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APPLICATIONS OF WHITE NOISE CALCULUS TO THE COMPUTATION OF GREEKS

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ABSTRACT. We apply white noise calculus to the computation of Greeks for contingent claims priced both in a diffusion model and in a pure jump model. More precisely, we show how the option price can be represented in terms of the Donsker delta function. Using this representation we prove how one can explicitly compute Greeks. In particular we give proofs for Δ.

1. Introduction

The theory of white noise analysis was first introduced in [7] and was originally applied in quantum physics. Subsequently new applications have been found in stochastic differential equations (see [9]). More recently, the white noise analysis has been applied to finance (see [1], [2], [4], [5], [7], [8], [11], [12], [13] and the references therein). The major application in the above mentioned papers is to calculate the replicating portfolio of a given contingent claim. Its application to the computation of Greeks is still rare.

This paper represents important applications of white noise calculus to the computation of Greeks. The major tool used is the Donsker delta function. Our result does not need Malliavin calculus which has been used recently to compute Greeks (see [6]). Similar results were obtained in [3] in the jump diffusion case. The authors in [3] obtain a functional representation formula for functionals of jump diffusions in terms of the Fourier transform from which they compute Greeks using the way related to the likelihood method.

The key idea in our case is to express the payoff function in terms of the Donsker delta function (to be defined later on) and then use the Wick chain rule to compute Greeks. We shall concentrate on the delta, denoted by Δ, which is defined as the derivative of the option price with respect to the initial price. Our result gives a generalization of the computation of Greeks.

This paper is organized as follows: In Section 2 we recall the construction of Brownian motion W_t from a measure μ by means of the Bochner-Minlos theorem. A chaos expansion in terms of the Hermite polynomials is given. We review the definitions of stochastic test functions and stochastic distribution functions. We then define the white noise $\dot{W}_t$ of the Brownian motion associated with the Brownian motion W_t and Wick product, and give some useful results. Section 3 deals with the definition of the Hermite transform and its characterization theorem.

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Some important properties of Hermite transform are also given. In Section 4 we recall the Donsker delta function of a Gaussian process and some useful results. In Section 5 we use the white noise theory and the Donsker delta function to derive Δ. The result is new. In Section 6 we recall the construction of a Lévy process η_t from a given Lévy measure μ by means of the Bochner-Minlos theorem. A known chaos expansion for Lévy processes is used to define white noise $\dot{\widetilde{N}}(t,z)$ of the compensated Poisson random measure $\widetilde{N}(dt,dz)$ associated with η_t. Sections 7 and 8 review the Lévy Hermite transform and the Donsker delta function of a Lévy process, respectively. In Section 9 we give a formula to calculate Δ. The result is also new.

2. White Noise Framework

We set our framework and we briefly state the relevant results that will be useful to this paper. For a detailed review of white noise calculus we refer to [12]. Let $L^2(\mathbb{R})$ denote the set of measurable functions satisfying

$$\| f \|^2_{L^2(\mathbb{R})} := \int_{\mathbb{R}} f^2(t)dt < \infty. \tag{2.1}$$

We will work with the probability space $\Omega = S'(\mathbb{R})$, which is the space of tempered distributions, equipped with its Borel σ-algebra $\mathbb{F} = \mathbb{B}(\Omega)$. The space $S'(\mathbb{R})$ is the dual of the Schwartz space $S(\mathbb{R})$ of test functions, that is, the rapidly decreasing smooth functions on $\mathbb{R}$. By the Bochner-Minlos theorem (see [9] page 14) there exists a probability measure μ on Ω such that

$$\int_{\Omega} e^{i\langle \omega, f\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|f\|^2_{L^2(\mathbb{R})}}, \; f \in S(\mathbb{R}), \tag{2.2}$$

where $i = \sqrt{-1}$ and $\langle \omega, f\rangle = \omega(f)$ denotes the action of $\omega \in \Omega = S'(\mathbb{R})$ applied to $f \in S(\mathbb{R})$. The measure μ is called the *White noise probability measure*. The triple $(\Omega, \mathbb{B}(\Omega), \mu)$ is called the white noise probability space.

We have the following lemma from [9].

Lemma 2.1. *Let $f \in S(\mathbb{R})$. Then*

$$\mathbb{E}[\langle \cdot, f\rangle] = 0, \; f \in S(\mathbb{R}). \tag{2.3}$$

Moreover, we have the Itô isometry

$$\mathbb{E}[\langle \cdot, f\rangle^2] = \| f \|^2_{L^2(\mathbb{R})} \; \textit{ for all } f \in S(\mathbb{R}). \tag{2.4}$$

In [4] it was shown that the definition of $\langle \omega, f\rangle$ can be extended, by using Lemma 2.1, from $f \in S(\mathbb{R})$ to any $f \in L^2(\mathbb{R})$. In particular, this makes

$$\widetilde{W}_t := \widetilde{W}(t,\omega) = \langle \omega, \chi_{[0,t]}(\cdot)\rangle \tag{2.5}$$

well-defined since $\chi_{[0,t]}$ is in $L^2(\mathbb{R})$ for all $t \in \mathbb{R}$. By means of Kolmogorov's continuity theorem, the process $\widetilde{W}_t$ can be shown to have a continuous version which we will denote by W_t, $t \in \mathbb{R}$, that is, $W(t,\omega)$ is continuous in t for all ω, $P(\widetilde{W}_t = W_t) = 1$.

From now on we work with the Brownian motion W_t, $t \in \mathbb{R}$ on the white noise probability space $(S'(\mathbb{R}), \mathbb{B}(S'(\mathbb{R})), \mu)$. W_t is a Brownian motion with respect to the probability law μ.

Let $\mathbb{J}$ denote the set of all multi-indices $\alpha = (\alpha_1, \alpha_2, \cdots)$ of finite length $l(\alpha) = \max\{i, \alpha_i \neq 0\}$ with non-negative integers $\alpha_i \in \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$ for all i. Then for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{J}$ we put $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$ and
$\mid \alpha \mid = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. We can construct an orthogonal $L^2(\mu)$ basis $\{H_\alpha(\omega)\}_{\alpha \in \mathbb{J}}$ given by

$$H_\alpha(\omega) := h_{\alpha_1}(\langle \omega, \xi_1 \rangle) h_{\alpha_2}(\langle \omega, \xi_2 \rangle) \cdots h_{\alpha_n}(\langle \omega, \xi_n \rangle), \ \omega \in \Omega \tag{2.6}$$

where $\langle \omega, \cdot \rangle$ and ξ_j (resp. $h_j, \ j = 1, 2, \ldots, n$) are Hermite functions (resp. Hermite polynomials).

