Communications on Stochastic Analysis
Vol. 7, No. 4 (2013) 573-589
Serials Publications
www.serialspublications.com

GENERALIZATION OF THE ANTICIPATIVE GIRSANOV THEOREM

HUI-HSIUNG KUO, YUN PENG, AND BENEDYKT SZOZDA*

Abstract. We study the Itô formula and Girsanov theorem in the anticipative setting using the stochastic integral of adapted and instantly independent processes. The results of the present paper extend several of the previously known theorems. The generalization presented here can be summarized as a domain extension as we allow for a more general class of processes to be treated by the Itô formula and more general shifts to be used in the change of measure in the Girsanov theorem. Finally, we apply our results to present a toy problem of the Black–Scholes formula for a market that knows the future but not the past.

1. Introduction

In the present paper, we extend and generalize the results of [10] to obtain a version of the Girsanov theorem for a Brownian motion translated by a mixture of adapted and backward-adapted terms. Our setup and notation follow closely those of [10]. We let $(\Omega, \mathcal{F}, P)$ be a complete probability space, B_t be a Brownian motion defined on $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t \colon 0 \leq t \leq T\}$, be its natural filtration, that is $\mathcal{F}_t = \sigma\{B_s \colon 0 \leq s \leq t\}$. Since we are only interested in a finite time horizon, we fix it to be T. For the sake of brevity, we will write $\{\mathcal{F}_t\}$ for $\{\mathcal{F}_t \colon 0 \leq t \leq T\}$ and $\{f_t\}$ for $\{f_t \colon 0 \leq t \leq T\}$. If $\{f_t\}$ is a square-integrable stochastic process adapted to $\{\mathcal{F}_t\}$, we denote by $\mathcal{E}_t(f)$ the *stochastic exponential* associated to f defined by

$$\mathcal{E}_t(f) = \exp\left\{\int_0^t f_s \, dB_s - \frac{1}{2}\int_0^t f_s^2 \, ds\right\}.$$

First, we recall several theorems that are building blocks in many areas of application of stochastic analysis, *e.g.* financial mathematics. Namely, the Itô formula and the Girsanov theorem. We are concerned with generalization of these theorems to an anticipative setting that is based on a new stochastic integral introduced by Ayed and Kuo in [1, 2] and later developed by Kuo, Sae-Tang and Szozda in [11, 12, 13] and by Khalifa et al. in [8].

Received 2013-11-12; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60H05; Secondary 60H20.
Key words and phrases. Brownian motion, Itô integral, Itô formula, adapted stochastic processes, instantly independent stochastic processes, anticipating stochastic processes, stochastic integral, anticipating integral, Girsanov theorem, Black–Scholes formula.
* Benedykt Szozda acknowledges support from the T.N. Thiele Centre for Applied Mathematics in Natural Science and from CREATES (DNRF78), funded by the Danish National Research Foundation.

The first statement of the Girsanov theorem in the setting of the new stochastic integral appears in [10]. In the present paper we present a generalization of the results of [10] as well as generalization of some of the results on Itô formula from [12]. We review the relevant results on the new stochastic integral in Section 2 and recall the results that we intend to generalize in Section 3 where we also present the first extensions.

Section 4 contains an extension of the results of [12] while Section 5 contains the main results of the present paper, that is the generalization of the Girsanov theorem to the Brownian motion shifted by mixture of adapted and backward-adapted stochastic processes. Finally, in Section 6 we apply the results of Section 3 to obtain a Black–Scholes type formula for stock whose price is driven by backward-adapted processes.

To conclude the introduction, let us state two classic results that are generalized in the forthcoming sections. Namely the Itô formula and the Girsanov theorem.

Theorem 1.1 (Itô Formula – adapted). *Suppose that* $\{X_t^{(i)} : i = 1, 2, \ldots, n\}$ *are continuous martingales with respect to* $\{\mathcal{F}_t : 0 \le t \le T\}$ *and* $f(x_1, x_2, \ldots, x_n)$ *is a twice continuously differentiable real function on* $\mathbb{R}^n$*. Then*

$$\begin{aligned} f\big(X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n)}\big) = f\big(X_0^{(1)}, X_0^{(2)}, \ldots, X_0^{(n)}\big) \\ + \int_0^t \sum_{i=1}^n \frac{\partial}{\partial x_i} f\big(X_s^{(1)}, X_s^{(2)}, \ldots, X_s^{(n)}\big)\, dX_i \\ + \frac{1}{2} \int_0^t \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} f\big(X_s^{(1)}, X_s^{(2)}, \ldots, X_s^{(n)}\big)\, d\langle X_i, X_j\rangle_t, \end{aligned}$$

where $\langle X, Y\rangle_t$ *stands for predictable covariation of* $\{X_t\}$ *and* $\{Y_t\}$*.*

Theorem 1.2 (Girsanov, 1960, [6]). *Let* $\{f_t\}$ *be a square-integrable stochastic process adapted to* $\{\mathcal{F}_t\}$ *such that* $\mathbb{E}_P\big[\mathcal{E}_t(f)\big] < \infty$ *for all* $t \in [0, T]$*. Then,*

$$\widetilde{B}_t = B_t - \int_0^t f_s\, ds$$

is a Brownian motion with respect to an equivalent probability measure Q*, given by*

$$dQ = \mathcal{E}_T(f)\, dP.$$

2. The New Integral

As we have mentioned in the introduction, Ayed and Kuo [1, 2] introduced a new approach to stochastic integration of anticipating stochastic processes. Below we briefly recall their construction.

A stochastic process $\{\varphi_t\}$ is said to be *instantly independent* of the filtration $\{\mathcal{F}_t\}$ if for each $t \in [0, T]$, the random variable φ_t and the σ-field $\mathcal{F}_t$ are independent. For example $\varphi(B_1 - B_t)$ is instantly independent of $\{\mathcal{F}_t : t \in [0, 1]\}$ for any real measurable function $\varphi(x)$. Observe that for $t \ge 1$, $\varphi(B_1 - B_t)$ is adapted to $\{\mathcal{F}_t\}$.

Definition 2.1. Suppose that $\{f_t\}$ is a stochastic process adapted to the filtration $\{\mathcal{F}_t\}$ and $\{\varphi_t\}$ is instantly independent of the same filtration. We define the *stochastic integral* of $f_t\varphi_t$ as

$$\int_0^T f_t\varphi_t\, dB_t = \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n f_{t_{i-1}}\varphi_{t_i}\Delta B_i,$$

where $\Delta_n = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ is a partition of the interval $[0, T]$ and $\Delta B_i = B_{t_i} - B_{t_{i-1}}$ and $\|\Delta_n\| = \max\{t_i - t_{i-1} \colon i = 1, 2, \ldots n\}$, provided the limit exists in probability.

