

VERTICAL MARTINGALES, STOCHASTIC CALCULUS AND HARMONIC SECTIONS

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ABSTRACT. This work is about a new class of martingales: the vertical martingales. We construct vertical martingales for smooth submersions and we develop a stochastic calculus for them. Furthermore, a stochastic characterization for harmonic sections is given.

1. Introduction

Let $\pi : E \rightarrow M$ be a Riemannian submersion with the totally geodesic fibers property and denote by ∇^g the Levi-Civita connection on E . Since π is a submersion, it is possible to define the vertical bundle $VE = \bigcup_{p \in E} V_p E$, where $V_p E = \ker(\pi_{*p})$, $p \in E$. We also define the vertical connection ∇^v on E by the vertical projection of the Levi-Civita connection ∇^g into VE . With this hypothesis, in [11], C.Wood defines a harmonic section being a section σ of π such that

$$\tau_\sigma^v = \text{tr} \nabla^v \sigma_*^v = 0,$$

where σ_*^v is the vertical projection of σ_* into VE . In fact, Wood shows that this definition is consistent with a minimal solution of the vertical energy functional.

The main idea of this article was motivated by harmonic sections. It is well-known that there is a stochastic characterization for harmonic maps (see for example [3] or [4]). In a nutshell, if M is a Riemannian manifold, N is a smooth manifold with a symmetric connection and $\phi : M \rightarrow N$ is a smooth map, then ϕ is a harmonic map if and only if ϕ sends Brownian motions in M into martingales in N . However, harmonic sections require the vertical connection on the target manifold. Therefore, in order to construct a stochastic characterization for harmonic sections it is necessary to define a new concept, which we call the vertical martingales.

The environment of our work is more general than that used by Wood. In fact, we adopt two smooth manifolds E and M such that there exists a submersion $\pi : E \rightarrow M$. Moreover, we also endow E with a symmetric connection and M with a Riemannian metric. Thus, a vertical connection on E is adopted and, hence, a vertical stochastic calculus on E is constructed. More specifically, we

Received 2012-10-19; Communicated by the editors.

2010 *Mathematics Subject Classification*. 53C43, 58E20, 58J65, 60G48.

Key words and phrases. Vertical martingales, harmonic sections, stochastic analysis on manifolds.

construct a vertical integral of Itô, a vertical integral of Stratonovich, a formula of convert to one to another both and a vertical geometric formula of Itô.

From the vertical stochastic calculus on E we get our main result: a section $\sigma : M \rightarrow E$ of π is a harmonic section if and only if σ sends Brownian motions into vertical martingales.

As an application we study the vertical martingales in the tangent space TM endowed with the complete lift connection or the Sasaky metric. Moreover, as a result of our main Theorem we conclude that every harmonic section into TM has its vertical part null. Furthermore, vertical martingales in the Riemannian principal fiber bundle is studied. Here, we follow the characterization of semimartingales in Riemannian principal fiber bundle due to M. Arnaudon and S. Paycha [2].

2. Preliminaries

We begin by recalling some fundamental facts on Schwartz Theory and stochastic calculus on manifolds. We shall freely use concepts and notations from S. Kobayashi and N. Nomizu [7], L. Schwartz [10], P.A. Meyer [8] and M. Emery [4].

Firstly, we introduce the second order tangent fiber bundle. The idea is to follow the construction of the tangent vector bundle of a smooth manifold M , which is the vector space of velocity fields. However, there are different ways to view the velocity fields. In this article the adequate form is to approach by differential operators. Namely, a vector tangent space at x , $x \in M$, is the vector space of all differential operators on M at x of order at most one without a constant term.

Since we are considering velocity fields it is reasonable to think in a vector space of acceleration fields. In this case, let M be a smooth manifold and $x \in M$. The second order tangent space to M at x , which is denoted by $\tau_x M$, is the vector space of all differential operators on M at x of order at most two without a constant term. Taking a local coordinate system (x_1, \dots, x_n) around x we can write every $L \in \tau_x M$, in a unique way, as

$$L = a_{ij} D_{ij} + a_i D_i,$$

where $a_{ij} = a_{ji}$, $D_i = \frac{\partial}{\partial x^i}$ and $D_{ij} = \frac{\partial^2}{\partial x^i \partial x^j}$ are differential operators at x (we shall use the convention of summing over repeated indices). The elements of $\tau_x M$ are called second order tangent vectors at x and those of the dual vector space $\tau_x^* M$ are called second order forms at x .

The disjoint union $\tau M = \bigcup_{x \in M} \tau_x M$ (respectively, $\tau^* M = \bigcup_{x \in M} \tau_x^* M$) is canonically endowed with a vector bundle structure over M , which is called the second order tangent fiber bundle (respectively, second order cotangent fiber bundle) of M .

Our next step is to consider the maps between acceleration fields. Let M, N be smooth manifolds, $F : M \rightarrow N$ a smooth map and $L \in \tau_x M$. The differential of F , $F_*(x) : \tau_x M \rightarrow \tau_{F(x)} N$, is given by

$$F_*(x)L(f) = L_x(f \circ F),$$

where $f \in C^\infty(N)$.

A fact is that the vector structure of the second tangent order fiber bundle yields morphisms that are not obtained from smooth maps. Following we refine

the class of morphisms. Let L be a second order vector field on M . The square operator of L , denoted by QL , is the symmetric tensor given by

$$QL(f, g) = \frac{1}{2}(L(fg) - fL(g) - gL(f)),$$

where $f, g \in C^\infty(M)$. Let $x \in M$. We consider $Q_x : \tau_x M \rightarrow T_x M \odot T_x M$ as the linear application defined by

$$Q_x(L = a_{ij}D_{ij} + a_iD_i) = a_{ij}D_i \odot D_j.$$

Push-forward of the second order vectors by smooth maps is related to the so called Schwartz morphisms between second order tangent vector bundles.