The family $\{H_\alpha\}_{\alpha \in \mathbb{J}}$ is an orthogonal sequence that constitutes a basis for the Hilbert space $L^2(\mu)$. The unit vectors

$$\epsilon^{(k)} = (0, 0, \ldots 0, 1, 0, \ldots, 0) \tag{2.7}$$

with 1 on the k^{th} entry, 0 otherwise, $k = 1, 2, \ldots$ are important special cases of multi-indices (see [4]). We note that

$$H_{\epsilon^{(k)}}(\omega) = h_1(\langle \omega, \xi_k \rangle) = \langle \omega, \xi_k \rangle = \int_{\mathbb{R}} \xi_k(t) dW_t. \tag{2.8}$$

More generally, by a fundamental result of Itô [10], we have

$$I_n(\xi^{\hat{\otimes}\alpha}) = H_\alpha(\omega) \tag{2.9}$$

with $H_0 := 1$. $\otimes$ and $\hat{\otimes}$ denote the tensor product and the symmetrized tensor product, respectively. We now state the chaos decomposition for the elements of $L^2(\mu)$ (see [4], Theorem 5.2).

Theorem 2.2. *Let $F \in L^2(\mu)$, be an $\mathbb{F}_T$-measurable random variable. Then there exists a unique family $\{a_\alpha\}_{\alpha \in \mathbb{J}}$ of constants $a_\alpha \in \mathbb{R}$ such that*

$$F(\omega) = \sum_{\alpha \in \mathbb{J}} a_\alpha H_\alpha. \tag{2.10}$$

Moreover, the Itô isometry is valid:

$$\| F \|^2_{L^2(\mu)} = \sum_{\alpha \in \mathbb{J}} a_\alpha^2 \| H_\alpha \|^2_{L^2(\mu)} = \sum_{\alpha \in \mathbb{J}} a_\alpha^2 \alpha! \tag{2.11}$$

Example 2.3. For each $t \in \mathbb{R}$, the random variable $W_t \in L^2(\mu)$ has the expansion

$$\begin{aligned} W_t &= \langle \omega, \chi_{[0,t]}(\cdot) \rangle = \left\langle \omega, \sum_{k=1}^{\infty} (\chi_{[0,t]}, \xi_k)_{L^2(\mathbb{R})} \xi_k(\cdot) \right\rangle \\ &= \int_{\mathbb{R}} \sum_{k=1}^{\infty} (\chi_{[0,t]}, \xi_k)_{L^2(\mathbb{R})} \xi_k(s) dW_s = \sum_{k=1}^{\infty} (\chi_{[0,t]}, \xi_k)_{L^2(\mathbb{R})} \int_{\mathbb{R}} \xi_k(s) dW_s \\ &= \sum_{k=1}^{\infty} \left(\int_0^t \xi_k(s) ds \right) \int_{\mathbb{R}} \xi_k(s) dW_s = \sum_{k=1}^{\infty} \left(\int_0^t \xi_k(s) ds \right) H_{\epsilon^{(k)}}(\omega), \end{aligned} \tag{2.12}$$

where, in general, $(f, g)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(t) g(t) dt$.

Using Theorem 2.2 we define the following spaces.

Definition 2.4. For $0 \leq \rho \leq 1$ the *Kondratiev test function space* $(S)_\rho$ consists of all
$f = \sum_{\alpha \in \mathbb{J}} a_\alpha H_\alpha \in L^2(\mu)$, $a_\alpha \in \mathbb{R}$ such that

$$\| f \|^2_{\rho,k} := \sum_{\alpha \in \mathbb{J}} (\alpha!)^{1+\rho} a_\alpha^2 (2N)^{k\alpha} < \infty \text{ for all } \ k \in \mathbb{N} \tag{2.13}$$

where $(2N)^{k\alpha} := (2 \cdot 1)^{k\alpha_1} (2 \cdot 2)^{k\alpha_2} \cdots (2 \cdot j)^{k\alpha_j}$ if $k\alpha = (k\alpha_1, \ldots, k\alpha_j) \in \mathbb{J}$.

Definition 2.5. For $0 \leq \rho \leq 1$ the *Kondratiev distribution space* $(S)_{-\rho}$ consists of all formal series $F = \sum_{\alpha \in \mathbb{J}} b_\alpha H_\alpha \in L^2(\mu)$, $b_\alpha \in \mathbb{R}$ such that

$$\| F \|^2_{-\rho,-q} := \sum_{\alpha \in \mathbb{J}} (\alpha!)^{1-\rho} b_\alpha^2 (2N)^{-q\alpha} < \infty \text{ for some } q \in \mathbb{N}. \tag{2.14}$$

$(S)_\rho$ is endowed with the projective limit topology and $(S)_{-\rho}$ is endowed with limit topology induced by the above seminorms. We note that for any
$f = \sum_{\alpha \in \mathbb{J}} a_\alpha H_\alpha \in (S)_\rho$ and $F = \sum_{\alpha \in \mathbb{J}} b_\alpha H_\alpha \in (S)_{-\rho}$ the action

$$\langle F, f \rangle := \sum_{\alpha \in \mathbb{J}} a_\alpha b_\alpha \alpha! \tag{2.15}$$

is well defined and thus the space $(S)_{-\rho}$ is the dual of $(S)_\rho$. For general $0 \leq \rho \leq 1$ we have the following inclusions

$$(S)_1 \subset (S)_\rho \subset (S)_0 \subset L^2(\mu) \subset (S)_{-0} \subset (S)_{-\rho} \subset (S)_{-1}. \tag{2.16}$$

The spaces $(S)_0$ and $(S)_{-0}$ coincide with the Hida spaces (S) and $(S)^*$, respectively.

We can, in a natural way, define $(S)^*$-valued integrals as follows (see [2], [8] and [9]).

Definition 2.6. Suppose $Z : \mathbb{R} \to (S)^*$ has the property that

$$\langle Z_t, f \rangle \in L^1(\mathbb{R}) \ for \ all \ f \in (S). \tag{2.17}$$

Then $\int_{\mathbb{R}} Z_t dt$ is defined to be the unique element of $(S)^*$ such that

$$\left\langle \int_{\mathbb{R}} Z_t dt, f \right\rangle = \int_{\mathbb{R}} \langle Z_t, f \rangle dt \ for \ all \ f \in (S). \tag{2.18}$$

We can show that Equation (2.18) defines $\int_{\mathbb{R}} Z_t dt$ as an element of $(S)^*$ (see [8] Proposition 8.1). If expression (2.17) holds, we say that Z_t is integrable in $(S)^*$.