One of the central concepts in stochastic analysis is that of a martingale. Below we recall some basic facts about martingales and their instantly independent counterpart, near-martingales. The latter kind of processes were introduced and studied by Kuo, Sae-Tang and Szozda in [11]. They also appear in [3], where the authors call them *increment martingales*.

Definition 2.2. A stochastic process $\{X_t\}$ is said to be a *martingale* with respect to a filtration $\{\mathcal{F}_t\}$ if $E\,|X_t| < \infty$ for all $t \in [0, T]$ and

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s,\ 0 \le s < t \le T.$$

Taking into consideration the definition of the conditional expectation, we immediately see that $\{X_t\}$ as defined above is adapted to $\{\mathcal{F}_t\}$. It is, therefore, not feasible in the anticipating setting. However, in [11], authors define a *near-martingale*, which, as we will see, serves as an instantly independent counterpart to martingales.

Definition 2.3. We say that a process $\{X_t\}$ is a *near-martingale* with respect to a filtration $\{\mathcal{F}_t\}$ if $E\,|X_t| < \infty$ for all $0 \le t \le T$ and $\mathbb{E}[X_t - X_s|\mathcal{F}_s] = 0$ for all $0 \le s < t \le T$.

It is not hard to see that an adapted near-martingale is a martingale. For more properties of near-martingales we refer to [11].

It is a well-known fact that the Itô integral is a martingale, that is $X_t = \int_0^t f_s\, dB_s$ is a martingale with respect to $\{\mathcal{F}_t\}$, for any adapted stochastic process $\{f_t\}$ that is integrable with respect to B_t on the interval $[0, T]$. Similarly, if f_t and φ_t are as in Definition 2.1, and $Y^{(t)} = \int_0^t f_s\varphi_s\, dB_s$ exists for all $t \in [0, T]$, then $Y^{(t)}$ is a near-martingale with respect to $\{\mathcal{F}_t\}$ (see [11, Theorem 3.5].) Furthermore, $Y^{(t)}$ is also a near-martingale with respect to a *natural backward filtration* $\{\mathcal{G}^{(t)}\}$ of B_t defined by

$$\mathcal{G}^{(t)} = \sigma\{B_T - B_s \colon t \le s \le T\}.$$

For details see [11, Theorem 3.7]. In general, a *backward filtration* is any decreasing family of σ-fields, *i.e.* $\{\mathcal{G}^{(t)}\}$ satisfies $\mathcal{G}^{(t)} \subseteq \mathcal{G}^{(s)}$ for any $0 \le s \le t \le T$. A similar concept is also used in [14]. A process adapted to the natural backward Brownian filtration will be called *backward-adapted*.

Before we proceed, let us introduce a backward Brownian motion $B^{(t)}$, that is a process given by

$$B^{(t)} = B_T - B_{T-t}.$$

It is in fact a Brownian motion in the filtration $\overline{\mathcal{G}}^{(t)}$ (see [10, Proposition 3.2]) given by

$$\overline{\mathcal{G}}^{(t)} = \mathcal{G}^{(T-t)}.$$

Notice that this is a forward filtration induced by the backward filtration $\mathcal{G}^{(t)}$ of the underlying Brownian motion B_t.

3. From instantly independent to Backward-Adapted Processes

In this section we present a generalization of several results from [10]. Before we proceed with proofs of the new results, let us recall the versions of the theorems from [10].

Theorem 3.1 (Itô formula, [10, Theorem 3.4]). *Suppose that*

$$Y_i^{(t)} = \int_t^T h_i(B_t - B_s)\, dB_s + \int_t^T g_i(B_t - B_s)\, ds, \qquad i = 1, 2, \ldots, n,$$

where h_i, g_i, $i = 1, 2, \ldots, n$ are continuous, square-integrable functions. Then for any i, Y_i is instantly independent with respect to $\{\mathcal{F}_t\}$. Furthermore, let $f(x_1, x_2, \ldots, x_n)$ be a function in $C^2(\mathbb{R}^n)$. Then

$$df\big(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)}\big) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\big(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)}\big)\, dY_i^{(t)} - \frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}\big(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)}\big)\big(dY_i^{(t)}\big)\big(dY_j^{(t)}\big).$$

Theorem 3.2 ([10, Theorem 4.4]). *Suppose that $\{B_t\}$ is a Brownian motion on $(\Omega, \mathcal{F}_T, P)$ and $\varphi(x)$ is a square-integrable function on $\mathbb{R}$. Let*

$$\widetilde{B}_t = B_t + \int_0^t \varphi(B_T - B_s)\, ds.$$

Then $\widetilde{B}_t$ is a continuous near-martingale with respect to the probability measure Q given by

$$dQ = \exp\left\{-\int_0^T \varphi(B_T - B_s)\, dB_s - \frac{1}{2}\int_0^T \varphi^2(B_T - B_s)\, ds\right\} dP.$$

Theorem 3.3 ([10, Theorem 4.5]). *Suppose that the assumptions of Theorem 3.2 hold and*

$$\widehat{B}_t = B_T - B_t + \int_t^T \varphi(B_T - B_s)\, ds.$$

Then $\widehat{B}_t^2 - (T - t)$ is a continuous Q-near-martingale.

Theorem 3.4 ([10, Theorem 4.6]). *Suppose that the assumptions of Theorem 3.2 hold. Then the Q-quadratic variation of $\tilde{B}$ on the interval $[0, t]$ is equal to t.*

In the present paper, we weaken the assumptions of Theorems 3.1–3.4. In general, the main improvement lies in the fact that we drop the explicit dependence on the tail of Brownian motion in favor of adaptedness to the backward filtration.

That is, instead of representing the underlying function as $f(B_T - B_t)$, we assume that f_t is a backward-adapted stochastic process.

Notice that this is in fact a generalization for if f is any measurable real-valued function, it follows that $f(B_T - B_t)$ is adapted to the natural backward Brownian filtration $\{\mathcal{G}^{(t)}\}$. Moreover, it is a nontrivial generalization. A simple example that is not in the scope of the theory of [10] is the following. For a square-integrable real-valued function g define

$$\theta_t = \int_t^T g(B_T - B_s)\, dB_s.$$

Then $\{\theta_t\}$ is backward-adapted and (in general) cannot be expressed as $\theta_t = f(B_T - B_t)$. Of course, backward-adapted processes are instantly independent, but not all instantly independent stochastic processes are backward-adapted.

