t). \tag{2.1}$$

Here $W_0(t)$ is a standard Brownian motion. Assume that the premium is paid continuously at the rate of

$$c(t) := (1+\theta)\alpha_0(t)$$

with the safety loading $\theta > 0$. Therefore the surplus process $R(t)$ of the insurance company is governed by the following stochastic differential equation

$$dR(t) = c(t)dt - dC(t) = \theta\alpha_0(t)dt + \sigma_0(t)dW_0(t).$$

Assume that the insurance company adopted the proportional reinsurance to reduce the risk and let $q(t) \in [0,1]$ denote the proportional reinsured level at time t. The proportional reinsurance is available for a safety loading $\eta > \theta$. Thus the surplus of the insurance company after adopting the proportional reinsurance is given by:

$$dR(t) = (\theta - \eta q(t))\alpha_0(t)dt + (1-q(t))\sigma_0(t)dW_0(t).$$

In addition to the proportional reinsurance, the insurance company also want to invest its surplus in a financial market to increase the profit. In this paper, the financial market consists of d stocks and one bond whose prices are stochastic processes $S(t) := (S_1(t), \cdots, S_d(t))'$ and $S_0(t)$ respectively, where S' stands for the transpose of the matrix S. The bond price $S_0(t)$ is assume to satisfy the ordinary differential equation

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0(0) = s_0,$$

where $r(t)$ is a nonnegative bounded function. The stock prices processes are governed by the following stochastic differential equations,

$$dS_i(t) = S_i(t)[\alpha_i(t, \mathbf{X}(t))dt + \sum_{j=1}^{m} \tilde{\sigma}_{ij}(t)d\tilde{W}_j(t)], \quad S_i(0) = s_i,$$

where $\tilde{W}(t) := (\tilde{W}_1(t), \cdots, \tilde{W}_m(t))'$ is standard Brownian motion defined on a filtered complete probability space $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t\geq 0})$ and independent of the Brownian motion W_0; $\alpha_i(t, \mathbf{X}(t)), i = 1, 2, \cdots, d$, are the appreciation rate processes of the stocks; and the $d \times m$ matrix valued process $\tilde{\Sigma}(t) := (\tilde{\sigma}_{ij}(t))$ is the volatility process. Here $\{\mathbf{X}(t)\}$ is a continuous-time Markov chain with a finite state space $E := \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N\}$, where $\mathbf{e}_i \in \Re^N$ and the j^{th} component of $\mathbf{e}_i$ is the Kronecker delta, for each $i, j = 1, 2, \ldots, N$. More specifically, the chain is also right-continuous, irreducible and independent of $\tilde{W}(t)$. To specify statistical or probabilistic properties of the chain, we define the generator $\mathbf{\Lambda} := [\lambda_{ij}]_{i,j=1,2,\ldots,N}$ of the chain $\mathbf{X}$ under P. This is also called rate matrix or Q-matrix. Here, for each $i, j = 1, 2, \ldots, N$, λ_{ij} is the constant intensity of the transition of the chain $\mathbf{X}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$ at time t. Note that $\lambda_{ij} \geq 0$, for $i \neq j$ and $\sum_{j=1}^{N} \lambda_{ij} = 0$, so $\lambda_{ii} \leq 0$. In what follows for each $i, j = 1, 2, \ldots, N$ with $i \neq j$, we suppose that $\lambda_{ij} > 0$.

Set

$$\mathcal{G}_t := \sigma(S_i(s) : s \leq t, i = 1, 2, \cdots, d) \vee \sigma(W_0(s) : s \leq t), t \geq 0. \tag{2.2}$$

In our model $\mathcal{G}_t$, rather than $\mathcal{F}_t^{\tilde{W}} \vee \mathcal{F}_t^{W_0}$ (the filtration generated by $\tilde{W}$ and W_0), is the only information available to the investor at time t. That is the insurance company can observed the volatility of its own surplus process but it can not observed the risk of the stocks. The only observed for the investment in the market is the stocks' prices.