One of the important features of the Hida space $(S)^*$ is that it contains the singular white noise $\dot{W}_t$ for all t (see [13]). By formally differentiating (2.12) we arrive at the following definition.

Definition 2.7. The *white noise process* $\dot{W}(t)$ is defined by the following formal expansion

$$\dot{W}_t = \sum_{k=1}^{\infty} \xi_k(t) H_{\epsilon^{(k)}}(\omega), \ t \in \mathbb{R} \tag{2.19}$$

where $\xi_k(t)$ is the Hermite function and $\epsilon^{(k)}$ is given in Equation (2.7).

The following lemma says that the white noise $\dot{W}_t$ belongs to $(S)^*$.

Lemma 2.8. *For each $t \in \mathbb{R}$, $\dot{W}_t$ is a generalized function, that is, $\dot{W}_t \in (S)^*$.*

Proof. The proof is given in [5]. □

A process $B : \mathbb{R} \to (S)^*$ is differentiable in $(S)^*$ if the limit

$$\lim_{h\to 0} \frac{B_{t+h} - B_t}{h} \tag{2.20}$$

exists in $(S)^*$ for all t. We denote this limit by $\frac{d}{dt}B_t$.
$\frac{d}{dt}B_t = B'_t$ if and only if there exists a $q \in \mathbb{N}$ such that

$$\lim_{h\to 0} \parallel \frac{B_{t+h} - B_t}{h} - B'_t \parallel_{-q} = 0.$$

Lemma 2.9.

$$\frac{d}{dt}W_t \textit{ exists in } (S)^* \quad \textit{for all } t \in \mathbb{R}. \tag{2.21}$$

$(S)^*$ is too small for the purpose of solving stochastic ordinary and partial differential equations. However, we can find a unique solution in $(S)_{-1}$.

Next, we introduce a Wick product $\diamond$ on the space $(S)_{-1}$ (see [4]).

Definition 2.10. The *Wick product* $F \diamond G$ of $F = \sum_{\alpha\in\mathbb{J}} a_\alpha H_\alpha \in (S)_{-1}$ and $G = \sum_{\beta\in\mathbb{J}} b_\beta H_\beta \in (S)_{-1}$ with $a_\alpha, b_\beta \in \mathbb{R}$ is defined by

$$(F \diamond G)(\omega) = \sum_{\alpha,\beta\in\mathbb{J}} a_\alpha b_\beta H_{\alpha+\beta}. \tag{2.22}$$

The Wick product is a commutative, associative and distributive binary operation on each of the spaces $(S)_1, (S), (S)^*$ and $(S)_{-1}$ (see [9] page 47). The Wick exponential $\exp^\diamond X$ of $X \in (S)_{-1}$ is defined by

$$\exp^\diamond X := \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}, \tag{2.23}$$

where $X^{\diamond 0} := 1$, and $X^{\diamond n} := X \diamond X \diamond \cdots \diamond X$ (n factors) provided the series converges in $(S)_{-1}$. The following lemma is found in [9].

Lemma 2.11. *We have*

$$\exp^\diamond(\langle\omega, f\rangle) = \exp\{\langle\omega, f\rangle - \frac{1}{2} \parallel f \parallel^2\} \tag{2.24}$$

for $f \in L^2(\mathbb{R})$.

3. The Hermite Transform

The Hermite transform or $\mathcal{H}$-transform (see [9] Section 2.6) transforms an element $F \in (S)_{-1}$ into deterministic functions $\mathcal{H}F(z_1, z_2, \ldots)$ of complex variables $z_j \in \mathbb{C},\ j = 1, 2, \ldots$ with values in $\mathbb{C}$.

Definition 3.1. Let $F(\omega) = \sum_{\alpha\in\mathbb{J}} c_\alpha H_\alpha(\omega) \in (S)_{-1}$. Then the *Hermite transform* of F, denoted by $\mathcal{H}F$ or $\widetilde{F}$, is defined by

$$\mathcal{H}F(z) = \widetilde{F}(z) = \sum_{\alpha\in\mathbb{J}} c_\alpha z^\alpha \in \mathbb{C} \tag{3.1}$$

where $z = (z_1, z_2, \dots,) \in \mathbb{C}^{\mathbb{N}}$ (the set of all sequences of complex numbers) and $z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \cdots$ if $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathbb{J}$ where $z_j^0 = 1$.

One can show that the sum in Equation (3.1) converges on the infinite dimensional neighborhood

$$\mathbb{K}_q(R) = \{(z_1, z_2, \dots) \in \mathbb{C}^N ; \sum_{\alpha \in \mathbb{J}} (2N)^{q\alpha} \mid z^\alpha \mid^2 < R^2\} \tag{3.2}$$

for some $0 < q, R < \infty$ (see [9] Proposition 2.6.5).

The following proposition is an immediate consequence of Definitions 2.10 and 3.1.

Proposition 3.2. *If $F, G \in (S)_{-1}$, then*

$$\mathcal{H}(F \diamond G)(z) = \mathcal{H}F(z) \cdot \mathcal{H}G(z) \tag{3.3}$$

for all z such that $\mathcal{H}F(z)$ and $\mathcal{H}G(z)$ exist. In general

$$\mathcal{H}(f^\diamond(F))(z) = f(\mathcal{H}F(z)) \ (\textit{when convergent}) \tag{3.4}$$

if $f : \mathbb{C} \to \mathbb{C}$ is entire, $f(\mathbb{R}) \subset \mathbb{R}$ and $f^\diamond(F) := \sum_{\alpha \in \mathbb{J}} c_\alpha F^{\diamond n} \in (S)_{-1}$.

One can use the Hermite transform to characterize distributions in $(S)_{-1}$, (see [9] page 68). The Hermite transform also serves as a useful tool to describe the topology of $(S)_{-1}$. In particular, the convergence of sequences of Hida distributions can be characterized (see [9] page 81).