We begin with a generalized version of the Itô formula in Theorem 3.1. Namely, we replace the integrands $h_i(B_T - B_s)$ and $g_i(B_T - B_s)$ by any processes that are backward-adapted.

To prove the Itô Formula, we need the following technical lemma. It is a direct generalization of [10, Lemma 3.3].

Lemma 3.5. *Suppose that $\{B_t\}$ is a Brownian motion and $\{B^{(t)}\}$ is its backward Brownian motion, that is $B^{(t)} = B_T - B_{T-t}$ for all $0 \le t \le T$. Suppose also that g_t is a square-integrable process adapted to $\{\mathcal{G}^{(t)}\}$,. Then the following two identities hold*

$$\int_t^T g_s\, ds = \int_0^{T-t} g_{T-s}\, ds \tag{3.1}$$

$$\int_t^T g_s\, dB_s = \int_0^{T-t} g_{T-s}\, dB^{(s)}. \tag{3.2}$$

Proof. Let us first show that Equation (3.1) holds. Note that application of a change of variables $\overline{s} = T - s$ in the right side of Equation (3.1) yields

$$\int_0^{T-t} g_{T-s}\, ds = -\int_T^t g_{\overline{s}}\, d\overline{s} = \int_t^T g_{\overline{s}}\, d\overline{s}.$$

Thus the validity of Equation (3.1) is proven.

Next, we show that Equation (3.2) holds. By the definition of the stochastic integral the right side of Equation (3.2) becomes

$$\begin{aligned} \int_0^{T-t} g_{T-s}\, dB^{(s)} &= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n g_{T-t_{i-1}} \big(B^{(t_i)} - B^{(t_{i-1})}\big) \\ &= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^n g_{T-t_{i-1}} (B_{T-t_{i-1}} - B_{T-t_i}), \end{aligned} \tag{3.3}$$

where Δ_n is a partition of the interval $[0, T-t]$ and the convergence is understood to be in probability on the space $\big(\Omega, \overline{\mathcal{G}}^{(T)}, P\big)$. A change of variables,$\bar{t}_i = T - t_i$,

$i = 1, 2, \ldots, n$ transforms Equation (3.3) into

$$\begin{aligned}\int_0^{T-t} g_{T-s}\, dB^{(s)} &= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} g_{\bar{t}_{i-1}} (B_{\bar{t}_{i-1}} - B_{\bar{t}_i}) \\ &= \lim_{\|\Delta_n\|\to 0} \sum_{i=1}^{n} g_{\bar{t}_{i-1}} (B_{\bar{t}_{i-1}} - B_{\bar{t}_i}).\end{aligned} \tag{3.4}$$

Since $T = \bar{t}_0 > \bar{t}_1 > \bar{t}_2 > \cdots > \bar{t}_n = t$ can be chosen arbitrarily, and the probability space $\left(\Omega, \overline{\mathcal{G}}^{(T)}, P\right)$ coincides with $(\Omega, \mathcal{F}_T, P)$, see [10, Theorem 3.1], by the definition of the new stochastic integral, the last term in Equation (3.4) converges in probability to the new stochastic integral

$$\int_t^T g_s\, dB_s.$$

Hence the Equation (3.2) holds. □

Now we are ready to prove the generalization of the Itô formula.

Theorem 3.6. *Suppose that*

$$Y_i^{(t)} = \int_t^T h_i^{(s)}\, dB(s) + \int_t^T g_i^{(s)}\, ds \quad i = 1, 2, \ldots, n,$$

where $h_i^{(s)}, g_i^{(s)}$ *for* $i = 1, 2, \ldots, n$ *are continuous square-integrable stochastic processes that are adapted to* $\{\mathcal{G}^{(t)}\}$. *Then for any* $i = 1, 2, \ldots, n$, Y_i *is instantly independent with respect to* $\mathcal{F}_t$. *Let furthermore* $f(x_1, x_2, \ldots, x_n)$ *be a function in* $C^2(\mathbb{R}^n)$, *we have following Itô Formula,*

$$\begin{aligned} df(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)}) = &\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)})\, dY_i^{(t)} \\ &- \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_1^{(t)}, Y_2^{(t)}, \ldots, Y_n^{(t)}) \left(dY_i^{(t)}\right)\left(dY_j^{(t)}\right).\end{aligned} \tag{3.5}$$

Proof. Since the only difference between the arguments establishing the one- and multi-dimensional cases is the amount of bookkeeping, we will only show that Equation (3.5) holds with $n = 1$. For the sake of clarity of notation, we let

$$Y^{(t)} = \int_t^T h_s\, dB_s + \int_t^T g_s\, ds.$$

Let us define

$$X_t = \int_0^t h_{T-s}\, dB^{(s)} + \int_0^t g_{T-s}\, ds.$$

Since h_s and g_s are adapted to $\{\mathcal{G}^{(t)}\}$, we can view X_t as an Itô integral on the probability space $(\Omega, \overline{\mathcal{G}}^{(0)}, P)$. Application of the classic Itô Formula and the Itô table yield

$$\begin{aligned} f(X_{T-t}) - f(X_0) &= \int_0^{T-t} f'(X_s)\, dX_s + \frac{1}{2}\int_0^{T-t} f''(X_s)\,(dX_s)^2 \\ &= \int_0^{T-t} f'(X_s) h_{T-s}\, dB^{(s)} + \int_0^{T-t} f'(X_s) g_{T-s}\, ds \\ &\quad + \frac{1}{2}\int_0^{T-t} f''(X_s) h_{T-s}^2\, ds. \end{aligned} \tag{3.6}$$

By Lemma 3.5 we have the following identities

$$\begin{aligned} X_{T-t} &= Y^{(t)}, \\ \int_0^{T-t} f'(X_s) h_{T-s}\, dB^{(s)} &= \int_t^T f'(X_{T-s}) h_s\, dB_s, \\ \int_0^{T-t} f'(X_s) g_{T-s}\, ds &= \int_t^T f'(X_{T-s}) g_s\, ds, \\ \int_0^{T-t} f''(X_s) h_{T-s}^2\, ds &= \int_t^T f''(X_{T-s}) h_s^2\, ds. \end{aligned} \tag{3.7}$$

Putting Equations (3.6) and (3.7) together gives

$$f(Y^{(t)}) - f(Y^{(T)}) = \int_t^T f'(Y^{(s)}) h_s\, dB_s + \int_t^T f'(Y^{(s)}) g_s\, ds \tag{3.8}$$

$$+ \frac{1}{2}\int_t^T f''(Y^{(s)}) h_s^2\, ds. \tag{3.9}$$

Notice that $dY^{(t)} = -h_t\, dB_t - g_t\, dt$ and $\big(dY^{(t)}\big)^2 = h_t^2\, dt$. Using the above in Equation (3.9) and changing to the differential notation yields

$$df\big(Y^{(t)}\big) = f'\big(Y^{(t)}\big)\, dY^{(t)} - \frac{1}{2} f''\big(Y^{(t)}\big)\,\big(dY^{(t)}\big)^2,$$

which ends the proof. □

Since it is not difficult to derive a corollary to Theorem 3.6 that covers the case when the function f depends explicitly on time, we state it without a proof.