Definition 2.1. Let M and N be smooth manifolds, $x \in M$ and $y \in N$. A linear application $F : \tau_x M \rightarrow \tau_y N$ is called *Schwartz morphism* if

- (1) $F(T_x M) \subset T_y N$;
- (2) for all $L \in \tau_x M$ we have $Q(FL) = (F \otimes F)(QL)$.

The class of Schwartz morphisms have the good property: A linear application $F : \tau_x M \rightarrow \tau_y N$ is a Schwartz morphism if and only if there exists a smooth map $\phi : M \rightarrow N$ with $\phi(x) = y$ such that $F = \phi_{x*}$ (see for example [5, Prop. 1]).

We observe that, for this work, the Schwartz morphisms are relevant on the construction of the vertical geometric Itô formula (see section 5).

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space which satisfies the usual conditions (see for instance [4, (1.1)]). We now deal with the nexus between stochastic calculus and differential topology. The idea arises from the Itô formula for real semimartingales. Namely, given a real function f that is C^2 and a real semimartingale X we have, in the differential form, that

$$df(X_t) = D_i f(X_t) dX_t^i + \frac{1}{2} D_{ij} f(X_t) d[X^i, X^j]_t. \tag{2.1}$$

In order to view the nexus between stochastic calculus and differential topology is necessary to define the class of semimartingales in a smooth manifold.

Definition 2.2. Let M be a smooth manifold and X_t a stochastic process with values in M . We call X_t a *semimartingale* if, for all f smooth on M , $f(X_t)$ is a real semimartingale.

Viewing the Itô formula (2.1), in [10], L. Schwartz noticed that if X_t is a continuous semimartingale in a smooth manifold M , the Itô's differentials dX_t^i and $d[X^i, X^j]_t$ (where (x_i) is a local coordinate system and X_t^i is the i th coordinate of X_t in this system) behave under a change of coordinates as the coefficients of a second order tangent vector. Consequently, the (purely formal) stochastic differential

$$d^2 X_t = dX_t^i D_i + \frac{1}{2} d[X^i, X^j]_t D_{ij},$$

is a linear differential operator on M , at X_t , of order at most two with no constant term. Therefore, the tangent object to X_t is formally a second order tangent. This fact is known as Schwartz principle.

From now on we assume that all *semimartingales are continuous*. Our next step is to introduce the stochastic calculus from the Schwartz principle. Let X_t

be a semimartingale in M , $\Theta_{X_t} \in \tau_{X_t}^* M$ an adapted stochastic second order form along X_t and (U, x^i) a local coordinate system in M . With respect to this chart the second order form Θ can be written as

$$\Theta_x = \Theta_i(x)d^2x^i + \Theta_{ij}(x)dx^i \cdot dx^j,$$

where Θ_i and $\Theta_{ij} = \Theta_{ji}$ are (\mathcal{C}^∞ say) functions on M . Here, $dx^i \cdot dx^j, i, j = 1, \dots, n$, is the product defined in [4, (6.6)], which gives a second order form. Then the integral of Θ along X_t is locally defined by

$$\int_0^t \Theta d^2X_s = \int_0^t \Theta_i(X_s)dX_s^i + \frac{1}{2} \int_0^t \Theta_{ij}(X_s)d[X^i, X^j]_s. \tag{2.2}$$

Let b be a section of $T_0^2(M)$, which is defined along X_t . The quadratic integral of b along X_t is locally defined by

$$\int_0^t b(dX, dX)_s = \int_0^t b_{ij}(X_s)d[X^i, X^j]_s,$$

where $b(x) = b_{ij}(x)dx^i \otimes dx^j$ and b_{ij} are smooth functions. Here, we observe that both integrals $\int_0^t \Theta d^2X_s$ and $\int_0^t b(dX, dX)_s$ are well defined. To see these facts we refer the reader to [4].

Our next step is to connect the stochastic calculus and the differential geometry. We start by rewriting the fundamental concept of differential geometry: the connection. Let M be a smooth manifold endowed with a symmetric connection ∇^M . In [8], P. Meyer showed that for ∇^M there exists a section Γ^M in $Hom(\tau M, TM)$ such that $\Gamma^M|_{TM} = Id_{TM}$ and $\Gamma^M(AB) = \nabla_A^M B$, where $A, B \in TM$. Thus, taking a semimartingale X_t in M and θ be a 1-form along X_t , $\theta_{X_t} \in T_{X_t}^* M$, we define the integral of Itô by $\int_0^t \theta d^M X_s := \int_0^t \Gamma^{M*} \theta d^2 X_t$, which is the nexus between the stochastic calculus and the differential geometry.

The Itô's integral gives the martingales in smooth manifolds. In fact, a semimartingale X_t is said a ∇^M -martingale if for every 1-form θ on M we have that $\int_0^t \theta d^M X_s$ is a real local martingale. Furthermore, from the Itô's integral we can define the Brownian motion in a Riemannian manifold.

Definition 2.3. Let M be a Riemannian manifold with a metric g . A semimartingale B_t in M is called a *Brownian motion* if $\int_0^t \theta d^g B_t$ is a real local martingale for all $\theta \in T^* M$, where ∇^g is the Levi-Civita connection, and for any section b in $T_0^2(M)$ we have

$$\int_0^t b(dB, dB)_s = \int_0^t \text{tr } b_{B_s} ds. \tag{2.3}$$

3. Vertical Connection

In this section we proceed with the study of the vertical connection. In fact, in order to construct the vertical stochastic calculus it is necessary to describe the vertical connection in the way of the second order theory.

We begin by introducing the environment to define the vertical connection. Let E and M be differential manifolds such that there is a smooth submersion $\pi : E \rightarrow M$. Let us denote the vertical distribution by $VE = \ker(\pi_*)$ and the vertical projection by $\mathbf{v} : TE \rightarrow VE$. A vector field X on E is called vertical

vector field if $X_p \in V_p E$ for all $p \in E$. Analogous, a 1-form θ is called vertical form if $\theta(p) \in V_p^* E$ for all $p \in E$.

