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STOCHASTIC CONTROL OF ITÔ-LÉVY PROCESSES WITH APPLICATIONS TO FINANCE

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Abstract. We give a short introduction to the stochastic calculus for Itô-Lévy processes and review briefly the two main methods of optimal control of systems described by such processes:
(i) Dynamic programming and the Hamilton-Jacobi-Bellman (HJB) equation
(ii) The stochastic maximum principle and its associated backward stochastic differential equation (BSDE).
The two methods are illustrated by application to the classical portfolio optimization problem in finance. A second application is the problem of risk minimization in a financial market. Using a dual representation of risk, we arrive at a stochastic differential game, which is solved by using the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, which is an extension of the HJB equation to stochastic differential games.

1. Introduction

This review paper is based on lecture notes from an intensive course which one of us (B.Ø) gave at the Buea School on Financial and Actuarial Mathematics, held in Buea, Cameroon on 22 -27 April 2013. The purpose of the course was to give the participants a quick introduction to some important tools in the modern research within mathematical finance, with emphasis on applications to portfolio optimization and risk minimization. The content of this paper is the following:

In Section 2 we review some basic concepts and results from the stochastic calculus of Itô-Lévy processes.

In Section 3 we present a *portfolio optimization* problem in an Itô-Lévy type financial market. We recognize this as a special case of a stochastic control problem and we present the first general method for solving such problems: *Dynamic programming* and the *HJB* equation. We show that if the system is Markovian we can use this method to solve the problem.

In Section 4 we study a *risk minimization* problem in the same market. By a general representation of convex risk measures, this problem may be regarded as

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a *stochastic differential game*, which also can be solved by dynamic programming (HJBI equation) if the system is Markovian.

Finally, in Section 5 we study the portfolio optimization problem by means of the second main stochastic control method: *The maximum principle.* The advantage with this method is that it also applies to non-Markovian systems.

2. Stochastic Calculus for Itô-Lévy Processes

In this section we give a brief survey of stochastic calculus for Itô-Lévy processes. For more details we refer to Chapter 1 in [4]. We begin with a definition of a Lévy process:

Definition 2.1. A *Lévy process* on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a process, $\eta(t) \equiv \eta(t, \omega)$ with the following properties

(i) $\eta(0) = 0$.
(ii) η has stationary, independent increments.
(iii) η is stochastically continuous

The jump of η at time t is $\Delta\eta(t) = \eta(t) - \eta(t-)$.

Remark 2.2. One can prove that η always has a càdlàg (i.e. left continuous with right sided limits) version. We will use this version from now on.

The jump measure $N([0,t], U)$ gives the number of jumps of η up to time t with jump size in the set $U \subset \mathbb{R}_0 \equiv \mathbb{R} \setminus \{0\}$. If we assume that $\overline{U} \subset \mathbb{R}_0$, then it can be shown that U contains only finitely many jumps in any finite time interval. The *Lévy measure* $\nu(\cdot)$ of η is defined by

$$\nu(U) = \mathbb{E}[N([0,1], U)], \tag{2.1}$$

and $N(\mathrm{d}t, \mathrm{d}\zeta)$ is the differential notation of the random measure $N([0,t], U)$. Intuitively, ζ can be regarded as generic jump size. Let $\tilde{N}(\cdot)$ denote the *compensated jump measure* of η, defined by

$$\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) \equiv N(\mathrm{d}t, \mathrm{d}\zeta) - \nu(\mathrm{d}\zeta)\mathrm{d}t. \tag{2.2}$$

For convenience we shall from now on impose the following additional integrability condition on $\nu(\cdot)$:

$$\int_{\mathbb{R}} \zeta^2 \nu(\mathrm{d}\zeta) < \infty, \tag{2.3}$$

which is equivalent to the assumption that for all $t \geq 0$

$$\mathbb{E}[\eta^2(t)] < \infty. \tag{2.4}$$

This condition still allows for many interesting kinds of Lévy processes. In particular, it allows for the possibility that a Lévy process has the following property:

$$\int_{\mathbb{R}} (1 \wedge |\zeta|)\nu(\mathrm{d}\zeta) = \infty. \tag{2.5}$$

This implies that there are infinitely many small jumps. Under the assumption (2.3) above the *Itô-Lévy decomposition theorem* states that any Lévy process has the form

$$\eta(t) = at + bB(t) + \int_0^t \int_{\mathbb{R}} \zeta \tilde{N}(\mathrm{d}s, \mathrm{d}\zeta), \tag{2.6}$$

where $B(t)$ is a Brownian motion, and a, b are constants.