In our paper, we allow $d < m$ as long as the following condition is satisfied: for any $t \geq 0$, the $d \times d$ matrix $A(t) := (a_{ij}(t))$ is almost full rank a.s.(almost surely). In other words, the market is allowed to be incomplete.

Assume that for each $i = 0, 1, 2, \cdots, d$, let $\pi_i(t)$ denote the amount invested in the i^{th} asset at time t. Here $\pi_0(t)$ represents the amount invested in the fixed interest security at time t. Suppose $\vartheta(t) := (q(t), \pi(t))$ denotes the reinsurance and investment strategy adopted by the insurance company at time t, where $\pi(t) := (\pi_1(t), \pi_2(t), \cdots, \pi_d(t))'$. Note that once $\boldsymbol{\pi}(t)$ is determined, the amount invested in the fixed interest security is completely specified as follows:

$$\pi_0(t) = R^{\vartheta}(t) - \sum_{i=1}^{d} \pi_i(t) \ . \tag{2.3}$$

Here $R^{\vartheta}(t)$ is the surplus of insurance company associated with the reinsurance and investment strategy ϑ. If $\pi_i(t) < 0$, for some $i = 1, 2, \cdots, d$, this means that the i^{th} stock is sold short. Whereas, $\sum_{i=1}^{d} \pi_i(t) > R^{\vartheta}(t)$ corresponds to a credit.

Denote by $L^2_{\mathcal{G}}(0, T, \Re^d)$ the set of $\Re^d$-valued, $\mathcal{G}_t$-adapted process $\xi(t)$ with $E \int_0^T |\xi(t)|^2 dt < \infty$ and we now define the class of admissible strategies.

Definition 2.1. The set of all admissible strategies over the planning period $[0, T]$ is denoted by $\mathcal{A}[0, T]$ and is given by

$$\mathcal{A}[0,T] := \Big\{ (\vartheta(s))_{0 \leq s \leq T} : \ \vartheta(s) = (q(s), \pi(s)) \in [0,1] \times \Re^d \text{ is } \mathcal{G}(s)\text{-adapted and } \pi(s) \in L^2_{\mathcal{G}}(0, T, \Re^d) \text{ a.s.} \Big\}.$$

It is easy to get the surplus process of the insurance company after adopting the admissible strategy ϑ with initial surplus z satisfies the following stochastic differential equation:

$$\begin{cases} dR^{\vartheta}(t) = & [r(t)R^{\vartheta}(t) + (\theta - \eta q(t))\alpha_0(t) + \pi'(t)(\alpha(t,\mathbf{X}(t)) - r(t)\mathbf{1})]dt \\ & (1-q(t))\sigma_0(t)dW_0(t) + \pi'(t)\tilde{\sigma}(t)d\tilde{W}(t) \\ R^{\vartheta}(0) = & z, \end{cases} \tag{2.4}$$

where we make use of the following notations

$$\begin{aligned} &\mathbf{1} := (1,1,\cdots,1)', \\ &\tilde{W}(t) := (\tilde{W}_1(t), \tilde{W}_2(t), \cdots, \tilde{W}_m(t))', \\ &\alpha(t,\mathbf{X}(t)) := (\alpha_1(t,\mathbf{X}(t)), \alpha_2(t,\mathbf{X}(t), \cdots, \alpha_d(t,\mathbf{X}(t))'. \end{aligned}$$

Let

$$g_{s,t} := \exp\{\int_s^t r(u)du\},$$

then from the standard stochastic differential equation theory we have the following closed form the expression of $R^{\vartheta}(t)$:

$$\begin{aligned} R^{\vartheta}(t) = g_{0,t}\Big\{z \quad &+ \quad \int_o^t g_{0,s}^{-1}[(\theta - \eta q(s))\alpha_0(s) + \pi'(s)(\alpha(s,\mathbf{X}(s)) - r(s)\mathbf{1})]ds \\ &+ \quad \int_0^t g_{0,s}^{-1}[(1-q(s))\sigma_0(s)dW_0(s) + \pi'(s)\tilde{\sigma}(s)d\tilde{W}(s)]\Big\}. \end{aligned}$$

Our problem is to find the optimal strategy to maximize the expected exponential utility of terminal wealth up to time T, namely

$$J(z,\vartheta,T) = \lambda - \frac{\gamma}{\kappa}E\exp\{-\kappa R^{\vartheta}(T)\} \tag{2.5}$$

where λ, κ, γ are positive constants and ϑ ranges over the set $A[0,T]$ of all admissible strategies.