We now define the vertical connection as such in the differential geometry. Given a symmetric connection ∇^E on E , in each fiber $\pi^{-1}(x)$, $x \in M$, it is possible to induce a connection ∇^x from the connection ∇^E . In this way, ∇^x is the vertical projection of ∇^E on $\pi^{-1}(x)$ for vertical vector fields. Our purpose is now generalized this concept: let U be a vertical vector fields on E and X a vector field on E , we define the vertical connection on E by $\nabla_X^v V = \mathbf{v}\nabla_X^E V$. According to theory of connections on vector bundles, the vertical connection is a connection on the vertical fiber bundle VE .

In order to write the vertical connection in the second order theory is necessary to construct the vertical spaces in the second order tangent fiber bundle. Let $p \in E$, by Rank Theorem, there exist a local coordinate system $(x_1, \dots, x_m, v_1, \dots, v_k)$ of some neighborhood $U \ni p$ such that

$$\pi(x_1, \dots, x_m, v_1, \dots, v_k) = (x_1, \dots, x_m), \tag{3.1}$$

where (x_1, \dots, x_m) is a local coordinate system of a neighborhood $V \ni \pi(p)$. The possibility of such a choice of coordinates is fundamental to construct the vertical martingale.

Taking the coordinates (3.1) we obtain in $T_p E$ the coordinate basis $\{\bar{D}_1(p), \dots, \bar{D}_m(p), D_1(p), \dots, D_k(p)\}$, where $\bar{D}_i = \partial/\partial x^i, i = 1, \dots, m$, and $D_\alpha = \partial/\partial v^\alpha, \alpha = 1, \dots, k$. It is clear that $\{D_1(p), \dots, D_k(p)\}$ are vertical vectors in $T_p E$ and it also spans the vertical space $V_p E$. Also, in coordinates (3.1), a second vector L in $\tau_p E$ is written as

$$L(p) = a_{\alpha\beta} D_{\alpha\beta}(p) + a_{ij} D_{ij}(p) + a_{\alpha j} D_{\alpha j}(p) + a_\alpha D_\alpha(p) + a_i D_i(p).$$

We denote by $\mathfrak{V}_p E$ the subspace spanned by $\{D_{\alpha\beta}, D_{\alpha j}, D_\alpha; \alpha, \beta = 1, \dots, k\}$. Here, we observe that $\mathfrak{V}_p M$ is the $\ker \pi_*(p)$. We denote by $\mathfrak{V}E = \bigcup_{p \in E} \mathfrak{V}_p E$ and by $\mathbf{v} : \tau E \rightarrow \mathfrak{V}E$ the vertical projection.

We are now in position to give a description of the vertical connection in the second order fiber bundle. In view of Γ^E be a linear homomorphism from τE into TE we can restrict it to $\mathfrak{V}E$. A little bit more, we take

$$\Gamma^v(L) = \mathbf{v}\Gamma^E(L), \tag{3.2}$$

where $L \in \mathfrak{V}E$. The connection Γ^v is the object that is associated with ∇^v . In fact, if V is a vertical vector field and X is a vector field on E , then XV is a second vector field in $\mathfrak{V}E$. Thus a simple account gives $\Gamma^v(XV) = \nabla_X^v V$.

4. Vertical Integral of Itô and Stratonovich

Writing the vertical connection in the second order fiber bundle leads to construct a integral vertical such as the Itô's integral. In consequence, vertical martingales are defined. This is the purpose of this section.

We now proceed with the construction of an integral of Itô for the vertical connection. Let X_t be a semimartingale in E . Adopting the coordinates (3.1), by Schwartz principle, we obtain

$$d^2 X_t = dX_t^\alpha D_\alpha + dX_t^i D_i + \frac{1}{2} d[X^\alpha, X^\beta]_t D_{\alpha\beta} + \frac{1}{2} d[X^i, X^j]_t D_{ij} + \frac{1}{2} d[X^i, X^\beta]_t D_{i\beta}.$$

Since that $d^2X_t \in \tau_{X_t}M$, we can project it into $\mathfrak{V}_{X_t}(E)$, that is,

$$\mathbf{v}(d^2X_t) = dX_t^\alpha D_\alpha + \frac{1}{2}d[X^\alpha, X^\beta]_t D_{\alpha\beta} + \frac{1}{2}d[X^\alpha, X^j]_t D_{\alpha j}.$$

Let Θ_{X_t} be an adapted stochastic second order form along X_t such that $\Theta_{X_t} \in \mathfrak{V}_{X_t}^*E$. With respect to the local coordinate system (3.1) the second order form Θ can be written as

$$\Theta_p = \Theta_\alpha d^2v^\alpha(p) + \Theta_{\alpha\beta} dv^\alpha \cdot dv^\beta(p) + \Theta_{\alpha j} dv^\alpha \cdot dx^j(p),$$

where Θ_α , $\Theta_{\alpha\beta} = \Theta_{\beta\alpha}$ and $\Theta_{\alpha j} = \Theta_{j\alpha}$ are (C^∞ say) functions in E . From definition of the integral (2.2) we can see that

$$\begin{aligned} \int_0^t \Theta d^2X_s &= \int_0^t \Theta_\alpha(X_s) dX_s^\alpha + \frac{1}{2} \int_0^t \Theta_{\beta\gamma}(X_s) d[X^\beta, X^\gamma]_s + \frac{1}{2} \int_0^t \Theta_{\beta j}(X_s) d[X^\beta, X^j]_s \\ &= \int_0^t \Theta \mathbf{v} d^2X_t. \end{aligned} \quad (4.1)$$

We now use the equality above to connect the vertical connection and stochastic integral. In fact, we begin by observing that $\mathbf{v}d^2X_t \in \mathfrak{V}_{X_t}(E)$. Consequently, $\Gamma^v(\mathbf{v}(d^2X_t)) \in VE$ by (3.2). Thus, taking a vertical form θ in E we have $\theta(\Gamma^v(\mathbf{v}(d^2X_t)))$. On the other hand, taking local coordinate system (3.1) we can write $\theta(p) = \theta_\alpha(p) dv^\alpha$ and, consequently,

$$\Gamma^{v*}\theta(p) = \theta^\alpha(p)(d^2v^\alpha + \Gamma_{\beta\gamma}^\alpha(p) dv^\beta \cdot dv^\gamma + \Gamma_{\beta j}^\alpha(p) dv^\beta \cdot dx^j).$$

This allows us to define a integral of Itô for the vertical connection.