More generally, we study the *Itô-Lévy processes*, which are the processes of the form

$$\begin{aligned} X(t) &= x + \int_0^t \alpha(s,\omega)\mathrm{d}s + \int_0^t \beta(s,\omega)\mathrm{d}B(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} \gamma(s,\zeta,\omega)\tilde{N}(\mathrm{d}s, \mathrm{d}\zeta), \end{aligned} \tag{2.7}$$

where $\int_0^t |\alpha(s)|\mathrm{d}s + \int_0^t \beta^2(s)\mathrm{d}s + \int_0^t \int_{\mathbb{R}} \gamma^2(s,\zeta)\nu(\mathrm{d}\zeta)\mathrm{d}s < \infty$ a.s., and $\alpha(t)$, $\beta(t)$, and $\gamma(t,\zeta)$ are predictable processes (predictable w.r.t. the filtration $\mathcal{F}_t$ generated by $\eta(s)$, for $s \leq t$).

In differential form we have

$$\mathrm{d}X(t) = \alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta). \tag{2.8}$$

We now proceed to the *Itô formula* for Itô-Lévy processes: Let $X(t)$ be an Itô-Lévy process defined as above. Let $f : [0,T] \times \mathbb{R}$ be a $\mathcal{C}^{1,2}$ function and put $Y(t) = f(t, X(t))$.

Then $Y(t)$ is also an Itô-Lévy process with representation:

$$\begin{aligned} \mathrm{d}Y(t) &= \frac{\partial f}{\partial t}(t,X(t))\mathrm{d}t + \frac{\partial f}{\partial x}(t,X(t))(\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t)) \\ &\quad + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X(t))\beta^2(t)\mathrm{d}t \\ &\quad + \int_{\mathbb{R}} \{f(t,X(t)+\gamma(t,\zeta)) - f(t,X(t))\}\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta) \\ &\quad + \int_{\mathbb{R}} \{f(t,X(t)+\gamma(t,\zeta)) - f(t,X(t)) - \frac{\partial f}{\partial x}(t,X(t))\gamma(x,\zeta)\}\nu(\mathrm{d}\zeta)\mathrm{d}t, \end{aligned} \tag{2.9}$$

where the last term can be interpreted as the quadratic variation of jumps.

The *Itô isometries* state the following:

$$\mathbb{E}\left[\left(\int_0^T \beta(s)\mathrm{d}B(s)\right)^2\right] = \mathbb{E}\left[\int_0^T \beta^2(s)\mathrm{d}s\right] \tag{2.10}$$

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}} \gamma(s,\zeta)\tilde{N}(\mathrm{d}s,\mathrm{d}\zeta)\right)^2\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} \gamma^2(s,\zeta)\nu(\mathrm{d}\zeta)\mathrm{d}s\right] \tag{2.11}$$

Martingale properties: If the quantities of (2.11) are finite, then

$$M(t) = \int_0^t \int_{\mathbb{R}} \gamma(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z) \tag{2.12}$$

is a martingale for $t \leq T$.

The *Itô representation theorem* states that any $F \in L^2(\mathcal{F}_T, \mathbb{P})$ has the representation

$$F = \mathbb{E}[F] + \int_0^T \varphi(s)\mathrm{d}B(s) + \int_0^T \int_{\mathbb{R}} \psi(s,\zeta)\tilde{N}(\mathrm{d}s, \mathrm{d}\zeta) \tag{2.13}$$

for suitable predictable (unique) L^2-processes $\varphi(\cdot)$ and $\psi(\cdot)$.

Remark 2.3. Using *Malliavin calculus* (see [1]), we get the representation

$$\varphi(s) = \mathbb{E}[D_s F|\mathcal{F}_t]$$

and

$$\psi(s,\zeta) = \mathbb{E}[D_{s,\zeta} F|\mathcal{F}_s],$$

where D_s and $D_{s,\zeta}$ are the Malliavin derivatives at s and (s,ζ) w.r.t. $B(\cdot)$ and $\tilde{N}(\cdot,\cdot)$, respectively.

