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MATHEMATICAL MODEL OF HEAVY DIFFUSION PARTICLES SYSTEM WITH DRIFT

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ABSTRACT. In the article we consider the model of coalescing diffusion particles which have some masses. At the moment of coalescing the masses of the particles are summed together and influence their motions. The system of processes that describes evolution of the particles is constructed by martingale methods. The Markov property of this system is stated and the asymptotic restriction on the mass growing of an individual particle is obtained.

1. Introduction

This paper is devoted to the construction a mathematical model of coalescing diffusion particles on $\mathbb{R}$. We assume that every particle has a mass, which influences its diffusion and drift. The particles start from a finite or countable set of points, move independently up to the moment of meeting, after which they coalesce and their masses are summed.

Systems of coalescing diffusion particles were studied by Arratia R. A. [1, 2], Le Jan Y. [15], Norris J. [18], Evans S. S. [8], Dawson D. A. [3, 4], Dorogovtsev A. A. [5, 6], Konarovskyi V. V. [13, 14, 12] and others. Particular attention is paid to a fairly wide class of coalescing particles systems, in which every subsystem may be described as a separate system [15, 17, 8, 1, 9]. On the one hand, such systems are widely used in turbulence theory and statistical mechanics [18, 10], on the other hand they represent an important interest in terms of mathematics itself. For example, the fact that the particles which start from an arbitrary compact set, instantly coalesce to the finite number [8], allows to integrate over a stochastic flow [5], and the latter, in turn, develops a new stochastic analysis. It should be noted that the ability to describe the motion of an arbitrary subsystem of the system, without taking into consideration all the particles of the system, allows to develop good methods for the study of appropriate mathematical models.

Often there is a need to assume that the particles transfer some mass. Models in which the particles have mass are actively studied. However, in some models a mass that is transferred does not influence their motion [3, 4, 21, 20], while in others, it influences but the particles don't coalesce (smooth interaction) [5].

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In the study of systems in which the particles transfer some mass, from the physical point of view it is natural to assume that in coalescing the mass is preserved (the mass of the new particle is equal to the sum of the masses of particles, from which it was formed) and influences their motion. A system of Brownian particles which have masses that are summed together at the moment of coalescing was constructively constructed in the papers [13, 14]. Moreover, the diffusion of particles changes only when particles are changing their masses. In this case, the random environment in which particles diffuse, is homogeneous. The desire to make the model that was under consideration earlier, more close to reality (to include heterogeneity and drift) leads to the fact that we have to consider the diffusion coefficients, which depend not only on the mass, but also on the position of the particles. So, it is assumed that the trajectory $x(t),\ t \geq 0$, of a particle satisfies the following stochastic differential equation

$$dx(t) = \frac{a(x(t))}{m(t)}dt + \frac{\sigma(x(t))}{\sqrt{m(t)}}dw(t),$$

where $m(t)$ is a mass of the particles at the moment t, $w(t),\ t \geq 0$, some Wiener process, $a,\ \sigma$ are bounded Lipschitz continuous functions and $\inf_{x\in\mathbb{R}} \sigma(x) > 0$. We call such system of the particles the heavy diffusion particles system with drift.

It should be noted that in this case the system can not be described by specifying its finite subsystems, as it was done in the work [15, 17, 8, 9]. So, first a mathematical model of a finite number of particles is constructed, after we do the passing to the limit as the number of particles tend to infinity. Ability of passing to the limit ensures that the particles which are far from an isolated subsystem of finite system of particles, have little effect on it (Lemma 4.6).

This work consists of two parts. In the first part it is constructed the mathematical model of a finite number of particles (Section 2) and its Markov property is stated (Section 3). In the second part the passing to the limit as the number of particles tend to infinity is done and some properties of the infinite system of particles are shown (Sections 5, 6).

2. Finite Particles System

In this section it is studied the case of finite number of particles. The set of processes that describe such motion, is constructed by coalescing and rescaling of the solutions of stochastic differential equations

$$dx_i(t) = a(x_i(t))dt + \sigma(x_i(t))dw_i(t),$$

where $w_i(t),\ t \geq 0,\ i = 1, \dots, N$ are independent Wiener processes.

Let $N \in \mathbb{N}$ is fixed. Denote $[N] = \{1, 2, \dots, N\}$.

Definition 2.1. A set $\pi = \{\pi_1, \dots, \pi_p\}$ of non-intersection subsets of $[N]$ is called *order partitioning* of $[N]$ if

1) $\bigcup_{i=1}^{p} \pi_i = [N]$;
2) if $l, k \in \pi_i$ then $\{l \wedge k, \dots, l \vee k\} \subseteq \pi_i$, for all $i = 1, \dots, p$.

The set of all order partitioning of $[N]$ is denoted by Π^N.

Every element $\pi = \{\pi_1, \dots, \pi_p\} \in \Pi^N$ generates equivalence between $[N]$ elements. We assume that $i \sim_\pi j$ if there exists a number k such that $i, j \in \pi_k$. Denote an equivalence class that contain the element $i \in [N]$ by $\widehat{i}_\pi$, i.e. $\widehat{i}_\pi = \{j \in [N] : \ j \sim_\pi i\}$.

Let $\gamma : [N] \to [N]$ be some bijection. Define

$$i_\pi^\gamma = \gamma^{-1} \left(\min_{j \in \widehat{i}_\pi} \gamma(j) \right).$$

Remark 2.2. The map γ will define range of particle. We will suppose that i-th particle has range $\gamma(i)$.

If (R, r) is some metric space then we denote by C_R the space of continuous functions from $[0, \infty)$ to $\mathbb{R}$ with metric

$$d_R(\xi, \eta) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left(\max_{t \in [0,k]} r(\xi(t), \eta(t)) \wedge 1 \right), \quad \xi, \eta \in \mathrm{C}_R.$$

Consider the subspace $E^N = \{x \in \mathbb{R}^N : \ x_i \le x_{i+1}, \ i = 1, \dots, N-1\}$ of the space $\mathbb{R}^N$ and the set $B^N = \{b \in \mathbb{R}^N : \ b_i > 0, \ i = 1, \dots, N\}$. Elements of space E^N and B^N will be used as start points and masses of particles, respectively.

Take $b \in B^N$ and construct a map Λ_γ^b from $\{\xi \in \mathrm{C}_{\mathbb{R}^N} : \ \xi(0) \in E^N\}$ to $\mathrm{C}_{\mathbb{R}^N}$. It will be used to define a system of processes that describes the joint motion of the N particles system. Let $\xi \in \mathrm{C}_{\mathbb{R}^N}$ and $\xi(0) \in E^N$. Construct an element $\zeta = \Lambda_\gamma^b \xi$ by induction.