Definition 4.1. Let E and M be smooth manifolds such that there is a smooth submersion $\pi : E \rightarrow M$. Let ∇^E be a symmetric connection, X_t a semimartingale on E and θ a vertical form on E . We define the *vertical integral* of Itô of θ along X_t as $\int_0^t \theta d^v X_s = \int_0^t \Gamma^{v*}\theta(\mathbf{v}d^2X_s)$. Let $(U, x_1, \dots, x_m, v_1, \dots, v_k)$ be the local coordinate system (3.1). Then the vertical integral of Itô of θ along X_t is locally given by

$$\int_0^t \theta d^v X_s = \int_0^t \theta_\alpha(X_s) dX_s^\alpha + \frac{1}{2} \int_0^t \Gamma_{\beta\gamma}^\alpha \theta_\alpha(X_s) d[X^\alpha, X^\beta]_s + \frac{1}{2} \int_0^t \Gamma_{\beta j}^\alpha \theta_\alpha(X_s) d[X^\alpha, X^j]_s.$$

One can observe that the vertical Itô integral is well posed because the right side of the above equality is an integral such as (4.1).

From Definition 4.1 we can define our main object: the vertical martingales.

Definition 4.2. A semimartingale X_t in E is called a *vertical martingale* if $\int_0^t \theta d^v X_s$ is a real local martingale for every vertical form θ on E .

Our next step is to define a vertical Stratonovich integral. Let θ be a vertical form on E . Thus in the local coordinate system (3.1) we have $\theta = \theta_\alpha dv^\alpha$. A second order form from 1-form θ is obtained by using the operator $d : T^*E \rightarrow \tau^*E$, which is locally given by

$$d\theta(p) = \theta_\alpha(p) d^2v^\alpha + D_\alpha \theta_\beta(p) dv^\alpha \cdot dv^\beta + D_j \theta_\alpha(p) dv^\alpha \cdot dx^j. \quad (4.2)$$

It is direct that $d\theta$ belongs to \mathfrak{V}^*E . Therefore, taking a semimartingale X_t in E there is sense in

$$d\theta(X_t)(\mathbf{v}d^2X_t) = \theta_\alpha(X_t)dX_t^\alpha + \frac{1}{2}D_\alpha\theta_\beta(X_t)d[X^\alpha, X^\beta]_t + \frac{1}{2}D_i\theta_\beta(X_t)d[X^i, X^\beta]_t.$$

Hence by (4.1), we can define a vertical integral such as the integral of Stratonovich.

Definition 4.3. Let E and M be smooth manifolds such that there is a smooth submersion $\pi : E \rightarrow M$. Let X_t be a semimartingale on E and θ a vertical form on E along X_t . We define the *vertical integral* of Stratonovich of θ along X_t as $\int \theta \delta^v X_t = \int d\theta(\mathbf{v}d^2X_t)$. Let $(U, x_1, \dots, x_m, v_1, \dots, v_k)$ be the local coordinate system (3.1). Then the vertical integral of Stratonovich of θ along X_t is locally given by

$$\int_0^t \theta \delta^v X_s = \int_0^t \theta_\alpha(X_s)dX_s^\alpha + \frac{1}{2} \int_0^t D_\alpha\theta_\beta(X_s)d[X^\alpha, X^\beta]_s + \frac{1}{2} \int_0^t D_i\theta_\beta(X_s)d[X^i, X^\beta]_s.$$

From definition of vertical integrals of Itô and Stratonovich we can show a formula to convert one to the other. In fact, a simple computation in coordinates gives

Proposition 4.4. For a vertical form θ on E and a semimartingale X_t on E we have

$$\int_0^t \theta \delta^v X_s = \int_0^t \theta d^v X_s - \frac{1}{2} \int_0^t \nabla^v \theta(dX, \mathbf{v}dX)_s,$$

where $\mathbf{v}dX$ is the vertical projection of dX into VE .

5. The Geometric Itô Formulas

In this section our main goal is to construct Itô's formulas for the vertical integral of Itô and vertical integral of Stratonovich. Our ideas are inspired in [3]. Let us first prove a version of Proposition 1.8 in [5].

Lemma 5.1. Let E and M be smooth manifolds such that there is a smooth submersion $\pi : E \rightarrow M$. Suppose that N is a smooth manifold, $\phi : N \rightarrow E$ is a smooth map and θ is a vertical form on E . Then

$$(\mathbf{v}\phi)^*d\theta = d((\mathbf{v}\phi)^*\theta),$$

where d is the operator given by (4.2).

Proof. We first adopt the local coordinate system (3.1). Since θ is a vertical form, it follows that $(\mathbf{v}\phi)^*\theta = \phi^*\theta$. Of course, $(\phi^*\theta) = (\theta^\alpha \circ \phi)d(v^\alpha \circ \phi)$. Applying the operator d at $(\phi^*\theta)$ we deduce that

$$\begin{aligned} d(\phi^*\theta) &= d((\theta^\alpha \circ \phi)d(v^\alpha \circ \phi)) \\ &= d(\theta^\alpha \circ \phi)d(v^\alpha \circ \phi) + (\theta^\alpha \circ \phi)d^2(v^\alpha \circ \phi) \\ &= \phi^*d\theta^\alpha \phi^*dv^\alpha + (\theta^\alpha \circ \phi)\phi^*d^2v^\alpha \\ &= \phi^*(d\theta^\alpha dv^\alpha + \theta^\alpha d^2v^\alpha) \end{aligned}$$

Now, in coordinates, the differential of θ^α is given by

$$d\theta^\alpha = \frac{\partial \theta^\alpha}{\partial x_i} dx^i + \frac{\partial \theta^\alpha}{\partial v_\beta} dv^\beta.$$

We thus obtain

$$d(\phi^*\theta) = \phi^*\left(\frac{\partial\theta^\alpha}{\partial x_i} dx^i dv^\alpha + \frac{\partial\theta^\alpha}{\partial v^\beta} dv^\beta dv^\alpha + \theta^\alpha d^2v^\alpha\right).$$

Since the right side is $\phi^*d\theta$, in coordinates, and $\phi^*d\theta = (\mathbf{v}\phi)^*d\theta$, because $d\theta \in \mathfrak{V}^*E$, we conclude that

$$d((\mathbf{v}\phi)^*\theta) = (\mathbf{v}\phi)^*d\theta.$$

□

Proposition 1.8 in [5] is responsible to the Itô's formula for the integral of Stratonovich (see for example [5] or [8]). Following the same idea, Lemma 5.1 was constructed specifically to show the same in the vertical case.