Example 2.4. Suppose $\eta(t) = \eta_0(t) = \int_0^t \int_{\mathbb{R}} \zeta \tilde{N}(\mathrm{d}s, \mathrm{d}\zeta)$, i.e. $\eta(t)$ is a pure-jump martingale. We want to find the representation of $F := \eta_0^2(T)$. By the Itô formula we get

$$\begin{aligned} \mathrm{d}(\eta_0^2(t)) &= \int_{\mathbb{R}} \{(\eta_0(t)+\zeta)^2 - (\eta_0(t))^2\}\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) \qquad (2.14) \\ &\quad + \int_{\mathbb{R}} \{(\eta_0(t)+\zeta)^2 - (\eta_0(t))^2 - 2\eta_0(t)\zeta\}\nu(\mathrm{d}\zeta)\mathrm{d}t \\ &= \int_{\mathbb{R}} 2\eta_0(t)\zeta\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) + \int_{\mathbb{R}} \zeta^2\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) + \int_{\mathbb{R}} \zeta^2\nu(\mathrm{d}\zeta)\mathrm{d}t \\ &= 2\eta_0(t)\mathrm{d}\eta_0(t) + \int_{\mathbb{R}} \zeta^2\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) + \int_{\mathbb{R}} \zeta^2\nu(\mathrm{d}\zeta)\mathrm{d}t. \qquad (2.15) \end{aligned}$$

This implies that

$$\eta_0^2(T) = T\int_{\mathbb{R}} \zeta^2\nu(\mathrm{d}\zeta) + \int_0^T 2\eta_0(t)\mathrm{d}\eta_0(t) + \int_0^T \int_{\mathbb{R}} \zeta^2\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta). \tag{2.16}$$

Note that it is <u>not</u> possible to write $F \equiv \eta_0^2(T)$ as a constant + an integral w.r.t. $\mathrm{d}\eta_0(t)$.

This has an interpretation in finance: It implies that in a normalized market with $\eta_0(t)$ as the risky asset price, the claim $\eta_0^2(T)$ is *not replicable*. This illustrates that markets based on Lévy processes are typically <u>not</u> complete.

Consider the following stochastic differential equation (SDE):

$$\begin{aligned} \mathrm{d}X(t) &= b(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}B(t) \qquad (2.17) \\ &\quad + \int_{\mathbb{R}} \gamma(t, X(t^-), \zeta)\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta); \quad X(0) = x. \qquad (2.18) \end{aligned}$$

Here $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$; $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$; and $\gamma : [0,T] \times \mathbb{R}^n \times \mathbb{R}_0^l \to \mathbb{R}^{n\times l}$ are given functions. If these functions are Lipschitz continuous with respect to x and with at most linear growth in x, uniformly in t, then a unique L^2 - solution to the above SDE exists.

Example 2.5. The (generalized) geometric Itô-Lévy process X is defined by:

$$\begin{aligned} \mathrm{d}X(t) &= X(t^-)\left[\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t)\right. \\ &\quad \left. + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta)\right]; \quad X(0) = x > 0. \end{aligned} \tag{2.19}$$

If $\gamma > -1$ then $X(t)$ can never jump to 0 or a negative value, and then the solution is

$$\begin{aligned} X(t) &= x\exp\left[\int_0^T \beta(s)\mathrm{d}B(s) + \int_0^t (\alpha(s) - \frac{1}{2}\beta^2(s))\mathrm{d}s \right. && (2.20) \\ &\quad + \int_0^t \int_{\mathbb{R}} \{\ln(1+\gamma(s,\zeta)) - \gamma(s,\zeta)\}\nu(\mathrm{d}\zeta)\mathrm{d}s \\ &\quad \left. + \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,\zeta))\tilde{N}(\mathrm{d}s,\mathrm{d}\zeta)\right]. && (2.21) \end{aligned}$$

If $b(t,x) = b(x)$, $\sigma(t,x) = \sigma(x)$, and $\gamma(t,x,\zeta) = \gamma(x,\zeta)$, i.e. $b(\cdot)$, $\sigma(\cdot)$, and $\gamma(\cdot,\cdot)$ do not depend on t, the corresponding SDE takes the form

$$\mathrm{d}X(t) = b(X(t))\mathrm{d}t + \sigma(X(t))\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(X(t),\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta). \tag{2.22}$$

Then $X(t)$ is called an Itô-Lévy diffusion or simply a *jump-diffusion*.