3. Separation Principle

In this section, we shall establish the separation principle for the partial information control problem by the classic filtering theory. Through the innovation process of the filtering theory, we reduce the partial information control problem to the corresponding full information control problem. Base on this, we shall derive a $\mathcal{G}_t$-adapted representation for the wealth process corresponding to the strategy ϑ.

Let $Y_i(t) := \log(S_i(t))$, then by Itô's formula, we have

$$dY_i(t) = \left(\alpha_i(t,\mathbf{X}(t)) - \frac{1}{2}a_{ii}(t)\right)dt + \sum_{j=1}^m \tilde{\sigma}_{ij}(t)d\tilde{W}_j(t), \tag{3.1}$$

where

$$a_{ij}(t) := \sum_{k=1}^m \tilde{\sigma}_{ik}(t)\tilde{\sigma}_{jk}(t), i,j = 1,2,\cdots,d. \tag{3.2}$$

Note that the quadratic covariation process between $Y_i(t)$ and $Y_j(t)$ is given by $\int_0^t a_{ij}(s)ds$. Therefore, the matrix value process $(a_{ij}(t))$ is $\mathcal{G}_t-$adapted. Let $\Sigma(t) \equiv (\sigma_{ij}(t))$ be the square root of $A(t) := (a_{ij}(t))$. Then, $\sigma_{ij}(t)$ is $\mathcal{G}_t$-adapted, i.e., it is completely observable.

In order to derive the corresponding full information control problem, we put

$$p_t^i := P(\mathbf{X}(t) = \mathbf{e}_i|\mathcal{G}_t), i = 1, 2, \cdots, N \tag{3.3}$$

and use the notation

$$g(s, p_s) := \sum_{i=1}^{N} g(s, \mathbf{e}_i)p_s^i.$$

Theorem 3.1. *Under any admissible strategy ϑ, the corresponding wealth process $R^\vartheta(t)$ satisfies the following stochastic differential equation:*

$$\begin{cases} dR^\vartheta(t) = & [r(t)R^\vartheta(t) + (\theta - \eta q(t))\alpha_0(t) + \pi'(t)(\alpha(t, p_t) - r(t)\mathbf{1})]dt \\ & +(1 - q(t))\sigma_0(t)dW_0(t) + \pi'(t)\Sigma(t)dW(t) \\ R^\vartheta(0) = & z, \end{cases} \tag{3.4}$$

where the stochastic process $p_t := (p_t^1, \cdots, p_t^N)'$ is governed by

$$dp_t = \mathbf{\Lambda} p_t dt + D(p_t)[\Sigma(t)^{-1}(\alpha(t) - \alpha(t, p_t)\mathbf{1}')]'dW(t) \tag{3.5}$$

and the innovation process $W(t) := (W_1(t), \cdots, W_d(t))'$ is a d-dimensional $\mathcal{G}_t$ Brownian motion, which is given by

$$dW(t) := \Sigma(t)^{-1}dY(t) - \Sigma(t)^{-1}[\alpha(t, p_t) - \frac{1}{2}\tilde{A}(t)]dt, \tag{3.6}$$

where

$$\alpha(t) := (\alpha(t, \mathbf{e}_1), \cdots, \alpha(t, \mathbf{e}_N)), \quad \mathbf{1} := (1, \cdots, 1)' \in \Re^N$$
$$Y(t) := (Y_1(t), \cdots, Y_d(t))', \quad \tilde{A}(t) := (a_{11}(t), \cdots, a_{dd}(t))',$$

and $D(p_t)$ is the diagonal matrix with diagonal elements given by the row vector $(p_t^1, \cdots, p_t^N)$.