Theorem 5.2. *Let E and M be smooth manifolds such that there is a smooth submersion $\pi : E \rightarrow M$. Suppose that N is a smooth manifold, $\phi : N \rightarrow E$ is a smooth map and θ is a vertical form on E . Then for a semimartingale X_t in N we have*

$$\int_0^t \theta \delta^v \phi(X_s) = \int_0^t (\mathbf{v}\phi^*\theta) \delta X_s.$$

Proof. Let X_t be a semimartingale in N and θ a vertical form on E . By definition of the vertical Stratonovich integral,

$$\begin{aligned} \int_0^t \theta \delta^v \phi(X_s) &= \int_0^t d\theta \mathbf{v} d^2 \phi(X_s) = \int_0^t (\mathbf{v}\phi)^* d\theta d^2 X_s = \int_0^t d((\mathbf{v}\phi)^*\theta) d^2 X_s \\ &= \int_0^t (\mathbf{v}\phi)^*\theta \delta X_s, \end{aligned}$$

where we used Lemma 5.1 in the third equality. □

Our next purpose is to construct the geometric Itô's formula for the vertical integral of Itô. We begin introducing the second fundamental form, the vertical tension field and the vertical harmonic maps.

Definition 5.3. Let E and M be smooth manifolds such that there is a smooth submersion $\pi : E \rightarrow M$ and $\phi : N \rightarrow E$ a smooth map. Suppose that E and N are equipped with symmetric connections ∇^E and ∇^N , respectively. Furthermore, denote the vertical connection on E by Γ^v and the connection ∇^N by Γ^N . The section α_ϕ^v of $\tau^*N \otimes \phi^*VE$ is given by

$$\alpha_\phi^v = \Gamma^v \mathbf{v}\phi_* - \mathbf{v}\phi_* \Gamma^N. \quad (5.1)$$

The *vertical second fundamental form* of ϕ , denoted by β_ϕ^v , is the unique section of $(TN \odot TN)^* \otimes \phi^*VE$ such that $\alpha_\phi^v = \beta_\phi^v \circ Q$.

In the case that N is a Riemannian manifold and Γ^N is the Levi-Civita connection, the *vertical tension field* of ϕ , $\tau_\phi^v : N \rightarrow VE$, is given by

$$\tau_\phi^v = \text{tr} \beta_\phi^v.$$

We call ϕ a *vertical harmonic map* if $\tau_\phi^v = 0$.

The following linear algebra Lemma shows that the second fundamental form β_F^v is well defined.

Lemma 5.4. *Let α_ϕ^v be the section of $\tau^*N \otimes \phi^*VE$ defined by (5.1). Then there exists an unique section β_ϕ^v of $(TN \odot TN)^* \otimes \phi^*VE$ such that $\alpha_\phi^v = \beta_\phi^v \circ Q$.*

Proof. Since $\text{Ker } Q = TN \subset \text{Ker } \alpha_\phi^v$, the lemma follows from the first isomorphism theorem (see [9] pp 67). \square

The following Lemma is necessary in the main Theorem of this section.

Lemma 5.5. *Under hypothesis of Definition 5.3, for each vertical form θ on E ,*

$$\int_0^t \alpha_\sigma^{v*} \theta d_2 X_s = \frac{1}{2} \int_0^t \beta_\sigma^{v*} \theta (dX, dX)_s.$$

Proof. By definition of β_ϕ^v , for each vertical form θ on E we have

$$\frac{1}{2} \int_0^t \beta_\sigma^{v*} \theta (dX, dX)_s = \int_0^t Q^* \beta_\sigma^{v*} \theta d^2 X_s = \int_0^t (\beta_\sigma^v \circ Q)^* \theta d^2 X_s = \int_0^t \alpha_\sigma^{v*} \theta d^2 X_s.$$

The first equality follows from Proposition 6.31 in [4]. \square

We are in a position to show a geometric Itô's formula for vertical the integral of Itô.

Theorem 5.6. *Let E and M be smooth manifolds such that there is a smooth submersion $\pi : E \rightarrow M$ and $\phi : N \rightarrow E$ a smooth map. Suppose that E and N are equipped with symmetric connections ∇^E and ∇^N , respectively. Furthermore, endowed E with the vertical connection ∇^v . If X_t is an N -valued semimartingale and if θ is a vertical form on E , then*

$$\int_0^t \theta d^v \phi(X_s) = \int_0^t \phi^* \theta d^N X_s + \frac{1}{2} \int_0^t \beta_\phi^{v*} \theta (dX, dX)_s.$$