Take $\pi^0 \in \Pi^N$ such that

$$i \sim_{\pi^0} j \Leftrightarrow \xi_i(0) = \xi_j(0).$$

Put $\tau_0 = 0$ and

$$\zeta_i^0 = \xi_{i_{\pi^0}^\gamma} \left(\frac{t}{\sum_{j \in \widehat{i}_{\pi^0}} b_j} \right), \quad t \ge 0, \ i = 1, \dots, N.$$

Let π^k, τ_k, ζ_i^k, $i = 1, \dots, N$ are defined. Denote

$$\tau_{k+1} = \inf\{t > \tau_k : \ \zeta_i^k(t) = \zeta_j^k(t), \ i \not\sim_{\pi^k} j, \ i, j = 1, \dots, N\}.$$

If $\tau_{k+1} = \infty$ then put $\pi^{k+1} = \pi^k$, else take an element $\pi^{k+1} \in \Pi^N$ such that

$$i \sim_{\pi^{k+1}} j \Leftrightarrow \zeta_i^k(\tau_{k+1}) = \zeta_j^k(\tau_{k+1}).$$

Define

$$\zeta_i^{k+1}(t) = \begin{cases} \zeta_i^k(t), & t < \tau_{k+1}, \\ \zeta_{i_{\pi^{k+1}}^\gamma} \left(\tau_{k+1} + \frac{(t - \tau_{k+1}) \sum_{j \in \widehat{i}_{\pi^k}} b_j}{\sum_{j \in \widehat{i}_{\pi^{k+1}}} b_j} \right), & t \ge \tau_{k+1}. \end{cases}$$

Put $\Lambda_\gamma^b \xi = \zeta^{N-1}$.

Remark 2.3. Λ_γ is measurable map from the space $(L_n, \mathcal{L}_n)$ to $(\mathrm{C}_{\mathbb{R}^N}, \mathcal{B}(\mathrm{C}_{\mathbb{R}^N}))$, where $L_n = \left\{ f \in \mathrm{C}_{\mathbb{R}^N} : f(0) \in E^N \right\}$, $\mathcal{L}_n = \mathcal{B}(\mathrm{C}_{\mathbb{R}^N}) \cap L_n$.

Let's state the main result of this section.

Theorem 2.4. *Let $\gamma : [N] \to [N]$ be some bijection, $x \in E^N$, $b \in B^N$ and $\xi_i(t)$, $t \geq 0$, $i = 1, \ldots, N$, be solutions of the stochastic differential equations*

$$\begin{cases} d\xi_i(t) = a(\xi_i(t))dt + \sigma(\xi_i(t))dw_i(t), \\ \xi_i(0) = x_i, \end{cases} \tag{2.1}$$

were $w_i(t)$, $t \geq 0$, $i = 1, \ldots, N$, are independent Wiener processes, a, σ are some bounded Lipschitz continuous functions on $\mathbb{R}$ and $\inf_{x \in \mathbb{R}} \sigma(x) > 0$. Then the random process

$$\zeta = \Lambda_\gamma^b \xi$$

satisfies the following conditions

1°) $\mathfrak{M}_i = \zeta_i(\cdot) - \int_0^{\cdot} \frac{a(\zeta_i(s))}{m_i(s)} ds$ *is a continuous square integrable martingale with respect to the filtration*
$$\mathcal{F}_t^\zeta = \sigma(\zeta_i(s),\ s \leq t,\ i = 1, \ldots, N),$$
where $m_i(t) = \sum_{j \in A_i(t)} b_j$, $A_i(t) = \{j : \ \exists s \leq t \ \zeta_j(s) = \zeta_i(s)\}$;

2°) $\zeta_i(0) = x_i$, $i = 1, \ldots, N$;

3°) $\zeta_i(t) \leq \zeta_j(t)$, $i < j$, $t \geq 0$;

4°) $\langle \mathfrak{M}_i \rangle_t = \int_0^t \frac{\sigma^2(\zeta_i(s))}{m_i(s)} ds$, $t \geq 0$;

5°) $\langle \mathfrak{M}_i, \mathfrak{M}_j \rangle_t \mathbb{I}_{\{t < \tau_{i,j}\}} = 0$, $t \geq 0$, *where* $\tau_{i,j} = \inf\{t : \zeta_i(t) = \zeta_j(t)\}$.

Remark 2.5. Further, unless otherwise stated we assume that a, σ are a bounded Lipschitz continuous functions on $\mathbb{R}$ and $\inf_{x \in \mathbb{R}} \sigma(x) > 0$.

The proof of the theorem follows from the construction of mapping Λ_γ^b and the next lemma.

Lemma 2.6. *Let $w_i(t)$, $t \geq 0$, $i = 1, \ldots, N$, be a set of independent Wiener processes, τ be a stopping time with respect to the filtration $\mathcal{F}_t^w = \sigma(w_i(s),\ s \leq t,\ i = 1, \ldots, N)$ and a random variable ξ be strictly positive measurable with respect to $\mathcal{F}_\tau^w$. Then*

$$\widehat{w}_i(t) = \begin{cases} w_i(t), & \textit{if } t < \tau, \\ w_i(\tau) + \frac{1}{\sqrt{\xi}}[w_i(\tau + (t - \tau)\xi) - w_i(\tau)], & \textit{else} \end{cases}$$

are independent Wiener processes, moreover, τ is a stopping time with respect to $\mathcal{F}_t^{\widehat{w}} = \sigma(\widehat{w}_i(s),\ s \leq t,\ i = 1, \ldots, N)$ and random variable ξ is measurable with respect to $\mathcal{F}_\tau^{\widehat{w}}$.

Theorem 2.4 describes evolution of the finite heavy diffusion particles system with drift. Let's prove that the conditions 1°)-5°) uniquely determine the distribution of such particles system.

Lemma 2.7. *Suppose that a system of processes $\zeta_i(t)$, $t \geq 0$, $i = 1, \ldots, N$, satisfies the condition 1°)-5°) of Theorem 2.4 and $\gamma : [N] \to [N]$ is some bijection. Then there exists a system of independent Wiener processes $w_i(t)$, $t \geq 0$, $i = 1, \ldots, N$, such that*

$$\zeta = \Lambda_\gamma^b \xi,$$

where $\xi_i(t)$, $t \geq 0$, $i = 1, \dots, N$, are solutions of the stochastic differential equations

$$\begin{cases} d\xi_i(t) = a(\xi_i(t))dt + \sigma(\xi_i(t))dw_i(t), \\ \xi_i(0) = x_i. \end{cases}$$

Proof. Suppose that $\zeta_i(t)$, $t \geq 0$, $i = 1, \dots, N$, satisfy the conditions 1°)-5°) of Theorem 2.4. We first show that the processes ζ_i and ζ_j coalesce at the moment of the meeting, for all $i, j = 1, \dots, N$, i.e.

$$\mathbb{P}\{\zeta_i(\tau_{i,j} + t) = \zeta_j(\tau_{i,j} + t),\ t \geq 0 \mid \tau_{i,j} < \infty\} = 1.$$

Since $\frac{\sigma(\zeta_i(s))}{\sqrt{m_i(s)}} > 0$, $i = 1, \dots, N$, than by the Doob theorem [16] there exists a system of Wiener processes $\widetilde{w}_i(t)$, $t \geq 0$, $i = 1, \dots, N$, adapted to the filtration $\mathcal{F}_t^\zeta$ such that

$$\zeta_i(t) = x_i + \int_0^t \frac{a(\zeta_i(s))}{m_i(s)} ds + \int_0^t \frac{\sigma(\zeta_i(s))}{\sqrt{m_i(s)}} d\widetilde{w}_i(s).$$