Proof. From (3.1), we see that

$$Y(t) - Y(0) - \int_0^t \left(\alpha(s, \mathbf{X}(s)) - \frac{1}{2}\tilde{A}(s)\right) ds = \int_0^t \tilde{\Sigma}(s)d\tilde{W}(s),$$

is martingales with a quadratic covariation process $\int_0^t A(s)ds = \int_0^t \Sigma(s)^2 ds$. Therefore, by the martingale representation theorem, there exists a standard Brownian motion $B := (B_1, \cdots, B_d)'$ such that

$$\tilde{\Sigma}(s)d\tilde{W}(s) = \Sigma(s)dB(s), \tag{3.7}$$

Thus,

$$dY(t) = \left(\alpha(t, \mathbf{X}(t)) - \frac{1}{2}\tilde{A}(t)\right) dt + \Sigma(t)dB(t). \tag{3.8}$$

Note that $Y_i(t) = \log(S_i(t))$ and therefore $Y(t)$ is completely observable. Set

$$d\tilde{Y}(t) := \Sigma(t)^{-1}dY(t),$$

Then we can write the observation equation (3.8) in the classic form:

$$\tilde{Y}(t) = \tilde{Y}(0) + \int_0^t \Sigma(s)^{-1}\left(\alpha(s, \mathbf{X}(s)) - \frac{1}{2}\tilde{A}(s)\right) ds + B(t). \tag{3.9}$$

Now we go on define the innovation process W by

$$\begin{aligned} dW(t) : &= d\tilde{Y}(t) - E\left[\Sigma(t)^{-1}\left(\alpha(t, \mathbf{X}(t)) - \frac{1}{2}\tilde{A}(t)\right)|\mathcal{G}_t\right] dt \\ &= \Sigma(t)^{-1} dY(t) - \Sigma(t)^{-1}\left(\alpha(t, p_t)) - \frac{1}{2}\tilde{A}(t)\right) dt, \end{aligned}$$

where we use the fact that $\Sigma(t)$ and $\tilde{A}(t)$ are $\mathcal{G}_t$-adapted and the definition of the process p_t. Thus, from the relationship between $Y(t)$ and $B(t)$, we have

$$\Sigma(t) dW(t) = \Sigma(t) dB(t) + [\alpha(t, \mathbf{X}(t) - \alpha(t, p_t))] dt. \tag{3.10}$$

From nonlinear filtering theory (see e.g. Liptser and Shiryayev [16]), we have the following results:

(1) $W(t)$ is a standard $\mathcal{G}_t$-Brownian motion;
(2) $dp_t = \mathbf{\Lambda} p_t dt + D(p_t)[\Sigma(t)^{-1}(\alpha(t) - \alpha(t, p_t)\mathbf{1'})]' dW(t)$.

Thus combining the equation (2.4) (3.7) and (3.10), we can easily obtain the wealth equation in (3.4). □

Now from the wealth process equation (3.4), we have the following explicit expression of the wealth process

$$\begin{aligned} R^{\vartheta}(t) = g_{0,t}\Big\{ z &+ \int_0^t g_{0,s}^{-1}[(\theta - \eta q(s))\alpha_0(s) + \pi'(s)(\alpha(s, p_s) - r(s)\mathbf{1})] ds \\ &+ \int_0^t g_{0,s}^{-1}[(1 - q(s))\sigma_0(s) dW_0(s) + \pi'(s)\sigma(s) dW(s)]\Big\}. \end{aligned} \tag{3.11}$$

Thus the original criterion (2.5) of incomplete information control problem reduces to the following criterion with full information:

$$J(z, \vartheta, T) = \lambda - \frac{\gamma}{\kappa} e^{-\kappa z g_{0,T}} EH(T). \tag{3.12}$$

where

$$\begin{aligned} H(T) \quad := \quad \exp\Big\{ &- \int_0^T \kappa g_{s,T}[(\theta - \eta q(s))\alpha_0(s) + \pi'(s)(\alpha(s, p_s) - r(s)\mathbf{1})] ds \\ &- \int_0^T \kappa g_{s,T}[(1 - q(s))\sigma_0(s) dW_0(s) + \pi'(s)\sigma(s) dW(s)]\Big\}; \end{aligned} \tag{3.13}$$

and $W(t)$ is $\mathcal{G}_t$-Brownian motion; p_t satisfies the stochastic differential equation (3.5).