Proof. We calculate

$$\begin{aligned} \int_0^t \theta d^v \phi(X_s) &= \int_0^t (\Gamma^{v*} \theta) \mathbf{v} d^2 \phi(X_s) \\ &= \int_0^t (\Gamma^{v*} \theta) \mathbf{v} \phi_* d^2(X_s) = \int_0^t \mathbf{v} \phi^* (\Gamma^{v*} \theta) d^2 X_s \\ &= \int_0^t \mathbf{v} \phi^* (\Gamma^{v*} \theta) d^2 X_s + \int_0^t \Gamma^{N*} (\phi^* \theta) d^2 X_s - \int_0^t \Gamma^{N*} (\phi^* \theta) d^2 X_s \\ &= \int_0^t \Gamma^{N*} (\phi^* \theta) d^2 X_s + \int \left(\mathbf{v} \phi^* (\Gamma^{v*} \theta) - \Gamma^{N*} (\phi^* \theta) \right) d^2 X_s \\ &= \int_0^t \phi^* \theta d^N X_s + \int \alpha_\phi^{v*} \theta d_2 X_s \\ &= \int_0^t \phi^* \theta d^N X_s + \frac{1}{2} \int_0^t \beta_\phi^{v*} \theta (dX, dX)_s, \end{aligned}$$

where we use Lemma 5.5 in the last equality. \square

Corollary 5.7. *Under assumptions of Theorem 5.6 with “ N is a smooth manifold equipped with a symmetric connection ∇^N ” replaced by “ (N, g) is a Riemannian manifold and ∇^g is its Levi-Civita connection”, if B_t is a Brownian motion in N and if θ is a vertical form on E , then*

$$\int_0^t \theta d^v \phi(B_s) = \int_0^t \phi^* \theta d^g B_s + \frac{1}{2} \int_0^t \tau_\phi^{v*} \theta (B_s) ds. \tag{5.2}$$

An important nexus between stochastic analysis and differential geometry is the stochastic characterization of harmonic maps, see for example [4] or [5].

Proposition 5.8. *Let E and M be smooth manifolds such that there is a smooth submersion $\pi : E \rightarrow M$. Suppose that E is equipped with a symmetric connection ∇^E and (N, g) is a Riemannian manifold. Furthermore, endowed E with the vertical connection ∇^v . A smooth map $\phi : N \rightarrow E$ is a vertical harmonic map if and only if ϕ sends Brownian motions B_t to vertical martingales $\phi(B_t)$.*

Proof. Let B_t be a Brownian motion in N and θ a vertical form on E . By formula (5.2),

$$\int_0^t \theta d^v \phi(B_s) = \int_0^t \phi^* \theta d^g B_s + \frac{1}{2} \int_0^t \tau_\phi^{v*} \theta(B_s) ds.$$

We observe that $\int_0^t \phi^* \theta d^g B_s$ is a real local martingale. Since B_t and θ are arbitrary, the Doob-Meyer decomposition assures that $\int_0^t \theta d^v \phi(B_s)$ is a real local martingale if and only if τ_ϕ^v vanishes. From the definitions of vertical martingales and vertical harmonic maps we conclude the proof. \square

6. Harmonic Section

In this section we work with object that motivate this study: harmonic section. Before, we will give the environment for the study of the harmonic sections.

Let E be a smooth manifold and (M, g) a Riemannian manifold such that there is a smooth submersion $\pi : E \rightarrow M$. We also adopt a symmetric connection ∇^E on E such that π has the totally geodesics fibers property and we denote by ∇^g the Levi-Civita connection on M . Write $VE = \ker(\pi_*)$ for the vertical distribution and HE for a smooth distribution in TE such that $TE = VE \oplus HE$. Let us denote by $\mathbf{v} : TE \rightarrow VE$ and $\mathbf{h} : TE \rightarrow HE$ the vertical and horizontal projectors, respectively. Other important tool is the horizontal lift associated to distribution HE , which is a family of isomorphism $H_p = (\pi_p|_{H_p E})^{-1} : T_x M \rightarrow T_p E$ for all $p \in E$, where $\pi(p) = x$. Finally, the submersion $\pi : E \rightarrow M$ is called affine submersion with horizontal distribution if $\mathbf{h} \nabla_{H(X)}^E H(Y) = H(\nabla_X^M Y)$ (see [1] for more details). A Riemmanian submersion is a classical example of an affine submersion with horizontal distribution.

In this section, unless otherwise stated, we assume that $\pi : E \rightarrow M$ is an affine submersion with horizontal distribution.

The following we extend the definition given by Wood [11] for harmonic sections.

Definition 6.1. A section σ of π is called a *harmonic section* if $\tau_\sigma^v = 0$.

An immediate consequence of Proposition 5.8 is the characterization of the harmonic sections in the following stochastic context.

Theorem 6.2. *Let E be a smooth manifold and (M, g) a Riemannian manifold such that there is a smooth submersion $\pi : E \rightarrow M$. Assume that E is equipped with a symmetric connection ∇^E such that π has totally geodesics fibers property. Then a section σ of π is harmonic section if and only if, for every Brownian motion B_t in M , $\sigma(B_t)$ is a vertical martingale in E .*

7. Applications

In this section some applications of results are developed. We focus in the study of vertical martingales and harmonic sections.

7.1. Tangent Bundle with complete lift connection. Let (M, g) be a Riemannian manifold and TM its tangent bundle. To study vertical martingales in TM is necessary to introduce a connection on it. Denoting by ∇^g the Levi-Civita connection on M we prolong ∇^g to the complete lift connection ∇^c on TM (see [12] for the definition of ∇^c). In a nutshell, if X, Y are vector fields on M , then ∇^c satisfies the following equations:

$$\begin{aligned} \nabla_{X^v}^c Y^v &= 0 \\ \nabla_{X^v}^c Y^h &= 0 \\ \nabla_{X^h}^c Y^v &= (\nabla_X^g Y)^v \\ \nabla_{X^h}^c Y^h &= (\nabla_X^g Y)^h + \gamma(R(-, X)Y), \end{aligned} \tag{7.1}$$

where $R(-, X)Y$ denotes a tensor field W of type $(1,1)$ on M such that $W(Z) = R(Z, X)Y$ for any $Z \in TM$ and γ is a lift of tensors, which is defined at page 12 in [12]. Furthermore, X^v, Y^v and X^h, Y^h are the vertical and horizontal lifts of the X, Y into TM , respectively. A direct account shows that $\pi_{TM} : TM \rightarrow M$ is an affine submersion with horizontal distribution.