Take $i < j$, $n \in \mathbb{N}$ and denote $\tau_{i,j}^n = \tau_{i,j} \wedge n$. From last equation we have

$$\zeta_k(t + \tau_{i,j}^n) = \zeta_k(\tau_{i,j}^n) + \int_0^t \frac{a(\zeta_k(s + \tau_{i,j}^n))}{m_k(s + \tau_{i,j}^n)} ds + \int_0^t \frac{\sigma(\zeta_k(s + \tau_{i,j}^n))}{\sqrt{m_k(s + \tau_{i,j}^n)}} d\widetilde{w}_k^n(s).$$

where $\widetilde{w}_k^n(t) = \widetilde{w}_k(t + \tau_{i,j}^n) - \widetilde{w}_k(\tau_{i,j}^n)$, $t \geq 0$, $k = i, j$. Using the Lipschitz continuity of the function a and equality $m_i(t + \tau_{i,j}^n)\mathbb{I}_{\{\tau_{i,j}^n < n\}} = m_j(t + \tau_{i,j}^n)\mathbb{I}_{\{\tau_{i,j}^n < n\}}$, $t \geq 0$, we obtain

$$(\zeta_j(t + \tau_{i,j}^n) - \zeta_i(t + \tau_{i,j}^n))\mathbb{I}_{\{\tau_{i,j}^n < n\}} \leq L \int_0^t (\zeta_j(s + \tau_{i,j}^n) - \zeta_i(s + \tau_{i,j}^n))\mathbb{I}_{\{\tau_{i,j}^n < n\}} ds$$
$$+ \left[\int_0^t \frac{\sigma(\zeta_j(s + \tau_{i,j}^n))}{\sqrt{m_j(s + \tau_{i,j}^n)}} d\widetilde{w}_j^n(s) - \int_0^t \frac{\sigma(\zeta_i(s + \tau_{i,j}^n))}{\sqrt{m_i(s + \tau_{i,j}^n)}} d\widetilde{w}_i^n(s) \right] \mathbb{I}_{\{\tau_{i,j}^n < n\}}.$$

Since the random variable $\mathbb{I}_{\{\tau_{i,j}^n < n\}}$ is measurable with respect to $\mathcal{F}_{\tau_{i,j}^n}^\zeta$,

$$\mathbf{E}\left[(\zeta_j(t + \tau_{i,j}^n) - \zeta_i(t + \tau_{i,j}^n))\mathbb{I}_{\{\tau_{i,j}^n < n\}}\right]$$
$$\leq \int_0^t \mathbf{E}\left[(\zeta_j(s + \tau_{i,j}^n) - \zeta_i(s + \tau_{i,j}^n))\mathbb{I}_{\{\tau_{i,j}^n < n\}}\right] ds.$$

From Gronwall's inequality we have

$$\mathbf{E}\left[(\zeta_j(t + \tau_{i,j}^n) - \zeta_i(t + \tau_{i,j}^n))\mathbb{I}_{\{\tau_{i,j}^n < n\}}\right] = 0.$$

By Fatou's lemma,

$$\mathbf{E}\left[(\zeta_j(t + \tau_{i,j}) - \zeta_i(t + \tau_{i,j}))\mathbb{I}_{\{\tau_{i,j} < \infty\}}\right] = 0.$$

Hence, by virtue of the continuity of the processes ζ_i and ζ_j, we obtain needed equality.

Next calculate

$$\langle \mathfrak{M}_i, \mathfrak{M}_j \rangle_t \mathbb{I}_{\{t<\tau_{i,j}\}} = \int_0^t \frac{\sigma(\zeta_i(s))\sigma(\zeta_j(s))}{\sqrt{m_i(s)}\sqrt{m_j(s)}} d\langle \widetilde{w}_i, \widetilde{w}_j \rangle_s \mathbb{I}_{\{t<\tau_{i,j}\}} = 0.$$

Hence

$$\langle \widetilde{w}_i, \widetilde{w}_j \rangle_t \mathbb{I}_{\{t<\tau_{i,j}\}} = 0.$$

Let's take a system of Wiener processes $w_i'(t),\ t \geq 0,\ i = 1, \ldots, N$, that are independent of $\widetilde{w}_i(t),\ t \geq 0,\ i = 1, \ldots, N$, and denote

$$\delta_j = \inf\{t:\ \zeta_{\gamma^{-1}(j)}(t) \in \{\zeta_{\gamma^{-1}(1)}(t), \ldots, \zeta_{\gamma^{-1}(j-1)}(t)\}\}, \quad j = 2, \ldots, N.$$

$$\widehat{w}_i(t) = \begin{cases} \widetilde{w}_i(t), & \text{if } t < \delta_{\gamma(i)}, \\ \widetilde{w}_i(\delta_{\gamma(i)}) + w_i'(t) - w_i'(\delta_{\gamma(i)}), & \text{else,} \end{cases}$$

where $\delta_1 = +\infty$ and $i = 1, \ldots, N$. By the Levi theorem (see Theorem 2.6.1 [11]) $\widehat{w}_i(t),\ t \geq 0,\ i = 1, \ldots, N$, are a system of independent Wiener processes. Let $\pi^0 \in \Pi^N$ such that $i \sim_{\pi^0} j \Leftrightarrow \zeta_i(0) = \zeta_j(0)$ and $\tau_0 = 0$. Set

$$\tau_k = \inf\{t > \tau_{k-1}:\ \zeta_i(t) = \zeta_j(t),\ i \not\sim_{\pi^{k-1}} j,\ i, j = 1, \ldots, N\}$$

and if $\tau_k = \infty$ then put $\pi^k = \pi^{k-1}$, else take an element $\pi^k \in \Pi^N$ such that

$$i \sim_{\pi^k} j \Leftrightarrow \zeta_i(\tau_k) = \zeta_j(\tau_k).$$

Using the system of the processes $\widehat{w}_i(t),\ t \geq 0,\ i = 1, \ldots, N$, the stopping times τ_k and the elements π^k, $k = 0, \ldots, N-1$, one can construct a system of independent Wiener processes $w_i(t),\ t \geq 0,\ i = 1, \ldots, N$, such that

$$\zeta = \Lambda_\gamma^b \xi,$$

where $\xi_i(t)$, $t \geq 0$, $i = 1, \ldots, N$, are solutions of the stochastic differential equations

$$\begin{cases} d\xi_i(t) = a(\xi_i(t))dt + \sigma(\xi_i(t))dw_i(t), \\ \xi_i(0) = x_i. \end{cases}$$

The lemma is proved. □

Corollary 2.8. *The conditions* 1°*)-*5°*) of Theorem 2.4 uniquely determine the distribution of the process in the space* $(\mathrm{C}_{\mathbb{R}^N}, \mathcal{B}(\mathrm{C}_{\mathbb{R}^N}))$.

Definition 2.9. A system of processes is called the *process of heavy diffusion particles* with drift in the space E^N if it satisfies the conditions 1°)-5°) of Theorem 2.4.