Corollary 3.7. *Suppose that*

$$Y_i^{(t)} = \int_t^T h_i^{(s)}\, dB(s) + \int_t^T g_i^{(s)}\, ds \quad i = 1, 2, \ldots, n,$$

where $h_i^{(s)}, g_i^{(s)}$ for $i = 1, 2, \ldots, n$ are continuous square-integrable stochastic processes that are adapted to $\big\{\mathcal{G}^{(t)}\big\}$,. Suppose also that $f(x_1, x_2, \ldots, x_n, t)$ is a function twice continuously differentiable in the first n variables and once continuously *differentiable in the last variable. Then,*

$$\begin{aligned} &df(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) \\ &\quad = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t)\, dY_i^{(t)} \\ &\qquad - \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t) \left(dY_i^{(t)}\right) \left(dY_j^{(t)}\right) \\ &\qquad + \frac{\partial f}{\partial t}(Y_1^{(t)}, Y_2^{(t)}, \dots, Y_n^{(t)}, t). \end{aligned}$$

Using Theorem 3.6, we can easily find the counterpart to the exponential process for any process θ_t adapted to the backward filtration $\{\mathcal{G}^{(t)}\}$.

Example 3.8. Suppose that θ_t is a square-integrable stochastic process adapted to $\{\mathcal{G}^{(t)}\}$ and let

$$\mathcal{E}^{(t)}(\theta) = \exp\left\{ -\int_t^T \theta_s \, dB_s - \frac{1}{2} \int_t^T \theta_s^2 \, ds \right\}.$$

Then

$$d\mathcal{E}^{(t)}(\theta) = \theta_t \mathcal{E}^{(t)}(\theta)\, dB_t.$$

The process $\mathcal{E}^{(t)}(\theta)$ is called an *exponential process* of the backward-adapted process θ_t.

Proof. Let $f(x) = e^x$ and define

$$Y_t = -\int_t^T \theta_s \, dB_s - \frac{1}{2} \int_t^T \theta_s^2 \, ds.$$

Since $f(x) = f'(x) = f''(x)$ and $f(Y_t) = \mathcal{E}^{(t)}(\theta)$, application of Theorem 3.6 to $f(Y_t)$, yields

$$\begin{aligned} d\mathcal{E}^{(t)}(\theta) &= df(Y_t) \\ &= f'(Y_t)\, dY_t - \tfrac{1}{2} f''(Y_t)\, (dY_t)^2 \\ &= e^{Y_t} \left(\theta_t \, dB_t + \tfrac{1}{2}\theta_t^2 \, dt\right) - \tfrac{1}{2} e^{Y_t} \theta_t^2 \, dt \\ &= \theta_t \mathcal{E}^{(t)}(\theta)\, dB_t. \end{aligned}$$

Above we have used the fact that $dY_t = \theta_t \, dB_t + \frac{1}{2}\theta_t^2 \, dt$. □

Next, we generalize Theorems 3.2–3.4 in the same spirit as Theorem 3.6 generalizes Theorem 3.1. We begin with a theorem that is an extension of Theorem 3.2.

Theorem 3.9. *Suppose that $\{B_t\}$ is a Brownian motion on $(\Omega, \mathcal{F}_T, P)$ and φ_t is a square-integrable real-valued stochastic process adapted to $\{\mathcal{G}^{(t)}\}$. Let*

$$\widetilde{B}_t = B_t + \int_0^t \varphi(B_T - B_s)\, ds. \tag{3.10}$$

Then $\widetilde{B}_t$ is a continuous near-martingale with respect to the probability measure Q given by

$$\begin{aligned} dQ &= \mathcal{E}^{(0)}(\varphi)\, dP \\ &= \exp\biggl\{-\int_0^T \varphi_s\, dB_s - \frac{1}{2}\int_0^T \varphi_s^2\, ds\biggr\}\, dP. \end{aligned} \tag{3.11}$$

The following theorem generalizes Theorem 3.3.

Theorem 3.10. *Suppose that $\{B_t\}$ is a Brownian motion on $(\Omega, \mathcal{F}_T, P)$. Suppose also that φ_t is a square-integrable real-valued stochastic process adapted to $\{\mathcal{G}^{(t)}\}$ and Q is the probability measure given by Equation* (3.11)*. Let*

$$\widehat{B}_t = B_T - B_t + \int_t^T \varphi(B_T - B_s)\, ds.$$

Then $\widehat{B}_t^2 - (T - t)$ is a continuous Q-near-martingale.

Finally, we state the generalization of Theorem 3.4.

Theorem 3.11. *Suppose that $\{B_t\}$ is a Brownian motion in the probability space $(\Omega, \mathcal{F}_T, P)$, Q is a measure given by Equation* (3.11) *and $\tilde{B}$ be given by Equation* (3.10)*. Then the Q-quadratic variation of $\tilde{B}$ on the interval $[0, t]$ is equal to t.*

Proofs of Theorems 3.9–3.11 follow the lines of the proofs of Theorems 4.4–4.6 of [10] with processes of the form $g(B_T - B_t)$ substituted for backward-adapted processes g_t. For the sake of brevity we omit the details and refer an interested reader to [10].

4. Itô Formula for Mixed Terms

In Section 3, we proved the Itô formula for the backward-adapted Itô processes. The obvious limitation of the aforementioned Itô formula is the fact that it can treat functions that depend on the backward-adapted processes only. In the present section, we prove a more general result that is applicable to function depending on adapted and backward-adapted Itô processes.

The *adapted Itô process* is a stochastic process of the form

$$X_t = \int_0^t h_s\, dB(s) + \int_0^t g_s\, ds, \tag{4.1}$$

where h_t, g_t are adapted square-integrable processes. The *backward-adapted Itô process* is a stochastic process of the form

$$Y^{(t)} = \int_t^T \eta_s\, dB(s) + \int_t^T \zeta_s\, ds, \tag{4.2}$$

where η_t, ζ_t are backward-adapted processes.