Proposition 7.1. *Let (M, g) be a Riemannian manifold, ∇^g the Levi-Civita connection on M and TM its tangent bundle equipped with the complete lift connection ∇^c . Suppose that X_t is a semimartingale in TM and $J_t = \pi_{TM}(X_t)$ is a semimartingale in M . Then X_t is a vertical martingale if and only if, for each vertical form θ on TM ,*

$$\int_0^t \theta \delta^v X_s - \int_0^t \theta(\delta J_s)^v + \int_0^t \theta^{v*} d^M J_s$$

is a real local martingale, where θ^{v} is the push-forward of θ by the vertical lift on TM . In the case that J_t is a ∇^g -martingale, X_t is a vertical martingale if and only if*

$$\delta^v X_t = (\delta J_t)^v.$$

Proof. Let X_t be a semimartingale in TM and θ a vertical form on TM . From Proposition 4.4 we see that

$$\int_0^t \theta d^v X_s = \int_0^t \theta \delta^v X_s + \frac{1}{2} \int_0^t \nabla^v \theta(dX, \mathbf{v}dX)_s.$$

We now calculate $\nabla^v \theta$. In fact, for a vector field A on TM we have

$$\begin{aligned} \nabla^v \theta(A, \mathbf{v}A) &= A\theta(\mathbf{v}A) - \theta(\nabla_A^v \mathbf{v}A) \\ &= \mathbf{v}A\theta(\mathbf{v}A) - \theta(\nabla_{\mathbf{h}A}^v \mathbf{v}A), \end{aligned}$$

where we used that $\nabla_{\mathbf{v}A}^v \mathbf{v}A = 0$ in the last equality. Denoting $B = \pi_{TM*}(A)$ it follows that $\mathbf{v}A = B^v$ and $\mathbf{h}A = B^h$. We thus obtain

$$\nabla^v \theta(A, \mathbf{v}A) = B^v \theta(B^v) - \theta(\nabla_B^g B)^v,$$

because third equality in (7.1) and definition of ∇^v on TM . Denoting the push-forward of θ by vertical lift as θ^{v*} we deduce that

$$\nabla^v \theta(A, \mathbf{v}A) = B(\theta^{v*} B) - \theta^{v*}(\nabla_B^g B) = \nabla^g \theta^{v*}(B, B).$$

Writing $J_t = \pi_{TM*}(X_t)$ it follows that

$$\int_0^t \theta d^v X_s = \int_0^t \theta \delta^v X_s + \frac{1}{2} \int_0^t \nabla^g \theta^{v*} (dJ, dJ)_s.$$

We now use the formula of convert the Stratonovich's integral to the Itô's integral to obtain

$$\int_0^t \theta d^v X_s = \int_0^t \theta \delta^v X_s - \int_0^t \theta^{v*} \delta J_s + \int_0^t \theta^{v*} d^g J_s,$$

and the proof follows. \square

Our next step is to use Proposition 7.1 and Theorem 6.2 to study the harmonic section of π_{TM} .

Proposition 7.2. *Let (M, g) be a Riemannian manifold, ∇^g the Levi-Civita connection on M and TM its tangent bundle equipped with the complete lift connection ∇^c . Then a section σ of π_{TM} is harmonic section if and only if $\mathbf{v}\sigma_*$ is null.*

Proof. If $\mathbf{v}\sigma_*$ is null, it is immediate that σ is a harmonic map. Suppose, contrary to our claim, that there exist a harmonic section σ of π_{TM} such that $\mathbf{v}\sigma_*$ is no null. Since σ is a harmonic section, Theorem 6.2 assures that for every Brownian motion B_t in M we have that $\sigma(B_t)$ is a vertical martingale on TM . Using Propositions 7.1 and Theorem 5.2 we get

$$\mathbf{v}\sigma_* \delta B_s = (\delta B_s)^v.$$

Thus, taking two harmonic sections σ_1 and σ_2 we have $\mathbf{v}\sigma_{1*} \delta B_s = \mathbf{v}\sigma_{2*} \delta B_s$, for any Brownian motion B_t in M . However, if σ_1 is harmonic section then so is $\sigma_2 = 2\sigma_1$. We thus get $\mathbf{v}\sigma_{1*} \delta B_s = 2\mathbf{v}\sigma_{1*} \delta B_s$, a contradiction. \square

7.2. Tangent bundle with Sasaki metric. Let (M, g) be a Riemannian manifold and TM the tangent bundle equipped with the Sasaki metric g_s . For a deeper discussion of the Sasaki metric we refer the reader to [6]. Let us denote by ∇^g the Levi-Civita connection on M and by ∇^s the Levi-Civita connection on TM for the Sasaki metric. If X, Y are vector fields on M , then ∇^s satisfies the following equations:

$$\begin{aligned} \nabla_{X^v}^s Y^v &= 0 \\ \nabla_{X^v}^s Y^h &= \frac{1}{2}(R(-, X)Y)^h \\ \nabla_{X^h}^s Y^v &= (\nabla_X^g Y)^v + \frac{1}{2}(R(-, X)Y)^h \\ \nabla_{X^h}^s Y^h &= (\nabla_X^g Y)^h - \frac{1}{2}(R(-, X)Y)^v, \end{aligned} \tag{7.2}$$

where $R(-, X)Y$ denotes a tensor field W of type (1,1) on M such that $W(Z) = R(Z, X)Y$ for any $Z \in TM$. Furthermore, $(\cdot)^v$ and $(\cdot)^h$ are the vertical and horizontal lifts, respectively.

A simple observation shows that the vertical connection on TM defined from ∇^s is equal to that defined from the complete lift ∇^c . Furthermore, $\pi_{TM} : TM \rightarrow M$ is an affine submersion with horizontal distribution. Thus by the same method as in section above we have the following results.