3. Strictly Markov Property of the Process of Heavy Diffusion Particles with Drift in the Space E^N.

In this section the strictly Markov property of the heavy diffusion particles with drift is stated. Let $\gamma : [N] \to [N]$ be some bijection, $x \in E^N$, $b \in B^N$, $\xi_i(t)$, $t \geq 0$, $i = 1, \ldots, N$, be solutions of the stochastic differential equations (2.1) and $\zeta = \Lambda_\gamma^b \xi$. Denote by $\mathbb{P}_x^\xi$ the distribution of the random process ξ in the space $\mathrm{C}_{\mathbb{R}^N}$. As is well known (see for instance [7]), $x \to \mathbb{P}_x^\xi(A)$ is Borel function, for all $A \in \mathcal{B}(\mathrm{C}_{\mathbb{R}^N})$. Let

$$\mathbb{P}_x^\zeta = \mathbb{P}_x^\xi \circ (\Lambda_\gamma^b)^{-1}.$$

Then the map $x \to \mathbb{P}_x^\zeta(A)$ is Borel function.

Theorem 3.1. *The set of the distributions* $\{\mathbb{P}_x^\zeta,\ x \in E^N\}$ *is strictly Markov system.*

Proof. Let $\mathcal{F}_t(\mathrm{C}_{\mathbb{R}^N}) = \bigcap\limits_{\varepsilon>0} \bigcap\limits_{x\in E^N} \overline{\mathcal{B}_{t+\varepsilon}(\mathrm{C}_{\mathbb{R}^N})}^{\mathbb{P}_x^\zeta}$, where $\overline{\mathcal{B}_t(\mathrm{C}_{\mathbb{R}^N})}^{\mathbb{P}_x^\zeta}$ denotes the σ-algebra of cylinder sets $\{y \in \mathrm{C}_{\mathbb{R}^N} :\ y(s) \in B\}$, $s \le t$, $B \in \mathcal{B}(\mathbb{R}^N)$, that is completed by all $\mathbb{P}_x^\zeta$-null sets. We will show that, for every bounded $\mathcal{F}_t(\mathrm{C}_{\mathbb{R}^N})$-stopping time τ,

$$\mathbb{P}_x^\zeta(A \cap \{y :\ y(t+\tau(y)) \in \Gamma\}) = \int\limits_A \mathbb{P}^\zeta_{y'(\tau(y'))}\{y :\ y(t) \in \Gamma\}\mathbb{P}_x^\zeta(dy'),$$

where $A \in \mathcal{F}_\tau(\mathrm{C}_{\mathbb{R}^N})$, $\Gamma \in \mathcal{B}(\mathbb{R}^N)$, $x \in E^N$. This will be sufficient to prove our theorem.

Fix $i = 1, \ldots, N$. Since $\mathfrak{N}_i = y_i(\cdot) - \int\limits_0^{\cdot} \frac{a(y_i(s))}{m_i(s)} ds$ is a $(\mathbb{P}_x^\zeta, \mathcal{B}_t(\mathrm{C}_{\mathbb{R}^N}))$-martingale for each $x \in E^N$ and $t \to \mathfrak{N}_i(t)$ is right continuous, $\mathfrak{N}_i$ is also a $(\mathbb{P}_x^\zeta, \mathcal{F}_t(\mathrm{C}_{\mathbb{R}^N}))$-martingale. By Doob's optional sampling theorem $\mathfrak{N}_i(\cdot+\tau)$ is a $(\mathbb{P}_x^\zeta, \mathcal{F}_{t+\tau}(\mathrm{C}_{\mathbb{R}^N}))$-martingale. In particular, for $t > s$ $A \in \mathcal{F}_{s+\tau}(\mathrm{C}_{\mathbb{R}^N})$ and $C \in \mathcal{F}_\tau(\mathrm{C}_{\mathbb{R}^N})$, we have

$$\mathbf{E}_x\left[(\mathfrak{N}_i(t+\tau) - \mathfrak{N}_i(s+\tau))\mathbb{I}_{\{A\cap C\}}\right] = 0.$$

This implies that

$$\mathbf{E}_x\left[(\mathfrak{N}_i(t+\tau) - \mathfrak{N}_i(s+\tau))\mathbb{I}_A | \mathcal{F}_\tau(\mathrm{C}_{\mathbb{R}^N})\right] = 0 \quad \text{for } \mathbb{P}_x^\zeta\text{-a.a. } y.$$

Therefore, if $\widetilde{P}(y, A) = \mathbb{P}_x^\zeta(\theta_\tau^{-1}(A)|\mathcal{F}_\tau(\mathrm{C}_{\mathbb{R}^N}))$, $A \in \mathcal{B}(\mathrm{C}_{\mathbb{R}^N})$ is the regular conditional probability with respect to $\mathcal{F}_\tau(\mathrm{C}_{\mathbb{R}^N})$ (it exists by Theorem 1.3.1 [11]), where $\theta_\tau : \mathrm{C}_{\mathbb{R}^N} \to \mathrm{C}_{\mathbb{R}^N}$ is defined by $(\theta_\tau y)(t) = y(t+\tau(y))$, then $\widetilde{P}(y, \{y' :\ y'(0) = y(\tau(y))\}) = 1$ for $\mathbb{P}_x^\zeta$-a.a. y and $\mathfrak{N}_i$ is a $(\widetilde{P}(y,\cdot), \mathcal{F}_t(\mathrm{C}_{\mathbb{R}^N}))$-martingale. Similarly, $\mathfrak{N}_i^2 - \int\limits_0^t \frac{\sigma^2(y_i'(s))}{m_i(s)} ds$ and $\mathfrak{N}_i(t\wedge\tau_{i,j})\mathfrak{N}_j(t\wedge\tau_{i,j})$ are $(\widetilde{P}(y,\cdot), \mathcal{F}_t(\mathrm{C}_{\mathbb{R}^N}))$-martingales. Hence, by Lemma 2.7, $\widetilde{P}(y,\cdot) = \mathbb{P}^\zeta_{y(\tau(y))}$. Thus, for every $A \in \mathcal{F}_\tau(\mathrm{C}_{\mathbb{R}^N})$ and $\Gamma \in \mathcal{B}(\mathbb{R}^N)$, we have

$$\int\limits_A \mathbb{P}^\zeta_{y'(\tau(y'))}\{y :\ y(t) \in \Gamma\}\mathbb{P}_x^\zeta(dy') = \int\limits_A \widetilde{P}(y', \{y :\ y(t) \in \Gamma\})\mathbb{P}_x^\zeta(dy')$$

$$= \int\limits_A \mathbb{P}_x^\zeta(\{y :\ y(t+\tau(y)) \in \Gamma\}|\mathcal{F}_\tau(\mathrm{C}_{\mathbb{R}^N}))\mathbb{P}_x^\zeta(dy') = \mathbb{P}_x^\zeta(A\cap\{y :\ y(t+\tau(y)) \in \Gamma\}).$$

The theorem is proved. □

4. Infinite Particle System

In this section the countable particles system is considered. The system of processes which describes the motion of the particles is constructed from a finite system of processes by passing to the limit. The following theorem holds.