The classic Itô formula is applicable to functions of X_t, while Theorem 3.6 is applicable to functions of $Y^{(t)}$. The next theorem constitutes an Itô formula for functions that depend on both types of processes. It is a first step towards a general Itô formula and it only applies to functions of the form $\theta(X_t, Y^{(t)})$, where

$\theta(x,y) = f(x)\varphi(y)$. The first Itô formula of this type was introduced in [12, Theorem 5.1], where authors treated only the case when η, ζ are deterministic functions. Thus, while our arguments are similar to those of [12], our result extends the result of [12] substantially.

Theorem 4.1. *Suppose that $\theta(x,y)$ is a function of the form $\theta(x,y) = f(x)\varphi(y)$, where f and φ are twice continuously differentiable real-valued functions. Suppose also that X_t and $Y^{(t)}$ are defined as in Equations* (4.1) *and* (4.2) *respectively. Then*

$$\theta(X_T, Y^{(T)}) = \theta(X_0, Y^{(0)}) + \int_0^t \frac{\partial\theta}{\partial x}(x_t, y^{(t)})\, dx_t + \frac{1}{2}\int_0^t \frac{\partial^2\theta}{\partial x^2}(x_t, y^{(t)})\,(dx_t)^2 + \int_0^t \frac{\partial\theta}{\partial y}(x_t, y^{(t)})\, dy^{(t)} - \frac{1}{2}\int_0^t \frac{\partial^2\theta}{\partial y^2}(x_t, y^{(t)})\,(dy^{(t)})^2.$$

Proof. We begin by writing out $\theta(x_t, y^{(t)}) - \theta(x_0, y^{(0)})$ as a telescoping sum. for any partition $\delta_n = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t\}$, we have

$$\begin{aligned}\theta(x_t, y^{(t)}) - \theta(x_0, y^{(0)}) &= \sum_{i=1}^n \big[\theta(x_{t_i}, y^{(t_i)}) - \theta(x_{t_{i-1}}, y^{(t_{i-1})})\big] \\ &= \sum_{i=1}^n \big[f(x_{t_i})\varphi(y^{(t_i)}) - f(x_{t_{i-1}})\varphi(y^{(t_{i-1})})\big]. \end{aligned} \tag{4.3}$$

now, we apply the taylor expansion to f and φ, to obtain

$$\begin{aligned} f(x_i) &= \sum_{k=0}^\infty \frac{1}{k!} f^{(k)}(x_{t_{i-1}})(\delta x_i)^k \\ \varphi(y^{(t_{i-1})}) &= \sum_{k=0}^\infty \frac{1}{k!}\varphi^{(k)}(y^{(t_i)})(-\delta y_i)^k, \end{aligned}$$

where $\Delta X_i = X_{t_i} - X_{t_{i-1}}$ and $\Delta Y_i = Y^{(t_i)} - Y^{(t_{i-1})}$. Using the standard approximation results for the Brownian motion and adapted Itô processes, we obtain the following approximations

$$\begin{aligned} \Delta X_i &\approx h_{t_{i-1}}\Delta B_i + g_{t_{i-1}}\Delta t_i. \\ (\Delta X_i)^2 &\approx h_{t_{i-1}}^2 \Delta t_i \\ (\Delta X_i)^k &= o(\Delta t_i) \qquad \text{for } k \geq 3 \end{aligned} \tag{4.4}$$

To obtain a result for ΔY_i analogous to the first of Equations (4.4), we employ Lemma 3.5

$$\begin{aligned} \Delta Y_i &= \int_{t_i}^T \eta_s\, dB_s + \int_{t_i}^T \zeta_s\, ds - \int_{t_{i-1}}^T \eta_s\, dB_s - \int_{t_{i-1}}^T \zeta_s\, ds \\ &= \int_0^{T-t_i} \eta_{T-s}\, dB^{(s)} - \int_0^{T-t_{i-1}} \eta_{T-s}\, dB^{(s)} - \int_{t_{i-1}}^{t_i} \zeta_s\, ds \\ &= -\int_{T-t_i}^{T-t_{i-1}} \eta_{T-s}\, dB^{(s)} - \int_{t_{i-1}}^{t_i} \zeta_s\, ds. \end{aligned} \tag{4.5}$$

Now, the first of the integrals in Equation (4.5) can be viewed as a standard Itô integral of an adapted process with respect to a Brownian motion $B^{(t)}$ in its natural filtration $\overline{\mathcal{G}}^{(t)}$. Notice that since $T - t_{i-1} > T - t_i$, Equation (4.5) can be approximated as

$$\Delta Y_i \approx -\eta_{T-(T-t_i)}\Delta B_i - \zeta_{t_i}\Delta t_i = -\eta_{t_i}\Delta B_i - \zeta_{t_i}\Delta t_i.$$

Thus,

$$(\Delta Y_i)^2 \approx \eta_{t_i}^2 \Delta t_i \qquad \text{and} \qquad (\Delta Y_i)^k \approx o(\Delta t_i) \text{ for } k \geq 3. \tag{4.6}$$

Putting Equations (4.3) and (4.5)–(4.6) together yields

$$\begin{aligned}
&\theta(X_T, Y^{(T)}) - \theta(X_0, Y^{(0)}) \\
&\quad = \sum_{i=1}^{n} \bigg\{ f'(X_{t_{i-1}})\varphi(Y^{(t_i)})\big[h_{t_{i-1}}\Delta B_i + g_{t_{i-1}}\Delta t_i\big] + \frac{1}{2} f''(X_{t_{i-1}})\varphi(Y^{(t_i)})h_{t_{i-1}}^2 \Delta t_i \\
&\qquad\qquad + f(X_{t_{i-1}})\varphi'(Y^{(t_i)})\big[-\eta_{t_i}\Delta B_i - \zeta_{t_i}\Delta t_i\big] - \frac{1}{2} f(X_{t_{i-1}})\varphi''(Y^{(t_i)})\eta_{t_i}^2 \Delta t_i \bigg\}.
\end{aligned}$$