Definition 3.3. An $(S)_{-1}$-process X is *strongly integrable* over an interval $[a, b]$ if

$$\int_a^b X(t, \omega) dt := \lim_{\triangle t_k \to 0} \sum_{k=0}^{n-1} X(t_k^*, \omega) \triangle t_k \tag{3.5}$$

exists in $(S)_{-1}$ for all partitions $a = t_0 < t_1 < \dots < t_n = b$ of $[a, b]$, $\triangle t_k = t_{k+1} - t_k$ and $t_k^* \in [t_k, t_{k+1}]$ for $k = 1, \dots, n-1$.

Taking the Hermite transform in Equation (3.5) we have the following result (see [9] page 86).

Lemma 3.4. *Let X_t be an $(S)_{-1}$ process. Suppose there exist $q < \infty$, $\delta > 0$ such that*

$$\sup\{\widetilde{X}(t, z) : t \in [a, b], \ z \in \mathbb{K}_q(\delta)\} < \infty \tag{3.6}$$

and $\widetilde{X}(t, z)$ is a continuous function of $t \in [a, b]$ for each $z \in \mathbb{K}_q(\delta)$. Then X_t is strongly integrable and

$$\mathcal{H}\left(\int_a^b X_t dt\right) = \int_a^b \widetilde{X}_t dt, \tag{3.7}$$

where the integral to the right is the Lebesgue integral.

We also have the following useful lemma.

Lemma 3.5. *Let G be a bounded open subinterval of $\mathbb{R}$. Suppose $X(t,\omega)$ and $F(t,\omega)$ are $(S)_{-1}$ processes such that*

$$\frac{d\tilde{X}(t,z)}{dt} = \tilde{F}(t,z) \tag{3.8}$$

for all $(t,z) \in G \times \mathbb{K}_q(R)$, for some $q < \infty$, $R > 0$. Furthermore assume that

$$\begin{aligned} &\frac{d}{dt}\tilde{F}(t,z) \text{ is a bounded function of } (t,z) \in G \times \mathbb{K}_q(\delta), \quad \text{continuous in} \\ &t \in G \text{ for each } z \in \mathbb{K}_q(R) \text{ and analytic with respect to } z \in \mathbb{K}_q(R) \\ &\text{for all } t \in G,\ q < \infty,\ R > 0. \end{aligned} \tag{3.9}$$

Then

$$\mathcal{H}\left(\frac{d}{dt}F\right) = \frac{d}{dt}\left(\mathcal{H}(F)\right) = \frac{d}{dt}\widetilde{F} \tag{3.10}$$

on $\mathbb{K}_q(R)$.

Proof. The mean value theorem implies that, for all $z \in \mathbb{K}_q(R)$, there exists $\varepsilon \in [0,1]$ such that

$$\frac{\widetilde{F}(t+h)(z) - \widetilde{F}(t)(z)}{h} = \frac{d}{dt}\widetilde{F}(t+\varepsilon h)(z).$$

So if (3.8) and (3.9) hold, then

$$\frac{\widetilde{F}(t+h)(z) - \widetilde{F}(t)(z)}{h} \to \frac{d}{dt}\widetilde{F}(t)(z) \text{ as } h \to 0 \tag{3.11}$$

pointwise boundedly for $z \in \mathbb{K}_q(R)$. Since $(S)_1$ is a nuclear space (see Lemma 2.8.2 in [9]) the statement (3.11) is equivalent to convergence in $(S)_{-1}$. That is

$$\frac{F(t+h,\omega) - F(t,\omega)}{h} \to \frac{d}{dt}F(t,\omega) \text{ in } (S)_{-1}$$

for all (t,ω). The result then follows since the Hermite transform is a continuous linear functional in $(S)_{-1}$. □

We have the following chain rule in $(S)_{-1}$.

Proposition 3.6. *Suppose that $t \to X_t : \mathbb{R} \to (S)_{-1}$ is continuously differentiable and let $f : \mathbb{C} \to \mathbb{C}$ be entire (analytic on $\mathbb{C}$) such that $f(\mathbb{R}) \subset \mathbb{R}$ and $f^\diamond(X_t) \in (S)_{-1}$ for all t. Then*

$$\frac{d}{dt}f^\diamond(X_t) = (f')^\diamond(X_t) \diamond \frac{d}{dt}X_t \text{ in } (S)_{-1}. \tag{3.12}$$

Proof.

$$\begin{aligned} \mathcal{H}\left(f'^\diamond(X_t) \diamond \frac{d}{dt}X_t\right)(z) &= \mathcal{H}(f'^\diamond(X_t))(z) \cdot \mathcal{H}\left(\frac{d}{dt}X_t\right)(z) \\ &= f'(\mathcal{H}(X_t)(z)) \cdot \frac{d}{dt}\mathcal{H}(X_t)(z) \\ &= \frac{d}{dt}f(\mathcal{H}(X_t)(z)) = \frac{d}{dt}\mathcal{H}(f^\diamond(X_t))(z) \\ &= \mathcal{H}\left(\frac{d}{dt}f^\diamond(X_t)\right)(z) \end{aligned}$$

where we have used (3.3), (3.4) and (3.10). The result follows by the uniqueness of the Hermite transform. □

Remark 3.7. The Hermite transform is closely related to the so called $\mathbb{S}$-transform (see [8] and [9] page 80). The $\mathbb{S}$-transform maps random variables into non-random functionals. The use of Hermite transform has some advantages, for instance it enables the application of methods of complex analysis.

4. The Donsker Delta Function

The Donsker delta function is a generalized white noise functional which has been studied in several monographs within white noise analysis (see [4], [8], [11], [12] and the references therein). Here we define the Donsker delta function in the white noise framework as follows.

Definition 4.1. Let $X : \Omega \to \mathbb{R}$ be a random variable which also belongs to $(S)_{-1}$. Then a continuous function

$$\delta_X(\cdot) : \mathbb{R} \to (S)_{-1}$$

is called a *Donsker delta function* of X if it satisfies

$$\int_{\mathbb{R}} \varphi(x)\delta_X(x)dx = \varphi(X) \ a.e \tag{4.1}$$

for all (measurable) $\varphi : \mathbb{R} \to \mathbb{R}$ such that the integral on the left hand side converges in $(S)_{-1}$.

The following result taken from [4] plays an important role in the main result of this paper.