Theorem 4.1. *Let a, σ be bounded Lipschitz continuity functions and $\inf_{x\in\mathbb{R}} \sigma(x) > 0$. Then for every non-decreasing sequence of real numbers $\{x_i,\ i \in \mathbb{Z}\}$ and sequence of strictly positive real numbers $\{b_i,\ i \in \mathbb{Z}\}$ such that*

$$\overline{\lim_{n\to\pm\infty}}\{(x_{n+1} - x_n) \wedge b_{n+1} \wedge b_n\} > 0, \tag{4.1}$$

there exists a set of processes $\zeta_i(t),\ t \geq 0,\ i \in \mathbb{Z}$, satisfying

1°) $\mathfrak{M}_i = \zeta_i(\cdot) - \int_0^{\cdot} \frac{a(\zeta_i(s))}{m_i(s)} ds$ *is a continuous square integrable martingale with respect to the filtration*

$$\mathcal{F}_t^{\zeta} = \sigma(\zeta_i(s),\ s \leq t,\ i \in \mathbb{Z}),$$

where $m_i(t) = \sum_{j\in A_i(t)} b_j$, $A_i(t) = \{j :\ \exists s \leq t\ \zeta_j(s) = \zeta_i(s)\}$;

2°) $\zeta_i(0) = x_i$, $i \in \mathbb{Z}$;

3°) $\zeta_i(t) \leq \zeta_j(t)$, $i < j$, $t \geq 0$;

4°) $\langle \mathfrak{M}_i \rangle_t = \int_0^t \frac{\sigma^2(\zeta_i(s))}{m_i(s)} ds$, $t \geq 0$;

5°) $\langle \mathfrak{M}_i, \mathfrak{M}_j \rangle_t \mathbb{I}_{\{t<\tau_{i,j}\}} = 0$, $t \geq 0$, *where* $\tau_{i,j} = \inf\{t : \zeta_i(t) = \zeta_j(t)\}$.

Remark 4.2. In case where $m_i(t) = \infty$ we assume that $\frac{1}{m_i(t)} = 0$.

In order to prove the theorem, we will state several auxiliary lemmas.

Let $\{n_i,\ i \in \mathbb{Z}\}$ be some strictly increasing sequence of real numbers. Fix $N \in \mathbb{N}$ and choose a bijection $\gamma^N : [2N + 1] \to [2N + 1]$ as follows. Denote

$$D_1 = \{n_i, n_i + 1,\ i \in \mathbb{Z}\} \cap [0, N] = \{p_1^1, \ldots, p_{k_1}^1\}, \quad p_1^1 < \ldots < p_{k_1}^1,$$
$$D_2 = \{n_i, n_i + 1,\ i \in \mathbb{Z}\} \cap [-N, 0) = \{p_1^2, \ldots, p_{k_2}^2\}, \quad p_1^2 < \ldots < p_{k_2}^2.$$

Let

$$\mathbb{Z} \cap [0, N] \setminus D_1 = \{p_1^3, \ldots, p_{k_3}^3\}, \quad p_1^3 < \ldots < p_{k_3}^3,$$
$$\mathbb{Z} \cap [-N, 0) \setminus D_2 = \{p_1^4, \ldots, p_{k_4}^4\}, \quad p_1^4 < \ldots < p_{k_4}^4.$$

Put

$$\gamma^N(N + 1 + p_i^1) = i, \quad i = 1, \ldots, k_1,$$
$$\gamma^N(N + 1 - p_i^2) = k_1 + i, \quad i = 1, \ldots, k_2,$$
$$\gamma^N(N + 1 + p_i^3) = k_1 + k_2 + i, \quad i = 1, \ldots, k_3,$$
$$\gamma^N(N + 1 - p_i^4) = k_1 + k_2 + k_3 + i, \quad i = 1, \ldots, k_4.$$

Lemma 4.3. *Let $\{n_i,\ i \in \mathbb{Z}\}$ be a strictly increasing sequence of real numbers, $\gamma^N : [2N + 1] \to [2N + 1]$ be the bijection defined above, $\{x_i,\ i \in \mathbb{Z}\}$ be a non-decreasing a sequence of real numbers, $\{b_i,\ i \in \mathbb{Z}\}$ be a sequence of strictly positive numbers and $\{f_k,\ k \in \mathbb{Z}\} \subset \mathrm{C}(\mathbb{R})$, $f_k(0) = x_k$. Denote*

$$(g_{-N}^N, \ldots, g_N^N) = \Lambda_{\gamma^N}^{(b_{-N},\ldots,b_N)}(f_{-N}, \ldots, f_N), \quad N \in \mathbb{N}.$$

(i) *If for some* $m \in \mathbb{N}$ *and* $T > 0$ *there exist* $C > 0$ *and* $\delta > 0$ *such that*

$$\max\left\{ \max_{t\in[0,T]} f_k\left(\frac{t}{b_k}\right),\ k \in \{n_i, n_j+1;\ i = 0,\dots,m,\ j = 0,\dots,m-1\}\right\} < C,$$

$$\min_{t\in[0,T]} f_{n_m+1}\left(\frac{t}{b_{n_m+1}}\right) > C+\delta,$$

then for all $N > n_m$ *and* $k = -N,\dots,n_m$

$$\max_{t\in[0,T]} g_k^N(t) \le \max_{t\in[0,T]} g_{n_m}^N(t) < C, \quad \min_{t\in[0,T]} g_{n_m+1}^N(t) > C+\delta.$$

(ii) *If for some* $-m \in \mathbb{N}$ *there exist* $C < 0$ *and* $\delta < 0$ *such that*

$$\min\left\{ \min_{t\in[0,T]} f_k\left(\frac{t}{b_k}\right);\ k \in \{n_i, n_j+1;\ i = m+1,\dots,0\ j = m,\dots,0\}\right\} > C,$$

$$\max_{t\in[0,T]} f_{n_m}\left(\frac{t}{b_{n_m}}\right) < C+\delta,$$

then for all $N > -n_m$ *and* $k = n_m+1,\dots,N$

$$\min_{t\in[0,T]} g_k^N(t) \ge \min_{t\in[0,T]} g_{n_m+1}^N(t) > C, \quad \max_{t\in[0,T]} g_{n_m}^N(t) < C+\delta.$$

The proof of the lemma immediately follows from the construction of the map $\Lambda_{\gamma^N}^{(b_{-N},\dots,b_N)}$, $N \in \mathbb{N}$, and the choice of the bijection γ^N, $N \in \mathbb{N}$.

Let $f : D \to \mathbb{R}$ be some bounded function. Define $\|f\| = \sup_{x\in D} |f(x)|$.

Lemma 4.4. *Let* a, σ *be a bounded Lipschitz continuous functions and* $\inf_{x\in\mathbb{R}} \sigma(x) > 0$. *Then for each* $\delta > 0$ *and* $T > 0$ *the solution of the equations*

$$\xi(t) = x_0 + \int_0^t a(\xi(s))ds + \int_0^t \sigma(\xi(s))dw(s) \tag{4.2}$$

satisfies following condition

$$\mathbb{P}\left\{\max_{t\in[0,T]} \xi(t) - x_0 < \delta\right\} \ge \mathbb{P}\left\{ w(t) < \delta - \frac{\|a\|}{\inf\limits_{x\in\mathbb{R}} \sigma(x)^2} t, \quad t \in \left[0, T\cdot\|\sigma\|^2\right]\right\}.$$

Lemma 4.5. *For every* $a \in \mathbb{R}$, $\delta > 0$ *and a Wiener process* $w(t),\ t \ge 0$,

$$\mathbb{P}\{w(t) + at < \delta, \quad t \in [0,T]\} > 0.$$

The proof easily follows from the Girsanov theorem.