Using Definition 2.1 of the new stochastic integral, the definition of the Itô integral and letting n go to ∞, we obtain

$$\begin{aligned}
&\theta(X_T, Y^{(T)}) - \theta(X_0, Y^{(0)}) \\
&\quad = \int_0^T f'(X_t)\varphi(Y^{(t)})h_t \, dB_t + \int_0^T f'(X_t)\varphi(Y^{(t)})g_t \, dt \\
&\qquad + \frac{1}{2}\int_0^T f''(X_t)\varphi(Y^{(t)})h_t^2 \, dt - \int_0^T f(X_t)\varphi'(Y^{(t)})\eta_t \, dB_t \\
&\qquad - \int_0^T f(X_t)\varphi'(Y^{(t)})\zeta_t \, dt - \frac{1}{2}\int_0^T f(X_t)\varphi''(Y^{(t)})\eta_t^2 \, dt \\
&\quad = \int_0^T \frac{\partial\theta}{\partial x}(X_t, Y^{(t)}) \, dX_t + \frac{1}{2}\int_0^T \frac{\partial^2\theta}{\partial x^2}(X_t, Y^{(t)}) \, (dX_t)^2 \\
&\qquad + \int_0^T \frac{\partial\theta}{\partial y}(X_t, Y^{(t)}) \, dY^{(t)} - \frac{1}{2}\int_0^T \frac{\partial^2\theta}{\partial y^2}(X_t, Y^{(t)}) \, (dY^{(t)})^2.
\end{aligned}$$

This proves our claim. □

5. Girsanov Theorem for Mixture of Adapted and Anticipative Shifts

The main result of the present paper follows from application of the classic Girsanov Theorem 1.2 as well as Theorems 3.9–3.11 that constitute an anticipative version of the Girsanov theorem. The improvement of Theorems 5.1–5.4 over Theorems 3.9–3.11 lays in the fact that we allow for the translations of Brownian motion that can be decomposed into a sum of processes that are either adapted to $\{\mathcal{F}_t\}$ or adapted to $\{\mathcal{G}^{(t)}\}$. In this setting, we find that the Girsanov type results have the exact same form as Theorem 1.2.

Theorem 5.1. *Suppose that $\{B_t\}$ is a Brownian Motion and $\{\mathcal{F}_t\}$ is its natural filtration on probability space $(\Omega, \mathcal{F}, P)$. let f_t and g_t be continuous square-integrable* *stochastic processes such that* f_t *is adapted to* $\{\mathcal{F}_t\}$ *and* g_t *is adapted to* $\{\mathcal{G}^{(t)}\}$, i.e. *the backward Brownian filtration. Let*

$$\widetilde{B}_t = B_t + \int_0^t (f_s + g_s)\, ds.$$

Then $\widetilde{B}_t$ *is a near-martingale with respect to* $(\Omega, \mathcal{F}, Q)$, *where*

$$dQ = \exp\,\{-\int_0^T (f_t + g_t)\, dB_t - \frac{1}{2}\int_0^T (f_t + g_t)^2\, dt\}\, dP. \tag{5.1}$$

Remark 5.2. This theorem can be proved with methods similar to the ones used in [10], that is by defining the exponential process for a sum of processes f and g adapted to $\{\mathcal{F}_t\}$ and $\{\mathcal{G}^{(t)}\}$ respectively, and using the results of the preceding sections to repeat the calculations done in [10]. However, since the results that are applicable to translations of Brownian motion by $\int_0^t f(s)\, ds$ and $\int_0^t g(s)\, ds$ separately already exist, we can apply them to obtain a shorter proof.

Proof. First, let us rewrite $\widetilde{B}_t$ as

$$\widetilde{B}_t = B_t + \int_0^t f_s\, ds + \int_0^t g_s\, ds$$

and define $W_t = B_t + \int_0^t f_s\, ds$. Thus $\widetilde{B}_t = W_t + \int_0^t g_s\, ds$. Since f_t is adapted, application of the original Girsanov theorem yields that W_t is a Brownian motion with respect to $(\Omega, \mathcal{F}_T, Q_1)$ where

$$dQ_1 = \exp\bigl\{-\int_0^T f_t\, dB_t - \frac{1}{2}\int_0^T f_t^2\, dt\bigr\}\, dP \tag{5.2}$$

Now, since W_t is a Brownian motion on $(\Omega, \mathcal{F}_T, Q_1)$ and g_t is adapted to the backward filtration $\{\mathcal{G}^{(t)}\}$, we can apply Theorem 3.11. Therefore, $\widetilde{B}(t)$ is a near-martingale with respect to $(\Omega, \mathcal{F}_T, Q)$, where

$$dQ = \exp\{-\int_0^T g_t\, dW_t - \frac{1}{2}\int_0^T g_t^2\, dt\}\, dQ_1 \tag{5.3}$$

with dQ_1 given by Equation (5.2).

It remains to show that the measure Q in Equation (5.3) coincides with the measure Q in Equation (5.1). To this end, we put together the identity $dW_t = dB_t + f_t dt$, Equation (5.2) and Equation (5.3) obtain

$$\begin{aligned}
dQ &= \exp\Bigl\{-\int_0^T g_t\, dW_t - \frac{1}{2}\int_0^T g_t^2\, dt\Bigr\} dQ_1 \\
&= \exp\Bigl\{-\int_0^T g_t\, dW_t - \frac{1}{2}\int_0^T g_t^2\, dt - \int_0^T f_t\, dB_t - \frac{1}{2}\int_0^T f_t^2\, dt\Bigr\} dP \\
&= \exp\Bigl\{-\int_0^T g_t\, dB_t - \int_0^T g_t f_t\, dt - \frac{1}{2}\int_0^T g_t^2\, dt - \int_0^T f_t\, dB_t - \frac{1}{2}\int_0^T f_t^2\, dt\Bigr\} dP \\
&= \exp\Bigl\{-\int_0^T (g_t + f_t)\, dB_t - \frac{1}{2}\int_0^T (g_t + f_t)^2\, dt\Bigr\} dP.
\end{aligned}$$

Thus the theorem holds. □

Next we state generalization of Theorem 3.10.

Theorem 5.3. *Suppose that assumptions of Theorem 5.1 hold. Let*

$$\widehat{B}_t = B_T - B_t + \int_t^T \big(f_s + g_s\big)\, ds.$$

Then $\widehat{B}_t^2 - (T-t)$ *is a continuous* Q*-near-martingale.*

Finally, we give the generalization of Theorem 3.11.

Theorem 5.4. *Suppose that the assumptions of Theorem 5.1 hold. Then the* Q*-quadratic variation of* $\tilde{B}$ *on the interval* $[0,t]$ *is equal to* t.

Note that the proofs of Theorems 5.3 and 5.4 follow the same reasoning as the proof of Theorem 5.1, that is one first applies the adapted version of the Girsanov theorem (see Theorem 1.2) and then applies one of Theorems 3.10 or 3.11. We omit these proofs for the sake of brevity.