Proposition 4.2. *Suppose X is a normally distributed random variable with variance $v > 0$. Then δ_X is unique and is given by the expression*

$$\delta_X(x) = \frac{1}{\sqrt{2\pi v}} \exp^{\diamond}\left(-\frac{(x-X)^{\diamond 2}}{2v}\right) \in (S)_{-1}. \tag{4.2}$$

5. Greeks (1)

We consider the following model with two securities (see [1] and [13]):

(1) A risk-free asset (for example a bank account) where the price A_t at time t is given by
$$dA_t = r_t A_t dt, \ A_0 = 1. \tag{5.1}$$
(2) A risky asset (for example a stock) where the price S_t at time t is given by
$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \ S_0 = x > 0, \tag{5.2}$$

where r_t, μ_t and σ_t are deterministic functions satisfying the property

$$\int_0^T \left(\mid r_t \mid + \mid \mu_t \mid + \sigma_t^2\right) ds < \infty.$$

We assume that σ is bounded away from zero. The exact solution of the differential Equations (5.2) is given by

$$S_t = x \exp\left\{\int_0^t \left(\mu_s - \frac{1}{2}\sigma_s^2\right) ds + \int_0^t \sigma_s dW_s\right\}. \tag{5.3}$$

Then we consider Δ of a digital option. The digital option payoff then takes the form

$$\chi_{[K,\infty)}(S_T) \tag{5.4}$$

with strike price K and S_T is the value of a stock price at final time T. Following the constructions in the preceding sections, we apply the concept of white noise analysis together with the Donsker delta function to compute Δ for the digital option. Here we only illustrate the computation of Δ.

We define ν by:

$$d(\log S) = (\mu_t - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW := \nu_t dt + \sigma_t dW. \tag{5.5}$$

Let

$$v_T = \int_0^T \sigma_u^2 du. \tag{5.6}$$

Then

$$\log S_T \sim N\left(\int_0^T \nu_u du, v_T\right).$$

So we may apply (4.1) to get a.s.

$$f(S_T) = f(e^{\log S_T}) = \frac{1}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \exp^{\diamond}\left(-\frac{(y-\log S_T)^{\diamond 2}}{2v_T}\right) dy. \tag{5.7}$$

We note that for $f \in L^1(\mathbb{R})$ and with compact support the integral belongs to the distribution space $(S)_{-1}$ (by Lemma 3.4). The option price with the payoff function of the form (5.4) is given by

$$u(x) \quad = \quad \mathbb{E}[e^{-rT} f(S_T)] = \mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \exp^{\diamond}\left(-\frac{(y-\log S_T)^{\diamond 2}}{2v_T}\right) dy\right].$$

Theorem 5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function of polynomial growth. Then*

$$\begin{aligned}&\frac{\partial}{\partial x}\mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \exp^{\diamond}\left(-\frac{(y-\log S_T)^{\diamond 2}}{2v_T}\right) dy\right] \\ &\quad = \mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \exp^{\diamond}\left(-\frac{(y-\log S_T)^{\diamond 2}}{2v_T}\right) \diamond \frac{(y-\log S_T)}{v_T}\frac{1}{x} dy\right].\end{aligned} \tag{5.8}$$

Proof. Let

$$u(x) \quad = \quad \mathbb{E}[e^{-rT} f(S_T)] = \mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \exp^{\diamond}\left(-\frac{(y-\log S_T)^{\diamond 2}}{2v_T}\right) dy\right].$$

First we assume that $f \in L^2(\mathbb{R})$ and has compact support. Then by Lemma 3.4 the strong integral exists in $(S)_{-1}$. Taking the Hermite transform on both sides and using Lemma 3.4 since condition (3.6) holds we obtain the following deterministic equation

$$\widetilde{u}(x)(z) \quad = \quad \mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\mathcal{H}\left\{\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)\right\}dy\right]$$

where $\widetilde{u}$ denote the Hermite transforms of u and the expectation is taken in the generalized sense. The use of Proposition 3.2 gives

$$\widetilde{u}(x)(z) \quad = \quad \mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\exp\left(-\frac{(y-\log \widetilde{S}_T(x)(z))^2}{2v_T}\right)dy\right].$$

where $\widetilde{S}_T$ is the Hermite transform of S_T. Writing the derivative as a limit of difference quotients we get , for $\varepsilon \neq 0$,

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\widetilde{u}(x+\varepsilon)(z)-\widetilde{u}(x)(z)}{\varepsilon} \quad &= \quad \lim_{\varepsilon\to 0}\mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\right.\\ &\times\frac{1}{\varepsilon}\left\{\exp\left(-\frac{(y-\log \widetilde{S}_T(x+\varepsilon)(z))^2}{2v_T}\right)\right.\\ &\left.\left.-\exp\left(-\frac{(y-\log \widetilde{S}_T(x)(z))^2}{2v_T}\right)\right\}dy\right].\end{aligned}$$

Put

$$\begin{aligned}\widetilde{Z}_\varepsilon(y)(z) \quad :=& \quad \frac{1}{\varepsilon}\left\{\exp\left(-\frac{(y-\log \widetilde{S}_T(x+\varepsilon)(z))^2}{2v_T}\right)\right.\\ &\left.-\exp\left(-\frac{(y-\log \widetilde{S}_T(x)(z))^2}{2v_T}\right)\right\}.\end{aligned}$$

Using Taylor expansions and letting $\varepsilon \to 0$, we have

$$\widetilde{Z}_\varepsilon(y)(z) \quad \to \quad \exp\left(-\frac{(y-\log \widetilde{S}_T(x)(z))^2}{2v_T}\right)\left(\frac{y\log \widetilde{S}_T(x)(z)-\log \widetilde{S}_T^2(x)(z)}{v_T}\right)$$

in $L^2(\mu)$ as $\varepsilon \to 0$ since

$$\mid S_T(x+\varepsilon)\mid \quad \leq \quad (x+1)\exp\left(\int_0^T\left(\mu-\frac{1}{2}\sigma^2\right)ds+\int_0^T\sigma dW_t\right)\in L^1(du).$$

Set

$$\begin{aligned}\widetilde{K}_\varepsilon(y)(z) \quad :=& \quad \frac{e^{-rT}}{\sqrt{2\pi v_T}}f(e^y)\frac{1}{\varepsilon}\left\{\exp\left(-\frac{(y-\log \widetilde{S}_T(x+\varepsilon)(z))^2}{2v_T}\right)\right.\\ &\left.-\exp\left(-\frac{(y-\log \widetilde{S}_T(x)(z))^2}{2v_T}\right)\right\}.\end{aligned}$$