Proposition 7.3. *Let (M, g) be a Riemannian manifold, ∇^g the Levi-Civita connection on M and TM its tangent bundle equipped with the Sasaki metric g_s . Let*

X_t be a semimartingale in TM and $J_t = \pi_{TM}(X_t)$ a semimartingale in M . Then X_t is a vertical martingale if and only if, for each vertical form θ on TM ,

$$\int_0^t \theta \delta^v X_s - \int_0^t \theta(\delta J_s)^v + \int_0^t \theta^{v*} d^g J_s$$

is a real local martingale, where θ^{v*} is the push-forward of θ by the vertical lift on TM . In the case that J_t is a ∇^g -martingale, X_t is a vertical martingale if and only if

$$\delta^v X_t = (\delta J_t)^v.$$

Proposition 7.4. *Let (M, g) be a Riemannian manifold, ∇^g the Levi-Civita connection on M and TM its tangent bundle equipped with the Sasaky metric g_s . Then a section σ of π_{TM} is harmonic section if and only if $\mathbf{v}\sigma_*$ is null.*

7.3. Principal Riemannian fiber bundle. In [2], M. Arnaudon and S. Paycha show that semimartingales in a principal fiber bundle $P(M, G)$ with G -invariant Riemannian metric k can be decomposed into G - and M - valued semimartingales. More precisely, a semimartingale X_t with values in $P(M, G)$ splits in a unique way into a horizontal semimartingale X_t^h and a semimartingale V_t with values in G_t such that

$$X_t = X_t^h \cdot V_t.$$

Moreover, V_t is the stochastic exponential

$$V_t = \epsilon\left(\int \omega \delta X_t\right)$$

and X_t^h is the solution of the Itô equation

$$d^{\nabla^k} X_t^h = H_{\tilde{X}_t}^k d^{\nabla}(\pi \circ X_t).$$

Here ω is the connection form on P associated to Levi-Civita connection on M .

We now induce at each fiber $\pi^{-1}(x)$, $x \in M$, a metric k_x from the metric k such that $\pi_P : P \rightarrow M$ has the totally geodesic fibers property. We also endow G with a metric h such that $p : G \rightarrow \pi^{-1}(x) \subset P$ is an isometric map for each $p \in P$, $x = \pi_P(p)$. Let us denote by ∇^k , ∇^x and ∇^h the Levi-civita connections on P , $\pi^{-1}(x)$ and G , respectively. We observe that ∇^k induces a connection ∇^x at each fiber and that the vertical connection ∇^v coincides with latter in the fibers. We thus conclude that p is an affine map with respect to ∇^h and ∇^v for every $p \in P$.

Our next goal is to study the vertical martingales in the principal fiber bundle as described above.

Lemma 7.5. *If θ is a vertical form on $P(M, G)$ and if A is a vector field on $P(M, G)$, then for each $p \in P$*

$$\nabla^v \theta(A, \mathbf{v}A)_p = \nabla^h u^* \theta(\xi, \xi)_g,$$

where ξ is the vector field on G yielded by $\omega(A)$, by left translations, and $p = u \cdot g$ with $u \in \pi_P^{-1}(x)$, $x = \pi_P(p)$, and $g \in G$.

Proof. Let A be a vector field on $P(M, G)$. It is possible to write A as $A = A^h + \omega(A)^*$, where A^h is the horizontal lift of $\pi_*(A)$ and $(\omega(A))^*$ is the fundamental vector field associated to A . For abbreviation, we write ζ instead of $\omega(A)^*$. For each $p \in P$ we have compute

$$\begin{aligned}\nabla^{v*}\theta(A, \mathbf{v}A)_p &= A_p\theta(\mathbf{v}A)_p - \theta_p(\nabla_A^v \mathbf{v}A(p)) \\ &= (A^h + \zeta)_p\theta(\zeta)_p - \theta_p(\nabla_{A^h+\zeta}^v \zeta(p)) \\ &= \zeta_p\theta(\zeta)_p - \theta_p(\nabla_{A^h}^v \zeta(p) + \nabla_\zeta^v \zeta(p)).\end{aligned}$$

Since π_P has the totally geodesic fibers property and θ is a vertical form, it follows that $\theta(\nabla_{A^h}^v \zeta) = 0$. Therefore

$$\nabla^{v*}\theta(A, \mathbf{v}A)_p = \zeta_p\theta(\zeta)_p - \theta_p(\nabla_\zeta^v \zeta(p)).$$

Write $p = u \cdot g$. Because each $u \in P$ is an affine map from fiber $\pi^{-1}(x)$ and G , $x = \pi_P(u)$, we conclude that

$$\begin{aligned}\nabla^{v*}\theta(A, \mathbf{v}A)_p &= u_*\xi_g u^*\theta(\xi_g) - u^*\theta(\nabla_\xi^h \xi(g)) \\ &= \nabla^G(u^*\theta)(\xi, \xi)(g),\end{aligned}$$

where ξ is the vector field on G yielded by $\omega(A)$, by left translations, and the proof is complete. \square

From the Lemma above we can see vertical martingales in the $P(M, G)$.

Proposition 7.6. *Under the above assumptions, a semimartingale X_t in $P(M, G)$ is a vertical martingale if and only if V_t is a ∇^h -martingale.*

Proof. Let X_t be a semimartingale in P such that $X_t = X_t^h \cdot V_t$ as it was explained above. Take a vertical form θ on $P(M, G)$. By Proposition 4.4,

$$\int_0^t \theta d^v X_s = \int_0^t \theta \delta^v (X_s^h \cdot V_s) - \frac{1}{2} \int_0^t \nabla^{v*}\theta(dX, \mathbf{v}dX)_s.$$

A simple account yields

$$\int_0^t \theta \delta^v (X_s^h \cdot V_s) = \int_0^t R_{V_s}^* \theta \delta^v X_s^h + \int_0^t X_s^{h*} \theta \delta V_s = \int_0^t X_s^{h*} \theta \delta V_s,$$

because X_t^h is a horizontal semimartingale. From this and Lemma above we conclude that

$$\int_0^t \theta d^v X_s = \int_0^t X_s^{h*} \theta \delta V_s - \frac{1}{2} \int_0^t \nabla^{h*}(X_s^{h*} \theta)(dV, dV) = \int_0^t X_s^{h*} \theta d^h V_s,$$

where we use at last equality a formula to convert Stratonovich's integral to the Itô's integral. Since θ is arbitrary, it follows that X_t is a vertical martingale if and only if V_t is a ∇^h -martingale. \square

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