Remark 5.5. Using the relationship between probability measures Q and Q_1 given by Equation (5.3) from the proof of Theorem 5.1 we can deduce an interesting stochastic differential equation. To this end we will follow the lines of Example 3.8. From Equation (5.3) we have

$$dQ = \exp\{-\int_0^T g_t\, dW_t - \frac{1}{2}\int_0^T g_t^2\, dt\}\, dQ_1.$$

Let us define

$$\theta^{(t)}(g) = \exp\{-\int_t^T g_s\, dW_s - \frac{1}{2}\int_t^T g_s^2\, ds\},$$

Clearly, according to Example 3.8, $\theta^{(t)}(g)$ is a backward exponential process for the backward-adapted stochastic process g_t in the space $(\Omega, \mathcal{F}_T, Q_1)$. Thus we have the following SDE

$$\begin{aligned} d\theta^{(t)}(g) &= g_t\theta^{(t)}(g)\, dW_t \\ &= g_t\theta^{(t)}(g)\, (dB_t + f_t dt) \\ &= g_t\theta^{(t)}(g)\, dB_t + f_t g_t \theta^{(t)}(g)\, dt. \end{aligned}$$

The above equation may give some insight into Itô formulas for processes that are adapted to neither $\{\mathcal{F}_t\}$ nor $\{\mathcal{G}^{(t)}\}$ as the last term in the above equation is a stochastic process of the form

$$X_t = \int_0^t f_s \varphi_s\, ds,$$

with f and φ being adapted to $\{\mathcal{F}_t\}$ and $\{\mathcal{G}^{(t)}\}$ respectively.

We conclude this section with an example.

Example 5.6. Let

$$X_t = B_t + \int_0^t B_1\, dB_s,$$

where B_s is a Brownian motion on the probability space $(\Omega, \mathcal{F}_T, P)$. Define the equivalent probability measure Q by

$$dQ = \exp\{-\int_0^T B_1\, dB_t - \frac{1}{2}\int_0^T B_1^2\, dt\}.$$

Using Theorems 5.1–5.4, we conclude that X_t is a near-martingale in the probability space $(\Omega, \mathcal{F}_T, Q)$, its quadratic variation on the interval $[0, t]$ is equal to t and if

$$\widetilde{X}_t = X_T - X_t = B_T - B_t + \int_t^T B_1\, dB_s,$$

then $\widetilde{X}_t^2 - (T - t)$ is a near martingale on $(\Omega, \mathcal{F}_T, Q)$.

Note that the conclusions of this example cannot be obtained with the classic Girsanov Theorem 1.2 as the B_1 is not adapted to $\{\mathcal{F}_t\}$. It is also not possible to approach this example with results of [10] or of Section 3 of the present paper because B_1 is not adapted to $\{\mathcal{G}^{(t)}\}$. However, we can rewrite B_1 as

$$B_1 = (B_1 - B_t) + B_t,$$

where $(B_1 - B_t)$ is adapted to $\{\mathcal{G}^{(t)}\}$ and B_t is adapted to $\{\mathcal{F}_t\}$. In the view of the above equation, Theorems 5.1–5.4 are applicable.

6. Black–Scholes Equation in the Backward Case

In this section we discuss a simple scenario of Black–Scholes model in the backward-adapted setting. The outline of this approach comes from [4, Chapter 7]. In our setting, the market is composed of two assets. The first asset is a risk-free bond whose price D_t is driven by a deterministic differential equation

$$dD_t = rD_t\, dt,$$

where r is the risk-free interest rate. The second asset is a stock (or some security) S_t, whose price is dependent on the information right after time t and driven by a stochastic differential equation

$$dS_t = S_t\alpha_t\, dt + S_t\sigma_t\, dB_t,$$

where α_t and σ_t are both adapted to $\{\mathcal{G}^{(t)}\}$. This can be viewed as a special case of "insider information", where the "insider" uses only the knowledge unavailable to the rest of the market as the processes α_t and σ_t are completely out of the scope of the natural forward Brownian filtration, but instead are adapted to the natural backward Brownian filtration. The backward Brownian filtration describes exactly the future information generated by the driving Brownian process that is independent of the current or past state of the market.

We assume there is a contingent claim $\Phi(S_T)$, which is tradable on the market and whose price process is given by

$$\Pi_t = F(t, S_t)$$

for some smooth function $F(x, y)$. Our goal is to find a function F such that the market is arbitrage-free. Using Corollary 3.7, we have

$$\begin{aligned}
d\Pi_t &= dF(t, S_t) \\
&= \frac{\partial F}{\partial y}(t, S_t)\, dS_t - \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t, S_t)\,(dS_t)^2 + \frac{\partial F}{\partial x}(t, S_t)\, dt \\
&= \frac{\partial F}{\partial y}(t, S_t)(S_t \alpha_t\, dt + S_t \sigma_t\, dB_t) - \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t, S_t)(S_t^2 \sigma_t^2\, dt) + \frac{\partial F}{\partial x}(t, S_t)\, dt \\
&= \left(\frac{\partial F}{\partial y}(t, S_t) S_t \alpha_t - \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t, S_t) S_t^2 \sigma_t^2 + \frac{\partial F}{\partial x}(t, S_t)\right) dt \\
&\quad + \left(\frac{\partial F}{\partial y}(t, S_t) S_t \sigma_t\right) dB_t \\
&= \alpha_t^{\Pi} \Pi_t\, dt + \sigma_t^{\Pi} \Pi_t\, dB_t,
\end{aligned}$$

where

$$\begin{aligned}
\alpha_t^{\Pi} &= \frac{\frac{\partial F}{\partial y}(t, S_t) S_t \alpha_t - \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t, S_t) S_t^2 \sigma_t^2 + \frac{\partial F}{\partial x}(t, S_t)}{F(t, S_t)}, \\
\sigma_t^{\Pi} &= \frac{\frac{\partial F}{\partial y}(t, S_t) S_t \sigma_t}{F(t, S_t)}.
\end{aligned} \tag{6.1}$$