In order to maximize $J(z,\vartheta,T)$, we introduce the change of measure with the Girsanov density defined by

$$\begin{aligned}\frac{dQ}{dp}|\mathcal{G}_T &= L_T\\ &= \exp\Big\{-\int_0^T \kappa g_{s,T}[(1-q(s))\sigma_0(s)dW_0(s)+\pi'(s)\sigma(s)dW(s)]\\ &\qquad -\frac{1}{2}\int_0^T \kappa^2 g_{s,T}^2\left((1-q(s))^2\sigma_0^2+\pi(s)'\Sigma(s)^2\pi(s)\right)ds\Big\}. \end{aligned}\tag{3.14}$$

Thus under the probability measure Q, we have that

$$\begin{aligned}W_0^Q(t) &:= W_0(t)+\int_0^t \kappa g_{s,T}(1-q(s))\sigma_0(s)ds,\\ W^Q(t) &:= W(t)+\int_0^t \kappa g_{s,T}\pi(s)'\Sigma(s)ds\end{aligned}$$

are $\mathcal{G}_t$-Brownian motion and therefore the stochastic process p_t under Q can be expressed as

$$\begin{aligned}dp_t &= \mathbf{\Lambda}p_t dt-\kappa g_{t,T}D(p_t)[\pi(t)'(\alpha(t)-\alpha(t,p_t)\mathbf{1}')]'dt\\ &\quad +D(p_t)[\Sigma(t)^{-1}(\alpha(t)-\alpha(t,p_t)\mathbf{1}')]'dW^Q(t).\end{aligned}\tag{3.15}$$

Since

$$EH(T)=E^Q\exp\{-\kappa\int_0^T\chi(s,p_s,\vartheta(s))ds\},\tag{3.16}$$

where

$$\begin{aligned}\chi(s,p_s,\vartheta(s)): &= g_{s,T}(\theta-\eta q(s))\alpha_0(s)-\frac{1}{2}\kappa g_{s,T}^2(1-q(s))^2\sigma_0(s)^2\\ &\quad +g_{s,T}\pi(s)'(\alpha(s,p_s)-r(s)\mathbf{1})-\frac{1}{2}\kappa g_{s,T}^2\pi(s)'\Sigma(s)^2\pi(s).\end{aligned}\tag{3.17}$$

and p_t satisfies the stochastic differential equation (3.15). Thus to maximize the criterion $J(z,\vartheta,T)$, we only need to minimize the $E^Q\exp\{-\kappa\int_0^T\chi(s,p_s,\vartheta(s))ds\}$ subject to the controlled stochastic differential equations p_t being governed by the (3.15) defined on the filtered probability space $(\Omega,\mathcal{F},\mathcal{G}_t,Q)$. The solution of this complete observation problem forms the subject of the next section.

4. HJB-equation Approach

Note that our objective function in (2.5) and in (3.12) that are all defined with the initial time $t=0$. In this section, we shall extend to the general case for any time $0\le t\le T$ and for $R^\vartheta(t)=z, p_t=p$. Set

$$J(t,z,p,\vartheta,T)=\lambda-\frac{\gamma}{\kappa}e^{-\kappa z g_{t,T}}G(t,p,\vartheta),\tag{4.1}$$

where we define

$$G(t,p,\vartheta)=E^Q\left\{\exp\left[-\kappa\int_t^T\chi(s,p_s,\vartheta(s))ds\right]\Big|p_t=p\right\}.\tag{4.2}$$

In view of the HJB equation, we let

$$w(t,p) := \inf_{\vartheta\in\mathcal{A}[t,T]} \log G(t,p,\vartheta), \tag{4.3}$$

where $\mathcal{A}[t,T]$ denotes the admissible strategies over the interval $[t,T]$. Therefore,

$$\sup_{\vartheta\in\mathcal{A}[0,T]} J(z,\vartheta,T) = \lambda - \frac{\gamma}{\kappa} e^{-\kappa z g_{t,T}} e^{w(0,p)}. \tag{4.4}$$