Lemma 4.6. *Let* $y_n,\ n \in \mathbb{N}$, *be a non-decreasing sequence of real numbers such that* $\inf_{n\ge1}(y_{n+1} - y_n) = \delta > 0$, $\xi_n(t),\ t \ge 0,\ n \in \mathbb{N}$, *be the solutions of the equations*

$$\xi_n(t) = y_n + \int_0^t a(\xi_n(s))ds + \int_0^t \sigma(\xi_n(s))dw_n(s),$$

where $w_n(t),\ t \geq 0,\ n \in \mathbb{N}$, *are a set of Wiener processes and*

$$\xi_n^{max} = \max_{t\in[0,T]} \xi_n(t), \quad \xi_n^{min} = \min_{t\in[0,T]} \xi_n(t).$$

Then for every $\delta_1 \in \left(0, \frac{\delta}{2}\right)$,

$$\mathbb{P}\left\{\overline{\lim_{n\to\infty}}\left\{\max_{k=1,\ldots,n} \xi_k^{max} \leq y_n + \frac{\delta}{2},\ \xi_{n+1}^{min} > y_{n+1} - \delta_1\right\}\right\} = 1.$$

The proof of the lemma is similar to the proof of Lemma 5 [13], considering Lemmas 4.5 and 4.4.

Proof of Theorem 4.1. Since the sequences $\{x_i,\ i \in \mathbb{Z}\}$ and $\{b_i,\ i \in \mathbb{Z}\}$ satisfy the inequality (4.1), there exists a strictly increasing sequence of real numbers $\{n_i,\ i \in \mathbb{Z}\}$ such that

$$\inf_{i\in\mathbb{Z}}(x_{n_i+1} - x_{n_i}) = \delta > 0, \quad \sup_{i\in\mathbb{Z}}\left\{\frac{1}{b_{n_i}}, \frac{1}{b_{n_i+1}}\right\} < \infty.$$

Let $\gamma^N : [2N+1] \to [2N+1]$ be a bijection constructed by the sequence $\{n_i,\ i \in \mathbb{Z}\}$ and $\xi_n(t),\ t \geq 0,\ n \in \mathbb{Z}$, be the solutions of the equations

$$\xi_n(t) = x_n + \int_0^t a(\xi_n(s))ds + \int_0^t \sigma(\xi_n(s))dw_n(s),$$

where $w_n(t),\ t \geq 0,\ n \in \mathbb{Z}$, are independent Wiener processes.

Put

$$(\zeta_{-N}^N, \ldots, \zeta_N^N) = \Lambda_{\gamma^N}^{(b_{-N},\ldots,b_N)}(\xi_{-N}, \ldots, \xi_N),$$

for each $N \in \mathbb{N}$. Fix $T > 0$. By Lemmas 4.3 and 4.6

$$\mathbb{P}\left\{\exists N\ \forall n \geq N\ \zeta_k^n(t) = \zeta_k^N(t),\ t \in [0,T]\right\} = 1,$$

for any $k \in \mathbb{Z}$, i.e. the sequence $\{\zeta_k^n(t),\ t \in [0,T]\}_{n\geq k}$ is stabilized with probability 1, for all integer k. Denote the limit of $\{\zeta_k^n,\ t \in [0,T]\}_{n\geq k}$ by $\zeta_{k,T}$. From the stabilization of $\{\zeta_k^n,\ t \in [0,T]\}_{n\geq k}$ it follows that

$$\mathbb{P}\left\{\exists N\ \forall n \geq N\ m_k^n(t) = m_k^N(t),\ t \in [0,T]\right\} = 1,$$

where $m_k^n(t) = \sum_{j\in A_k^n(t)} b_j$, $A_k^n(t) = \{j :\ \exists s \leq t\ \zeta_j^n(s) = \zeta_k^n(s)\}$, $t \geq 0$. Let $m_{k,T}$ denote the limit of the sequence $\{m_k^n,\ t \in [0,T]\}_{n\geq k}$. Denote $\zeta_k(t) = \zeta_{k,T}(t)$ and $m_k(t) = m_{k,T}(t)$, for some $T \geq t$. It is clear that such definition is correct, moreover,

$$m_i(t) = \sum_{j\in A_i(t)} b_j, \quad A_i(t) = \{j :\ \exists s \leq t\ \zeta_j(s) = \zeta_i(s)\}.$$

From the stabilization it follows that one can construct a system of Wiener processes $\widetilde{w}_n(t),\ t \geq 0,\ n \in \mathbb{Z}$, such that $\langle \widetilde{w}_i, \widetilde{w}_j\rangle_t \mathbb{I}_{\{t<\tau_{i,j}\}} = 0$ and

$$\zeta_n(t) = x_n + \int_0^t \frac{a(\zeta_n(s))}{m_n(s)}ds + \int_0^t \frac{\sigma(\zeta_n(s))}{\sqrt{m_n(s)}}d\widetilde{w}_n(s), \quad n \in \mathbb{Z}.$$

This implies that the system of the processes $\zeta_n(t),\ t \geq 0,\ n \in \mathbb{Z}$ is found. The theorem is proved. $\square$

Lemma 4.7. *Let a sequence of random processes $\zeta_n,\ t \geq 0,\ n \in \mathbb{Z}$, satisfy the conditions 1°)-5°) of Theorem 4.1. Then $m_n(t) < \infty$, for all $t \geq 0$ and $n \in \mathbb{Z}$, i.e.*

$$\mathbb{P}\{\exists j_1, j_2 \quad \zeta_{j_1}(s) < \zeta_n(s) < \zeta_{j_2}(s),\ s \leq t\} = 1.$$

Proof. Let $\{n_i,\ i \in \mathbb{Z}\}$ be a strictly increasing sequence of integer number such that

$$\inf_{i \in \mathbb{Z}}(x_{n_i+1} - x_{n_i}) = \delta > 0, \quad \sup_{i \in \mathbb{Z}} \left\{ \frac{1}{b_{n_i}}, \frac{1}{b_{n_i+1}} \right\} = C < \infty. \tag{4.3}$$

Fix $t > 0$ and take $x > 2tC\|a\|$. Let's estimate following probability for $i < j$

$$\mathbb{P}\{\mathfrak{M}_{n_j}(s) - \mathfrak{M}_{n_i}(s) > x,\ s \leq t\} =$$
$$= \mathbb{P}\left\{ \zeta_{n_j}(s) - \int_0^s \frac{a(\zeta_{n_j}(r))}{m_{n_j}(r)} dr - \zeta_{n_i}(s) + \int_0^s \frac{a(\zeta_{n_i}(r))}{m_{n_i}(r)} dr > x,\ s \leq t \right\} \leq$$
$$\leq \mathbb{P}\left\{\zeta_{n_j}(s) - \zeta_{n_i}(s) + 2tC\|a\| > x,\ s \leq t\right\} \leq \mathbb{P}\left\{\zeta_{n_j}(s) - \zeta_{n_i}(s) > 0,\ s \leq t\right\}$$