Thus we have the following estimate

$$\begin{aligned}
\mid \widetilde{K}_\varepsilon(y)(z) \mid &= \left| \frac{e^{-rT}}{\sqrt{2\pi v_T}} f(e^y) \frac{1}{\varepsilon} \left\{ \exp\left(-\frac{(y - \log \widetilde{S}_T(x+\varepsilon)(z))^2}{2v_T} \right) \right.\right. \\
&\quad \left.\left. - \exp\left(-\frac{(y - \log \widetilde{S}_T(x)(z))^2}{2v_T} \right) \right\} \right| \\
&= \left| \frac{e^{-rT}}{\sqrt{2\pi v_T}} f(e^y) \exp\left(-\frac{(y - \log \widetilde{S}_T(x)(z))^2}{2v_T} \right) \right. \\
&\quad \left. \times \left(\frac{y \log \widetilde{S}_T(x)(z) - \log \widetilde{S}^2_T(x)(z)}{v_T} \right) \right| \\
&\leq A_1 \cdot \mid f(e^y) \mid e^{-A_2(y^2 - |y|)} (A_3 \mid y \mid + A_4) \in L^1(\mathbb{R}) \qquad (5.9)
\end{aligned}$$

for some positive constants A_1, A_2, A_3, A_4 for fixed z. Using a similar estimate as in (5.9) we obtain

$$\int_{\mathbb{R}} \mid \widetilde{K}_\varepsilon(y)(z) \mid dy \leq A_1 \int_{\mathbb{R}} \mid f(e^y) \mid e^{-A_2(y^2 - |y|)} (A_3 \mid y \mid + A_4) dy < \infty$$

for fixed z with constants independent of ε. Since $f(e^y)$ grows polynomially, we can use the dominated convergence theorem to interchange the order of taking the limit and expectation and obtain

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{\widetilde{u}(x+\varepsilon)(z) - \widetilde{u}(x)(z)}{\varepsilon} \\
&= \mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \lim_{\varepsilon \to 0} \int_{\mathbb{R}} f(e^y) \frac{1}{\varepsilon} \left\{ \exp\left(-\frac{(y - \log \widetilde{S}_T(x+\varepsilon)(z))^2}{2v_T} \right) \right.\right. \\
&\quad \left.\left. - \exp\left(-\frac{(y - \log \widetilde{S}_T(x)(z))^2}{2v_T} \right) \right\} dy \right].
\end{aligned}$$

Using the estimate (5.9) for some positive constants A_1, A_2, A_3, A_4 which are independent of ε and for fixed z we can use the dominated convergence theorem to interchange the order of taking the limit and the integral and obtain

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{\widetilde{u}(x+\varepsilon)(z) - \widetilde{u}(x)(z)}{\varepsilon} \\
&= \mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ \exp\left(-\frac{(y - \log \widetilde{S}_T(x+\varepsilon)(z))^2}{2v_T} \right) \right.\right. \\
&\quad \left.\left. - \exp\left(-\frac{(y - \log \widetilde{S}_T(x)(z))^2}{2v_T} \right) \right\} dy \right] \\
&= \mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \frac{d}{dx} \left(\exp\left(-\frac{(y - \log \widetilde{S}_T(x)(z))^2}{2v_T} \right) \right) dy \right] \\
&= \mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}} \int_{\mathbb{R}} f(e^y) \exp\left(-\frac{(y - \log \widetilde{S}_T(x)(z))^2}{2v_T} \right) \frac{(y - \log \widetilde{S}_T(x)(z))}{v_T} \frac{1}{x} dy \right]
\end{aligned}$$

where we have used the chain rule in the last equality. Thus we have

$$\begin{aligned}\frac{d}{dx}\widetilde{u}(x)(z) &= \mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\exp\left(-\frac{(y-\log\widetilde{S}_T(x)(z))^2}{2v_T}\right)\right.\\ &\qquad \left.\times\frac{(y-\log\widetilde{S}_T(x)(z))}{v_T}\frac{1}{x}dy\right].\end{aligned}$$

Since $S_T(x)\in(S)_{-1}$ the application of Proposition 3.2 yields

$$\begin{aligned}\frac{d}{dx}\widetilde{u}(x)(z) &= \mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\mathcal{H}\left\{\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)\right.\right.\\ &\qquad \left.\left.\diamond\frac{(y-\log S_T(x))}{v_T}\frac{1}{x}\right\}dy\right].\end{aligned}$$

By using Lemma 3.4 twice (recall that f has compact support) on the right hand side and Lemma 3.5 on the left hand side of the above equation we obtain

$$\begin{aligned}\mathcal{H}\left(\frac{d}{dx}u(x)\right)(z) &= \mathcal{H}\left\{\mathbb{E}\left[\frac{e^{-rT}}{\sqrt{2\pi v_T}}\int_{\mathbb{R}} f(e^y)\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)\right.\right.\\ &\qquad \left.\left.\diamond\frac{(y-\log S_T(x))}{v_T}\frac{1}{x}dy\right]\right\}.\end{aligned}$$

The result then follows by the uniqueness of the Hermite transform.

For the general case we consider, for f of polynomial growth, the sequence

$$f_n = f\chi_{[-n,n]}$$

whose functions have compact support so that $|f_n|\le|f|$. Then $f_n\to f$ in L^2 as $n\to\infty$. Let

$$u_n(x)=\mathbb{E}\left[e^{-rT}\int_{\mathbb{R}} f_n(e^y)\frac{1}{\sqrt{2\pi v_T}}\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)dy\right].$$

Then it is clear that

$$u_n(x)\to u(x) \text{ in } (S)_{-1}. \tag{5.10}$$

By Theorem 4.4.4 this is equivalent to

$$\mathcal{H}(u_n(x))(z)\to\mathcal{H}(u(x))(z).$$

Put

$$g(x)=\mathbb{E}\left[e^{-rT}\int_{\mathbb{R}} f(e^y)\frac{1}{\sqrt{2\pi v_T}}\exp^{\diamond}\left(-\frac{(y-\log S_T)^{\diamond 2}}{2v_T}\right)\diamond\frac{(y-\log S_T)}{v_T}\diamond\frac{1}{x}dy\right].$$