The *generator* A of a jump-diffusion $X(t)$ is defined by

$$(Af)(x) = \lim_{t\to 0} \frac{\mathbb{E}^x[f(X(t))] - f(x)}{t}, \tag{2.23}$$

if the limit exists. The form of the generator A of the process $X(\cdot)$ is given explicitly in the following lemma:

Lemma 2.6. *If $X(\cdot)$ is a jump-diffusion and $f \in \mathcal{C}_0^2(\mathbb{R})$, where $\mathcal{C}_0$ corresponds to f having compact support, then $(Af)(x)$ exists for all x and*

$$\begin{aligned} (Af)(x) &= \sum_{i=1}^n b_i(x)\frac{\partial f}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^n (\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\ &\quad + \sum_{k=1}^l \int_{\mathbb{R}} \{f(x+\gamma^{(k)}(x,\zeta)) - f(x) - \nabla f(x)\cdot\gamma^{(k)}(x,\zeta)\}\nu_k(\mathrm{d}\zeta) \end{aligned} \tag{2.24}$$

where $\gamma^{(k)}$ is column number k of the $n \times l$ matrix γ.

The generator gives a crucial link between jump diffusions and (deterministic) partial differential equations. We will exploit this when we come to the dynamic programming approach to stochastic control problems in the next section. One of the most useful expressions of this link is the following result, which may be regarded as a generalization of the classical mean-value theorem in classical analysis:

The Dynkin formula: Let X be a jump-diffusion process and let τ be a stopping time. Let $h \in \mathcal{C}^2(\mathbb{R})$ and assume that $\mathbb{E}^x\left[\int_0^\tau |Ah(X(t))|\mathrm{d}t\right] < \infty$ and $\{h(X(t))\}_{t\leq\tau}$ is uniformly integrable. Then

$$\mathbb{E}^x[h(X(\tau))] \quad = \quad h(x) + \mathbb{E}^x\left[\int_0^\tau Ah(X(t))\mathrm{d}t\right]. \tag{2.25}$$

3. Stochastic Control (1): Dynamic Programming

We start by a motivating example:

Example 3.1. (Optimal portfolio problem). Suppose we have a financial market with two investment possibilities:

(i) A risk-free asset with unit price $S_0(t) = 1$.
(ii) A risky asset with unit price $S(t)$ at time t given by

$$\begin{aligned}\mathrm{d}S(t) \quad &= \quad S(t^-)\left[\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t)\right.\\ &\qquad \left. + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta)\right], \quad \gamma > -1, \quad S(0) > 0.\end{aligned} \tag{3.1}$$

Let $\pi(t)$ denote a portfolio representing the fraction of the total wealth invested in the risky asset at time t. If we assume that $\pi(t)$ is *self-financing*, the corresponding wealth $X(t) = X_\pi(t)$ satisfies the state equation

$$\mathrm{d}X(t) = X(t^-)\pi(t)\left[\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta)\right]. \tag{3.2}$$

The problem is to maximize $\mathbb{E}[U(X_\pi(T))]$ over all $\pi \in \mathcal{A}$, where $\mathcal{A}$ denotes the set of all admissible portfolios and U is a given *utility function.*

This is a special case of the following *general stochastic control problem*:
The *state equation* is given by:

$$\begin{aligned}\mathrm{d}Y(t) = \mathrm{d}Y_u(t) \quad &= \quad b(Y(t),u(t))\mathrm{d}t + \sigma(Y(t),u(t))\mathrm{d}B(t)\\ &+ \quad \int_{\mathbb{R}} \gamma(Y(t),u(t),\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta), \quad Y(0) = y \in \mathbb{R}^k.\end{aligned} \tag{3.3}$$