Next, let $i \in \mathbb{Z}$ is fixed. For every $m \in \mathbb{N}$ take $y_m \in \mathbb{R}$ and a number n_{j_m} such that

$$\mathbb{P}\{\mathfrak{M}_{n_i}(s) < y_m,\ s \leq t\} \geq 1 - \frac{1}{2m^2}$$

and

$$\mathbb{P}\{\mathfrak{M}_{n_{j_m}}(s) > y_m + x,\ s \leq t\} \geq 1 - \frac{1}{2m^2}.$$

Write

$$\mathbb{P}\{\mathfrak{M}_{n_{j_m}}(s) - \mathfrak{M}_{n_i}(s) > x,\ s \leq t\}$$
$$\geq \mathbb{P}\{\{\mathfrak{M}_{n_{j_m}}(s) > y_m + x,\ s \leq t\} \cap \{\mathfrak{M}_{n_i}(s) < y_m,\ s \leq t\}\} \geq 1 - \frac{1}{m^2}.$$

Hence

$$\mathbb{P}\left\{\zeta_{n_{j_m}}(s) - \zeta_{n_i}(s) > 0,\ s \leq t\right\} \geq 1 - \frac{1}{m^2}.$$

By Borel-Cantelli lemma,

$$\mathbb{P}\left\{ \inf_{s \leq t}(\zeta_{n_{j_m}}(s) - \zeta_{n_i}(s)) = 0, \text{ for infinite numbers } m \right\} = 0.$$

So, we have

$$\mathbb{P}\left\{ \exists j \in \mathbb{Z}\ \inf_{s \leq t}(\zeta_{n_j}(s) - \zeta_{n_i}(s)) > 0 \right\} = 1.$$

Similarly

$$\mathbb{P}\left\{ \exists j \in \mathbb{Z}\ \inf_{s \leq t}(\zeta_{n_i}(s) - \zeta_{n_j}(s)) > 0 \right\} = 1.$$

This proves the lemma. $\square$

Theorem 4.8. *The conditions 1°)-5°) of Theorem 4.1 uniquely determine the distribution of the process in the space $(\mathrm{C}_{\mathbb{R}^\mathbb{Z}}, \mathcal{B}(\mathrm{C}_{\mathbb{R}^\mathbb{Z}}))$.*

Proof. Let a system of random processes $\zeta_n,\ t\geq 0,\ n\in\mathbb{Z}$, satisfy the conditions 1°)-5°) of Theorem 4.1. Then, by the Doob theorem [16], there exists a system of Wiener processes $\widetilde{w}_n(t)$, $t\geq 0$, $n\in\mathbb{Z}$, which are $\mathcal{F}_t^{\zeta}$-adapted, such that

$$\zeta_n(t)=x_n+\int_0^t\frac{a(\zeta_n(s))}{m_n(s)}ds+\int_0^t\frac{\sigma(\zeta_n(s))}{\sqrt{m_n(s)}}d\widetilde{w}_n(s).$$

From condition 5°) we have

$$\langle\widetilde{w}_i,\widetilde{w}_j\rangle_t\mathbb{I}_{\{t<\tau_{i,j}\}}=0,\quad t\geq 0,\quad i,j\in\mathbb{Z}.$$

Similarly to the proof of Lemma 2.7, it is easily seen that $\zeta_i(t)=\zeta_j(t)$ for $t\geq\tau_{i,j}$ and $i,j\in\mathbb{Z}$.

Next, let $\{n_i,\ i\in\mathbb{Z}\}$ be a strictly increasing sequence of integer numbers satisfying (4.3). Construct a bijection $\gamma_\infty:\mathbb{Z}\to\mathbb{N}$ as follows. For every $i\in\mathbb{Z}$ define

$$\gamma_\infty(i)=\lim_{N\to\infty}\gamma^N(i+N+1),$$

where the bijection γ^N, $N\in\mathbb{N}$, defined above. The existence of limit follows from the constructions of γ^N, $N\in\mathbb{N}$.

Take a system of independent Wiener processes $w_i'(t),\ t\geq 0,\ i\in\mathbb{Z}$, which are independent of $\widetilde{w}_i(t),\ t\geq 0,\ i\in\mathbb{Z}$ and denote

$$\delta_j=\inf\{t:\ \zeta_{\gamma_\infty^{-1}(j)}(t)\in\{\zeta_{\gamma_\infty^{-1}(1)}(t),\ldots,\zeta_{\gamma_\infty^{-1}(j-1)}(t)\}\},\quad j=2,3,\ldots.$$

Put

$$\widehat{w}_i(t)=\begin{cases}\widetilde{w}_i(t), & \text{if } t<\delta_{\gamma_\infty(i)},\\ \widetilde{w}_i(\delta_{\gamma_\infty(i)})+w_i'(t)-w_i'(\delta_{\gamma_\infty(i)}), & \text{else,}\end{cases}$$

where $\delta_1=+\infty$ and $i\in\mathbb{Z}$. By the Levi theorem (see Theorem 2.6.1 [11]), $\widehat{w}_i(t),\ t\geq 0,\ i\in\mathbb{Z}$, are independent Wiener processes.

Let $N\in\mathbb{N}$ and take $\pi^{0,N}\in\Pi^{2N+1}$ such that $i\sim_{\pi^{0,N}}j\Leftrightarrow\zeta_{i-N-1}(0)=\zeta_{j-N-1}(0)$ and $\tau_{0,N}=0$. Denote

$$\tau_{k,N}=\inf\{t>\tau_{k-1,N}:\ \zeta_{i-N-1}(t)=\zeta_{j-N-1}(t),\ i\not\sim_{\pi^{k-1,N}}j,\ i,j\in[2N+1]\}$$

and if $\tau_{k,N}=\infty$ then put $\pi^{k,N}=\pi^{k-1,N}$, else take $\pi^{k,N}\in\Pi^{2N+1}$ such that

$$i\sim_{\pi^{k,N}}j\Leftrightarrow\zeta_{i-N-1}(\tau_{k,N})=\zeta_{j-N-1}(\tau_{k,N}).$$

Using the system of the processes $\widehat{w}_i(t),\ t\geq 0,\ i=-N,\ldots,N$, stopping times $\tau_{k,N}$ and the elements $\pi^{k,N}$, $k=0,\ldots,2N$, in reverse order (similar to how it was done in the proof of Theorem 2.4), one can construct a system of independent Wiener processes $w_i^N(t),\ t\geq 0,\ i=-N,\ldots,N$.

From Lemma 4.7 and the construction of $w_i^N(t),\ t\geq 0,\ i=-N,\ldots,N,\ N\in\mathbb{N}$, it follows that for all $k\in\mathbb{Z}$ and $T>0$

$$\mathbb{P}\left\{\exists N'\ \forall N\geq N'\ w_k^N(t)=w_k^{N'}(t),\ t\in[0,T]\right\}=1.$$

Define $w_k=\lim_{N\to\infty}w_k^N$, $k\in\mathbb{Z}$. It is clear that $w_k(t),\ t\geq 0,\ k\in\mathbb{Z}$, are independent Wiener processes.