Using the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\left|\frac{d}{dx}u_n(x) - g(x)\right| \\
&= \left|\mathbb{E}\left[e^{-rT}\int_{\mathbb{R}}(f_n(e^y) - f(e^y))\frac{1}{\sqrt{2\pi v_T}}\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)\right.\right. \\
&\qquad \left.\left.\diamond\frac{(y-\log S_T(x))}{v_T}\diamond\frac{1}{x}dy\right]\right| \\
&\leq \mathbb{E}\left[e^{-rT}\int_{\mathbb{R}}|f_n(e^y) - f(e^y)|\left|\frac{1}{\sqrt{2\pi v_T}}\exp^{\diamond}\left(-\frac{(y-\log S_T(x))^{\diamond 2}}{2v_T}\right)\right.\right. \\
&\qquad \left.\left.\diamond\frac{(y-\log S_T(x))}{v_T}\diamond\frac{1}{x}\right|dy\right] \\
&\leq A_1\mathbb{E}\left[\int_{\mathbb{R}} \mid f_n(e^y) - f(e^y) \mid e^{-A_2(y^2-|y|)}(A_3 \mid y \mid + A_4)dy\right].
\end{aligned}$$

Since $f(e^y)$ grows polynomially the dominated convergence theorem implies that $\mid f_n(e^y) - f(e^y) \mid$ converges to 0 as $n \to \infty$. Therefore, using (5.9), the above inequality proves that

$$\frac{d}{dx}u_n(x) \to g(x) \text{ pointwise.} \tag{5.11}$$

From (5.10) and (5.11) we can deduce that $u(x)$ is continuously differentiable and that $\frac{d}{dx}u(x) = g(x)$. □

6. An Extension to Pure Lévy Jump Model

As in the Brownian motion case, we let $\Omega = S'(\mathbb{R})$ be the space of tempered distributions equipped with its Borel σ-algebra $\mathbb{F} = \mathbb{B}(\Omega)$. The space $S'(\mathbb{R})$ is the dual of the Schwartz space $S(\mathbb{R})$ of test functions, that is, the rapidly decreasing smooth functions on $\mathbb{R}$. Then we define the Lévy white noise probability measure μ, which exists by the Bochner-Minlos theorem (see [9] Appendix A), as the measure $d\mu$ defined on the Borel σ-algebra $\mathbb{B}(\Omega)$ of subsets of Ω by

$$\int_{\Omega} e^{i\langle\omega,f\rangle}d\mu(\omega) = e^{\int_{\mathbb{R}}\psi(f(y))dy},\ f \in S(\mathbb{R}), \tag{6.1}$$

where $i = \sqrt{-1}$ and $\langle\omega, f\rangle = \omega(f)$ denotes the action of $\omega \in \Omega = S'(\mathbb{R})$ applied to $f \in S(\mathbb{R})$, and ψ is given by

$$\psi(u) = \int_{\mathbb{R}}(e^{iuz} - 1 - iuz1_{\{|z|<1\}})\nu(dz). \tag{6.2}$$

Here ν is a Lévy measure on $\mathbb{R}_0$. We suppose that the Lévy measure $\nu(dz)$ satisfies the following condition. For every $\epsilon > 0$ there exists a $\lambda > 0$ such that

$$\int_{(-\epsilon,\epsilon)^c} e^{\lambda|z|}\nu(dz) < \infty. \tag{6.3}$$

The triple $(\Omega, \mathbb{B}(\Omega), \mu)$ is called the *Lévy white noise probability space.*

Lemma 6.1. *Let $f \in S(\mathbb{R})$. Then we have*

$$\mathbb{E}[\langle \cdot, f\rangle] = 0 \text{ and } \mathbb{E}[\langle \cdot, f\rangle^2] = M \int_{\mathbb{R}_0} f^2(y)dy, \tag{6.4}$$

where $M = \int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$.

In [4] it was shown that the definition of $\langle \omega, f\rangle$ can be extended, by using Lemma 6.1, from $f \in S(\mathbb{R})$ to any $f \in L^2(\mathbb{R})$. In particular, we can construct the Lévy process $\eta(t,\omega)$ as the càdlàg version of $\widetilde{\eta}(t,\omega)$ where

$$\widetilde{\eta}(t,\omega) = \langle \omega, \chi_{[0,t]}(\cdot)\rangle \tag{6.5}$$

which is defined since $\chi_{[0,t]}(\cdot)$ is in $L^2(\mathbb{R})$. This leads to the following theorem (see [4]).

Theorem 6.2. *The stochastic process $\{\widetilde{\eta}(t), 0 \le t \le T\}$ has a càdlàg version denoted by η. The process $\{\eta(t), 0 \le t \le T\}$ is a Lévy process with Lévy measure ν.*

The Lévy process η_t admits the following stochastic integral representation

$$\eta_t = \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds,dz),\ 0 \le t \le T, \tag{6.6}$$

where $\widetilde{N}(dt,dz)$ is a compensated Poisson random measure associated with η_t. This is called the pure jump Lévy process.

The Lévy white noise calculus is developed in a similar manner as in the Brownian motion case. We will refer to [4] for a detailed review. We mention that the results related to the Wick product and Hermite transform in the Brownian motion case are valid for the Lévy Wick product and Lévy Hermite transform, respectively, with minor modifications.

7. Donsker Delta Function of a Lévy Process

Put

$$\eta_t = \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds,dz). \tag{7.1}$$

The Donsker delta function $\delta_x(\eta_t)$ of η_t is a generalized white noise functional (see [4] and [8]).

Definition 7.1. Suppose that $X : \Omega \to \mathbb{R}$ is a random variable belonging to the Lévy-Hida distribution space $(S)_{-1}$. The *Donsker delta function* of X is a continuous function $\delta_\cdot(X) : \mathbb{R} \to (S)_{-1}$ such that

$$\int_{\mathbb{R}} h(x)\delta_x(X)dx = h(x) \tag{7.2}$$

for all measurable functions $h : \mathbb{R} \to \mathbb{R}$ for which the integral is well-defined in $(S)_{-1}$.

We want to represent a certain class of pure jump Lévy processes in terms of the Donsker delta function. We assume that the pure jump Lévy process satisfies the condition:
There exists $\varepsilon \in (0,1)$ such that, for $u \in \mathbb{R}$,

$$\lim_{|u|\to\infty} \mid u \mid^{-(1+\varepsilon)} \operatorname{Re}\left(\int_{\mathbb{R}} (e^{iuz} - 1 - iuz)\nu(dz)\right) = \infty \tag{7.3}$$

where $\operatorname{Re}\left(\int_{\mathbb{R}}(e^{iuz} - 1 - iuz)\nu(dz)\right)$ denotes the real part of $\int_{\mathbb{R}}(e^{iuz}-1-iuz)\nu(dz)$.