The *performance functional* is given by:

$$J_u(y) = \mathbb{E}^y\left[\int_0^{\tau_{\mathcal{S}}} \underbrace{f(Y(s),u(s))}_{\text{profit rate}}\mathrm{d}s + \underbrace{g(Y(\tau_{\mathcal{S}}))}_{\text{bequest function}} 1_{\{\tau_{\mathcal{S}}<\infty\}}\right], \tag{3.4}$$

where $\tau_{\mathcal{S}} = \inf\{t \geq 0 : Y(t) \notin \mathcal{S}\}$ (*bankruptcy time*), and $\mathcal{S}$ is a given *solvency region.*

Problem: Find $u^* \in \mathcal{A}$ and $\Phi(y)$ such that

$$\Phi(y) = \sup_{u\in\mathcal{A}} J_u(y) = J_{u^*}(y).$$

Theorem 3.2. *(Hamilton-Jacobi-Bellman (HJB) equation)*

(a) *Suppose we can find a function* $\varphi \in \mathcal{C}^2(\mathbb{R}^n)$ *such that*

(i) $A_v\varphi(y) + f(y,v) \leq 0$, *for all* $v \in \mathcal{V}$, *where* $\mathcal{V}$ *is the set of possible control values, and*

$$A_v\varphi(y) = \sum_{i=1}^{k} b_i(y,v)\frac{\partial\varphi}{\partial y_i}(y) + \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^T)_{ij}(y,v)\frac{\partial^2\varphi}{\partial y_i\partial y_j} \tag{3.5}$$

$$+\sum_{m}\int_{\mathbb{R}}\{\varphi(y+\gamma^{(k)}(y,v,\zeta)) - \varphi(y) - \nabla\varphi(y)\gamma^{(k)}(y,v,\zeta)\}\nu_k(\mathrm{d}\zeta)$$

(ii) $\lim_{t\to\tau_{\mathcal{S}}}\varphi(Y(t)) = g(Y(\tau_{\mathcal{S}}))1_{\{\tau_{\mathcal{S}}<\infty\}}$

(iii) *"growth conditions:"*

$$E^y\Big[|\varphi(Y(\tau))| + \int_0^{\tau_{\mathcal{S}}}\{|A\varphi(Y(t))| + |\sigma^T(Y(t))\nabla\varphi(Y(t))|^2$$

$$+\sum_{j=1}^{\ell}\int_{\mathbb{R}}|\varphi(Y(t)+\gamma^{(j)}(Y(t),u(t),\zeta_j))-\varphi(Y(t))|^2\nu_j(d\zeta_j)\}dt\Big] < \infty,$$

for all $u \in \mathcal{A}$ *and all stopping time* τ.

(iv) $\{\varphi^-(Y(\tau))\}_{\tau\leq\tau_{\mathcal{S}}}$ *is uniformly integrable for all* $u \in \mathcal{A}$ *and* $y \in \mathcal{S}$, *where, in general,* $x^- := max\{-x, 0\}$ *for* $x \in \mathbb{R}$.

Then

$$\varphi(y) \geq \Phi(y).$$

(b) *Suppose we for all* $y \in \mathcal{S}$ *can find* $v = \widehat{u}(y)$ *such that*

$$A_{\widehat{u}(y)}\varphi(y) + f(y,\widehat{u}(y)) = 0$$

and $\widehat{u}(y)$ *is an admissible feedback control (Markov control), i.e.* $\widehat{u}(y)$ *means* $\widehat{u}(Y(t))$. *Then* $\widehat{u}(y)$ *is an optimal control and*

$$\varphi(y) = \Phi(y).$$

Remark 3.3. This is a useful result because it, in some sense, basically reduces the original highly complicated stochastic control problem to a classical problem of maximizing a function of (possibly several) real variable(s), namely the function $v \mapsto A_v\varphi(y) + f(y,v); v \in \mathcal{V}$. We will illustrate this by examples below.

Sketch of proof: Using the "growth conditions" (iii) one can prove by an approximation argument that the Dynkin formula holds with $h = \varphi$ and $\tau = \tau_{\mathcal{S}}$, for any given $u \in \mathcal{A}$.