Based on the definition of $\chi(s,p_s,\vartheta)$ in (3.17) and the dynamics of p_t in (3.15), we can write the following HJB equation for $w(t,p)$

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{1}{2} tr[\Gamma\Gamma' D^2 w] + \frac{1}{2}(\nabla w)'\Gamma\Gamma'\nabla w + (\mathbf{\Lambda} p)'\nabla w \\ \quad + \inf_{q\in[0,1]} [\frac{1}{2}\kappa^2 g_{t,T}^2 (1-q)^2\sigma_0(t)^2 - \kappa g_{t,T}(\theta-\eta q)\alpha_0(t)] \\ \quad + \inf_\pi \Big[-\kappa g_{t,T}\pi'[(\alpha(t,p)-r(t)\mathbf{1}) + (\alpha(t)-\alpha(t,p)\mathbf{1}')D(p)\nabla w] \\ \qquad + \frac{1}{2}\kappa^2 g_{t,T}^2 \pi'\Sigma(t)^2\pi \Big] = 0 \\ w(T,p) = 0, \end{cases} \tag{4.5}$$

where $\Gamma := D(p)[\Sigma(t)^{-1}(\alpha(t)-\alpha(t,p)\mathbf{1}')]'$.

We first compute the infimum over the $q\in[0,1]$. Set

$$I(q) := \frac{1}{2}\kappa^2 g_{t,T}^2 (1-q)^2 \sigma_0(t)^2 - \kappa g_{t,T}(\theta - \eta q)\alpha_0(t),$$

then differentiating $I(q)$ with respect to q yields the infimum point $\bar{q}$ as follows:

$$\bar{q} = 1 - \frac{\eta\alpha_0(t)}{\kappa g_{t,T}\sigma_0(t)^2} < 1.$$

Note that we need to calculated the infimum over $q\in[0,1]$ and $I(q)$ is a second-order polynomial in q, therefore if $\bar{q}\geq 0$, i.e. $\frac{\alpha_0(t)}{\sigma_0(t)^2} \leq \frac{\kappa g_{t,T}}{\eta}$

$$\inf_{q\in[0,1]} I(q) = I(\bar{q}) = -\kappa g_{t,T}(\theta-\eta)\alpha_0(t) - \frac{1}{2}\frac{\eta^2\alpha_0(t)^2}{\sigma_0(t)^2},$$

otherwise, i.e. $\frac{\alpha_0(t)}{\sigma_0(t)^2} > \frac{\kappa g_{t,T}}{\eta}$

$$\inf_{q\in[0,1]} I(q) = I(0) = \frac{1}{2}\kappa^2 g_{t,T}^2 \sigma_0(t)^2 - \kappa g_{t,T}\theta\alpha_0(t).$$

Therefore

$$\begin{aligned} \inf_{q\in[0,1]} I(q) \quad &= \quad -\left[\kappa g_{t,T}(\theta-\eta)\alpha_0(t) + \frac{1}{2}\frac{\eta^2\alpha_0(t)^2}{\sigma_0(t)^2}\right] 1_{\{\frac{\alpha_0(t)}{\sigma_0(t)^2}\leq\frac{\kappa g_{t,T}}{\eta}\}} \\ &\quad + \left[\frac{1}{2}\kappa^2 g_{t,T}^2\sigma_0(t)^2 - \kappa g_{t,T}\theta\alpha_0(t)\right] 1_{\{\frac{\alpha_0(t)}{\sigma_0(t)^2}>\frac{\kappa g_{t,T}}{\eta}\}}. \end{aligned} \tag{4.6}$$