Remark 7.2. The condition (7.3) implies that the probability law of $\eta_t, t \geq 0$ is absolutely continuous with respect to the Lebesgue measure (see [4] page 226).

We need the following lemma (see [4]).

Lemma 7.3. *Let $u \in \mathbb{R}$ and $t \geq 0$. Then*

$$\exp(iu\eta_t) = \exp^{\diamond}\left(\int_0^t \int_{\mathbb{R}_0} (e^{iuz} - 1)\widetilde{N}(ds,dz) + t\int_{\mathbb{R}_0} (e^{iuz} - 1 - iuz)\nu(dz)\right).$$

The following result from [4] will be useful in our computations of Greeks.

Theorem 7.4. *The Donsker delta function $\delta_x(\eta_t)$ of η_t exists in $(S)_{-1}$ and admits a representation of the form*

$$\begin{aligned} \delta_x(\eta_t) \quad = \quad & \frac{1}{2\pi}\int_{\mathbb{R}_0} \exp^{\diamond}\left\{\int_0^t \int_{\mathbb{R}_0} (e^{iuz} - 1)\widetilde{N}(ds,dz)\right. \\ & \left. + t\int_{\mathbb{R}_0} (e^{iuz} - 1 - iuz)\nu(dz) - iux\right\} du \end{aligned}$$

for $u \in \mathbb{R}$ and $t \in [0,T]$.

Proof. The proof is based on an application of the Lévy-Hermite transform and the use of Fourier inversion formula. A detailed proof can be found in [4]. We omit the details. □

8. Greeks (2)

Suppose we have a financial market, where the bond price $S_0(t)$ and the stock price $S(t)$ are modeled as follows

(1) bond price:
$$S_0(t) = 1,\ 0 \leq t \leq T, \tag{8.1}$$
(2) stock price:
$$dS(t) = S(t)d\eta_t,\ S(0) = x > 0,\ 0 \leq t \leq T, \tag{8.2}$$

where η_t is a Lévy process of the form (7.1). Assume that $z > -1 + \epsilon$ for a.a z with respect to ν for some $\epsilon > 0$. This ensures that $S(t) > 0$ for all $0 \leq t \leq T$.

Using the Itô formula for Lévy processes the solution to Equation (8.2) is given by

$$S(t) = x\exp\left\{\int_0^t \int_{\mathbb{R}_0} (\log(1+z) - z)\nu(dz)ds + \int_0^t \int_{\mathbb{R}_0} \log(1+z)\widetilde{N}(ds,dz)\right\}. \tag{8.3}$$

Here we apply the concept of the white noise analysis together with the Donsker delta function to compute Δ of a digital option. We consider the digital option of the form

$$\chi_{[K,\infty)}(S_T) \tag{8.4}$$

with strike price K. Similar to the pure Brownian motion case, we apply the concepts of the white noise analysis together with the Donsker delta function of the Lévy process S_t. We will only illustrate the computation of Δ.

As in the pure Brownian motion case, we represent f in terms of the Donsker delta function

$$\begin{aligned}\delta_x(S_T) &= \frac{1}{2\pi}\int_{\mathbb{R}_0} \exp^{\diamond}\left\{\int_0^T\int_{\mathbb{R}_0}(e^{iuz}-1)\widetilde{N}(ds,dz)\right.\\ &\quad \left. +T\int_{\mathbb{R}_0}(e^{iuz}-1-iuz)\nu(dz)-iux\right\}du\end{aligned}$$

as

$$\begin{aligned}f(S_T) &= \int_{\mathbb{R}_0} f(y)\delta_y(S_T)dy\\ &= \int_{\mathbb{R}_0}\frac{1}{2\pi}\left(\int_{\mathbb{R}_0} f(y)\exp(-iuy)dy\right)\\ &\quad \times\exp^{\diamond}\left\{\int_0^T\int_{\mathbb{R}_0}(e^{iuz}-1)\widetilde{N}(ds,dz)\right.\\ &\quad \left. +T\int_{\mathbb{R}_0}(e^{iuz}-1-iuz)\nu(dz)\right\}du.\end{aligned}$$

We mention that, for $f\in L^1(\mathbb{R})$ with compact support the integral above converges in the distribution space $(S)_{-1}$. Thus the option price of the digital option takes the form

$$\begin{aligned}u(x) &= \mathbb{E}[e^{-rT}f(S_T)]\\ &= \mathbb{E}\left[e^{-rT}\int_{\mathbb{R}_0}\frac{1}{2\pi}\left(\int_{\mathbb{R}_0} f(y)\exp(-iuy)dy\right)\right.\\ &\quad \times\exp^{\diamond}\left\{\int_0^T\int_{\mathbb{R}_0}(e^{iuz}-1)\widetilde{N}(ds,dz)\right.\\ &\quad \left.\left. +\ T\int_{\mathbb{R}_0}(e^{iuz}-1-iuz)\nu(dz)\right\}du\right].\end{aligned}$$

Using Lemma 7.3 we can write the option price as follows

$$u(x)=\mathbb{E}\left[e^{-rT}\int_{\mathbb{R}_0}\frac{1}{2\pi}\left(\int_{\mathbb{R}_0} f(y)\exp(-iuy)dy\right)\exp(iuS_T)du\right]. \tag{8.5}$$

We now state the second main result of this paper.

Theorem 8.1. *Let f be a function of polynomial growth and let the integral $\int_{\mathbb{R}_0} f(y)\exp(-iuy)dy$ belong to L^1. Then*

$$\begin{aligned}&\frac{d}{dx}\mathbb{E}\left[e^{-rT}\int_{\mathbb{R}_0}\frac{1}{2\pi}\left(\int_{\mathbb{R}_0} f(y)\exp(-iuy)dy\right)\exp(iuS_T)du\right]\\ &= \mathbb{E}\left[e^{-rT}\int_{\mathbb{R}_0}\frac{1}{2\pi}\left(\int_{\mathbb{R}_0} f(y)\exp(-iuy)dy\right)\exp(iuS_T)iu\frac{S_T}{x}du\right].\end{aligned}$$

Proof. The proof follows the same arguments as in Theorem 5.1. We omit the details. □

9. Conclusion

In this paper we have shown how one can use white noise calculus to explicitly compute Greeks. In particular we obtained Δ for a diffusion model and a pure jump model. The approach is advantageous because we can handle discontinuous and path-dependent payoffs.

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