Let

$$(\zeta_{-N}^N,\ldots,\zeta_N^N)=\Lambda_{\gamma^N}^{(b_{-N},\ldots,b_N)}(\xi_{-N},\ldots,\xi_N),$$

where $\xi_k(t)$, $t \geq 0$, $k \in \mathbb{Z}$, are the solutions of the stochastic differential equations

$$\begin{cases} d\xi_k(t) = a(\xi_k(t))dt + \sigma(\xi_k(t))dw_k(t), \\ \xi_k(0) = x_k. \end{cases}$$

By Lemma 4.6, the sequence $(\zeta_{k_1}^N(t), \ldots, \zeta_{k_p}^N(t))$, $t \in [0, T]$, $p \geq \max\{k_1, \ldots, k_p\}$ is stabilized, when N grows. Moreover, $(\zeta_{k_1}, \ldots, \zeta_{k_p}) = \lim_{N \to \infty} (\zeta_{k_1}^N, \ldots, \zeta_{k_p}^N)$. Since the distribution of $(\zeta_{-N}^N, \ldots, \zeta_N^N)$ is unique in the space $\mathrm{C}_{\mathbb{R}^{2N+1}}$, we have the uniqueness of the distribution of $(\ldots, \zeta_{-n}, \ldots, \zeta_n, \ldots)$. The theorem is proved. □

Definition 4.9. Let $\zeta_n(t)$, $t \geq 0, n \in \mathbb{Z}$, satisfy the conditions 1°)-5°) of Theorem 4.1. The random process $(\zeta_n(t))_{n \in \mathbb{Z}}$, $t \geq 0$ is called the *process of heavy diffusion particles* with drift in the space $\mathbb{R}^{\mathbb{Z}}$.

5. Strictly Markov Property of the Process of Heavy Diffusion Particles with Drift in the Space $\mathbb{R}^{\mathbb{Z}}$.

Let $b \in (0, \infty)^{\mathbb{Z}}$ and $\overline{\lim}_{n \to \pm\infty} \{b_n \wedge b_{n+1}\} > 0$. Denote by K^b the set of elements $x \in \mathbb{R}^{\mathbb{Z}}$ satisfying

$$\overline{\lim_{n \to \pm\infty}} \{(x_{n+1} - x_n) \wedge b_{n+1} \wedge b_n\} > 0.$$

By Theorem 4.1, for every $x \in \mathrm{K}^b$, there exists the process of heavy diffusion particles with drift $\zeta(t)$, $t \geq 0$, such that $\zeta(0) = x$. From Lemma 4.6 and the proof of Theorem 4.8 we conclude that $\zeta(t) \in \mathrm{K}^b$, for all $t \geq 0$. In this section we show that the process of heavy diffusion particles with drift is the strictly Markov process in the space K^b. The following lemma holds.

Lemma 5.1. K^b *is measurable subset of* $\mathbb{R}^{\mathbb{Z}}$.

Denote $\mathcal{K}^b = \{A \cap \mathrm{K}^b,\ A \in \mathcal{B}\left(\mathbb{R}^{\mathbb{Z}}\right)\}$. Observe that K^b is a metric subspace of the space $\mathbb{R}^{\mathbb{Z}}$ with the metric

$$\rho(x, y) = \sum_{k \in \mathbb{Z}} \frac{1}{2^k} (|x_k - y_k| \wedge 1), \quad x, y \in \mathbb{R}^{\mathbb{Z}}.$$

Moreover, by Lemma 5.1, it follows that the Borel σ-algebra on K^b equals $\mathcal{K}^b$.

Let $\mathbb{P}_x^\zeta$ be the distribution of the process of heavy diffusion particles with drift $\zeta(t)$, $t \geq 0$, which starts from $x \in \mathrm{K}^b$.

Lemma 5.2. *For every set* $A \in \mathcal{B}(\mathrm{C}_{\mathrm{K}^b})$, *the map* $x \mapsto \mathbb{P}_x^\zeta(A)$ *is* $\mathcal{K}^b$*-measurable.*

Proof. To prove the lemma, it suffices to show that, for every bounded function $f \in \mathrm{C}_{\mathrm{K}^b}$, the map

$$x \mapsto \mathbf{E}_x f(\zeta)$$

is $\mathcal{K}^b$-measurable.

Let, for $x \in \mathrm{K}^b$, $\zeta^N(t)$, $t \geq 0$, $N \in \mathbb{N}$, be the set of the random processes that was constructed in the proof of theorem 4.1.

Put

$$\eta_k^N = \zeta_{(k \wedge N) \vee (-N)}^N, \quad k \in \mathbb{Z},\ N \in \mathbb{N}.$$

Then the sequence of the random processes $\eta^N,\ N \in \mathbb{N}$, converges with probability 1 in the space $\mathrm{C}_{\mathrm{K}^b}$ to the process of heavy diffusion particles with drift ζ. Hence

$$\mathbf{E}_x f(\eta^N) \to \mathbf{E}_x f(\zeta), \quad N \to \infty.$$

From this and the measurability of $x \mapsto \mathbb{P}_x^{\zeta^N}(C)$ for all $C \in \mathcal{B}(\mathbb{R}^{\mathbb{Z}})$ (see Section 3), the proof of the lemma follows . □

Theorem 5.3. *The set of the distributions* $\{\mathbb{P}_x^{\zeta},\ x \in \mathrm{K}^b\}$ *is a strictly Markov system.*

The proof of the theorem is analogous to the proof of Theorem 3.1, considering the existence of a regular conditional probability on space $(\mathrm{K}^b, \mathcal{K}^b)$ as it is standard measurable (see [19], Lemma 5.1 and [11]).

6. Properties of the Process of Heavy Diffusion Particles.

This section is devoted to the investigation of asymptotic properties of the process of heavy diffusion particles with drift. Specifically, an estimation of asymptotic growth of the mass is established. Next, assume that $b_k = 1$, $k \in \mathbb{Z}$, and $x_{k+1} - x_k > \delta$.

Lemma 6.1. *For every integer* k*,*

$$\mathbb{P}\left\{ \overline{\lim_{t\to+\infty}} \frac{\delta m_k(t)}{8\|\sigma\|\sqrt{t\ \ln\ln t}} \le 1 \right\} = 1.$$

The proof of the lemma is similar to the proof of the property 3° [13].

In case where $a \equiv 0$, $\sigma \equiv 1$, $b_k = 1$, $x_k = k$, $k \in \mathbb{Z}$, the following asymptotic properties of the process of heavy diffusion particles $\zeta_k(t),\ t \ge 0, k \in \mathbb{Z}$, are stated in [13].

Lemma 6.2. *For all* $k \in \mathbb{Z}$ *and* $p \in \mathbb{N}$*, the processes* $\zeta_k(\cdot)$ *and* $\zeta_{k+p}(\cdot)$ *coalesce in finite time, i.e.*

$$\mathbb{P}\{\tau_{k,k+p} < +\infty\} = 1.$$

Lemma 6.3. *For every* $\varepsilon \in (0, 1)$*,*

$$\mathbb{P}\left\{ \lim_{t\to+\infty} \frac{|\zeta_0(t)|}{\sqrt{2t\ \ln\ln t}} = 0 \right\} = 1,$$

$$\mathbb{P}\left\{ \overline{\lim_{t\to+\infty}} \frac{|\zeta_0(t)|}{\sqrt[4]{t^{1-\varepsilon}}} = \infty \right\} = 1.$$

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