Setting

$$\mathrm{II}(\pi) := \frac{1}{2}\kappa^2 g_{t,T}^2 \pi'\Sigma(t)^2\pi - \kappa g_{t,T}\pi'[(\alpha(t,p)-r(t)\mathbf{1}) + (\alpha(t)-\alpha(t,p)\mathbf{1}')D(p)\nabla w]$$

and differentiating with respect to π, we obtain that the infimum point $\bar{\pi}(t,p)$ is given by

$$\bar{\pi}(t,p) = (\kappa g_{t,T}\Sigma(t)^2)^{-1}\left[(\alpha(t,p)-r(t)\mathbf{1}) + (\alpha(t)-\alpha(t,p)\mathbf{1}')D(p)\nabla w\right].$$

Note that we have no constraint on π and therefore substituting $\bar{\pi}$ into II yields that

$$\begin{aligned}\inf_{\pi} \mathrm{II}(\pi) = \mathrm{II}(\bar{\pi}(t,p)) &= -\frac{1}{2}(\alpha(t,p) - r(t)\mathbf{1})'(\Sigma(t)^2)^{-1}(\alpha(t,p) - r(t)\mathbf{1}) \\ &\quad -(\alpha(t,p) - r(t)\mathbf{1})'(\Sigma(t)^2)^{-1}(\alpha(t) - \alpha(t,p)\mathbf{1}')D(p)\nabla w \\ &\quad -\frac{1}{2}(\nabla w)'\Gamma\Gamma'\nabla w.\end{aligned} \tag{4.7}$$

Thus substituting equation (4.6) and (4.7) into equation (4.5) and then the HJB equation becomes

$$\begin{cases} \frac{\partial w}{\partial t} + \frac{1}{2}tr[\Gamma\Gamma' D^2 w] + \Psi(t,p)'\nabla w + \Phi(t,p) = 0 \\ w(T,p) = 0, \end{cases} \tag{4.8}$$

where, to simplicity of notation, we have set

$$\Psi(t,p) : = \mathbf{\Lambda} p - D(p)(\alpha(t) - \alpha(t,p)\mathbf{1}')'(\Sigma(t)^2)^{-1}(\alpha(t,p) - r(t)\mathbf{1}), \tag{4.9}$$

$$\begin{aligned}\Phi(t,p) : &= -\frac{1}{2}(\alpha(t,p) - r(t)\mathbf{1})'(\Sigma(t)^2)^{-1}(\alpha(t,p) - r(t)\mathbf{1}) \\ &\quad - \left[\kappa g_{t,T}(\theta - \eta)\alpha_0(t) + \frac{1}{2}\frac{\eta^2\alpha_0(t)^2}{\sigma_0(t)^2}\right] 1_{\{\frac{\alpha_0(t)}{\sigma_0(t)^2} \le \frac{\kappa g_{t,T}}{\eta}\}} \\ &\quad + \left[\frac{1}{2}\kappa^2 g_{t,T}^2 \sigma_0(t)^2 - \kappa g_{t,T}\theta\alpha_0(t)\right] 1_{\{\frac{\alpha_0(t)}{\sigma_0(t)^2} > \frac{\kappa g_{t,T}}{\eta}\}}.\end{aligned} \tag{4.10}$$

Note that although the equation (4.8) is the linear partial equation, it is very difficult to obtain closed-form solution in our setting. However, the linearity partial equation leads to Feynman-Kac representation of the solution, which makes it possible to compute numerically by simulation.

Theorem 4.1. *The solution of equation (4.8) has the following Feynman-Kac representation:*

$$w(t,p) = E\left[\int_t^T \Phi(s,p_s)ds | p_t = p\right], \tag{4.11}$$

where p_s satisfies the following stochastic differential equation

$$\begin{cases} dp_s = \Psi(s,p_s)ds + \Gamma(s,p_s)dW(s) \\ p_t = p \end{cases} \tag{4.12}$$

and $W(s)$ is a Brownian motion. Moreover $w(t,p)$ defined by (4.11) is the unique viscosity solution of the equation (4.8).

Proof. The proof of the Feynman-Kac representation is standard, see e.g. Kloeden and Platen [13, p.153] and so we omit it. From Theorem 4.4 of Yong and Zhou [28], we can obtain that $w(t,p)$ is the viscosity of the equation 4.8. □

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