We now form a relative portfolio consisting of the stock and the contingent claim. We denote by u_t^S the percentage of stock in the portfolio at time t and by u_t^{Π} the percentage of the contingent claim in our portfolio. Thus the portfolio is given by (u_t^S, u_t^{Π}) with the restriction that $u_t^S + u_t^{\Pi} = 1$ for all t. Assuming that our portfolio is self-financing and without consumption or transaction costs, we obtain the following SDE for the dynamics of the value of the portfolio V

$$\begin{aligned}
dV_t &= V_t u_t^S \frac{sS_t}{S_t} + V_t u_t^{\Pi} \frac{d\Pi_t}{\Pi_t} \\
&= V_t \Big(u_t^S \big[\alpha_t\, dt + \sigma_t\, dB_t\big] + u_t^{\Pi} \big[\alpha_t^{\Pi}\, dt + \sigma_t^{\Pi}\, dB_t\big] \Big) \\
&= V_t \Big(\big[u_t^S \alpha_t + u_t^{\Pi} \alpha_t^{\Pi}\big] dt + \big[u_t^S \sigma_t + u_t^{\Pi} \sigma_t^{\Pi}\big] dB_t \Big).
\end{aligned}$$

In order to obtain a risk-free portfolio, we need to ensure that there is no stochastic part in the equation above. Moreover, in order to ensure that the new financial instrument does not introduce the arbitrage to the market, the interest rate of the value process of the risk-free portfolio needs to coincide with the interest rate of the risk-free bond, namely r. That is, together with the structural constraints on the portfolio, we have

$$u_t^S + u_t^{\Pi} = 1 \tag{6.2}$$

$$u_t^S \alpha_t + u_t^{\Pi} \alpha_t^{\Pi} = r \tag{6.3}$$

$$u_t^S \sigma_t + u_t^{\Pi} \sigma_t^{\Pi} = 0. \tag{6.4}$$

Equations (6.2) and (6.4) yield

$$u_t^S = -\frac{\sigma_t^\Pi}{\sigma_t - \sigma_t^\Pi}, \qquad u_t^\Pi = \frac{\sigma_t}{\sigma_t - \sigma_t^\Pi}. \tag{6.5}$$

Putting together Equations (6.5) and (6.1), we obtain

$$u_t^S = \frac{\frac{\partial F}{\partial y}(t, S_t)S_t}{\frac{\partial F}{\partial y}(t, S_t)S_t - F(t, S_t)}, \qquad u_t^\Pi = \frac{F(t, S_t)}{F(t, S_t) - \frac{\partial F}{\partial y}(t, S_t)S_t} \tag{6.6}$$

Now, together with Equation (6.3) and the terminal condition that comes from the form of the contingent claim Π, Equation (6.6) yields

$$\begin{cases} \frac{\partial F}{\partial x}(t, S_t) + \frac{\partial F}{\partial y}(t, S_t)rS_t - \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t, S_t)S_t^2\sigma_t^2 - F(t, S_t)r = 0 \\ F(T, s) = \Phi(s). \end{cases}$$

Observe that unlike with the classic Black–Scholes formula, in the above PDE we have a minus in front of the term with $\frac{\partial^2 F}{\partial y^2}$. This change of sign enters through the Itô formula for the backward-adapted processes. Intuitively this can be explained by the fact that the difference between the classic Black–Scholes model and our example is that of a different point of view. That is the former model looks forward with the information on the past and the latter looks backward with the information from the future. Thus the influence of the volatility (σ_t) will have opposite effects in the two models.

Of course, the above example is rather simple and not realistic on its own, however one might use it together with the classic Black–Scholes model to study the influence of the insider information on the market.

References

1. Ayed, W. and Kuo, H.-H.: An extension of the Itô integral, *Communications on Stochastic Analysis* **2**, no. 3 (2008) 323–333.
2. Ayed, W. and Kuo, H.-H.: An extension of the Itô integral: toward a general theory of stochastic integration, *Theory of Stochastic Processes* **16(32)**, no. 1 (2010) 1–11.
3. Basse-O'Connor, A., Graversen, S.-E., and Pedersen, J.: Stochastic integration on the real line *Theory of Probability and Its Applications*, (to appear).
4. Björk, T.: *Arbitrage Theory in Continuous Time* Oxford University Press, 2004.
5. Buckdahn, R.: Anticipative Girsanov transformations and Skorohod stochastic differential equations, Memoirs of the American Mathematical Society, Vol. 111, no. 533 (1994).
6. Girsanov, I. V.: On Transforming a Certain Class of Stochastic Processes by Absolutely Continuous Substitution of Measures, *Theory of Probability & Its Applications* **5**, no. 3 (1960) 285–301.
7. Itô, K.: Stochastic integral, *Proceedings of the Imperial Academy* **20**, no. 8 (1944) 519–524.
8. Khalifa, N., Kuo, H.-H., Ouerdiane, H., and Szozda, B.: Linear stochastic differential equations with anticipating initial conditions. *Communications on Stochastic Analysis* **7**, no. 2 (2013) 245–253.
9. Kuo, H.-H.: *Introduction to Stochastic Integration*, Universitext, Springer, 2006.
10. Kuo, H.-H., Peng, Y., and Szozda, B.: Itô formula and Girsanov theorem for anticipating stochastic integrals *Communications on Stochastic Analysis* **7**, no. 3 (2013) 441–458.
11. Kuo, H.-H., Sae-Tang, A., and Szozda, B.: A stochastic integral for adapted and instantly independent stochastic processes, in "*Advances in Statistics, Probability and Actuarial Science*" Vol. I, Stochastic Processes, Finance and Control: A Festschrift in Honour of Robert J. Elliott (eds.: Cohen, S., Madan, D., Siu, T. and Yang, H.), World Scientific, 2012, 53–71.

12. Kuo, H.-H., Sae-Tang, A., and Szozda, B.: The Itô formula for a new stochastic integral, *Communications on Stochastic Analysis* **6**, no. 4 (2012) 603–614.
13. Kuo, H.-H., Sae-Tang, A., and Szozda, B.: An isometry formula for a new stochastic integral, In "*Proceedings of International Conference on Quantum Probability and Related Topics*," May 29–June 4, 2011, Levico, Italy, *QP–PQ: Quantum Probability and White Noise Analysis* **29** (2013) 222–232.
14. Pardoux, É. and Protter, P.: A two-sided stochastic integral and its calculus, *Probability Theory and Related Fields*, **76**, no. 1 (1987) 15–49.

Hui-Hsiung Kuo: Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
E-mail address: kuo@math.lsu.edu
URL: http://www.math.lsu.edu/∼kuo

Yun Peng: Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA
E-mail address: ypeng8@tigers.lsu.edu

Benedykt Szozda: The T.N. Thiele Centre for Mathematics in Natural Science, Department of Mathematical Sciences, & CREATES, School of Economics and Management, Aarhus University, DK-8000 Aarhus C, Denmark
E-mail address: szozda@imf.au.dk
URL: http://home.imf.au.dk/szozda