This gives (if $\tau_{\mathcal{S}} < \infty$)

$$\mathbb{E}^y[\varphi(Y(\tau_{\mathcal{S}}))] = \varphi(y) + \mathbb{E}^y\left[\int_0^{\tau_{\mathcal{S}}} A\varphi(Y(t))\mathrm{d}t\right] \tag{3.6}$$

$$\leq_{(A\varphi+f\leq0)} \varphi(y) - \mathbb{E}^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y(t),u(t))\mathrm{d}t\right]. \tag{3.7}$$

This implies

$$\varphi(y) \geq \mathbb{E}^y\left[\int_0^{\tau_{\mathcal{S}}} f(Y(t),u(t))\mathrm{d}t + g(Y(\tau_{\mathcal{S}}))\right] \tag{3.8}$$

$$= J_u(y), \quad \text{for all} \quad u \in \mathcal{A}, \tag{3.9}$$

which means that

$$\varphi(y) \geq \sup_{u \in \mathcal{A}} J_u(y) = \Phi(y). \tag{3.10}$$

This proves (a).

To prove (b), observe that if we have a control $\hat{u}$ with *equality* above, i.e. $A\varphi + f = 0$, then by the argument in (a) we get

$$\varphi(y) = J_{\widehat{u}}(y).$$

Hence

$$\Phi(y) \leq \varphi(y) = J_{\widehat{u}}(y) \leq \Phi(y).$$

It follows that $\hat{u}$ is optimal. □

To illustrate this result, let us return to the optimal portfolio problem of Example 3.1:

Suppose $U(x) = \ln(x)$. Then the problem is to maximize $\mathbb{E}[\ln X_\pi(T)]$. Put

$$\begin{aligned} \mathrm{d}Y(t) &= \begin{bmatrix} \mathrm{d}t \\ \mathrm{d}X(t) \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ X(t)\pi(t)\alpha(t) \end{bmatrix} \mathrm{d}t + \begin{bmatrix} 0 \\ X(t)\pi(t)\beta(t) \end{bmatrix} \mathrm{d}B(t) && (3.11) \\ &\quad + \begin{bmatrix} 0 \\ X(t)\pi(t) \end{bmatrix} \int_{\mathbb{R}} \gamma(t,\zeta)\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) && (3.12) \end{aligned}$$

and

$$\begin{aligned} A_\pi\varphi(t,x) &= \frac{\partial\varphi}{\partial t}(t,x) + x\pi\alpha(t)\frac{\partial\varphi}{\partial x}(t,x) + \frac{1}{2}x^2\pi^2\beta^2(t)\frac{\partial^2\varphi}{\partial x^2}(t,x) && (3.13) \\ &\quad + \int_{\mathbb{R}} \{\varphi(t, x + x\pi\gamma(t,\zeta)) - \varphi(t,x) - \frac{\partial\varphi}{\partial x}(t,x)x\pi\gamma(t,\zeta)\}\nu(\mathrm{d}\zeta) \end{aligned}$$

Here $f = 0$ and $g(t,x) = \ln x$. We guess that $\varphi(x) = \ln x + \kappa(t)$, where $\kappa(t)$ is a deterministic function, and we maximize $A_\pi\varphi$ over all π.

Then we find, if we assume that $\alpha(t)$, $\beta(t)$, and $\gamma(t,z)$ are deterministic (this ensures that the system is Markovian; see Remark 3.4 below), that the optimal portfolio π^* is the solution of the equation

$$\pi^*(t)\beta^2(t) + \pi^*(t)\int_{\mathbb{R}} \frac{\gamma^2(t,\zeta)\nu(\mathrm{d}\zeta)}{1 + \pi^*(t)\gamma(t,\zeta)} = \alpha(t). \tag{3.14}$$

In particular, if $\nu = 0$ and $\beta^2(t) \neq 0$, then

$$\pi^*(t) = \frac{\alpha(t)}{\beta^2(t)}.$$

Remark 3.4. The assumption that $\alpha(t)$, $\beta(t)$, and $\gamma(t,z)$ are deterministic functions is used when applying the dynamic programming techniques in solving this type of stochastic control problems. More generally, for the dynamic programming/HJB method to work it is necessary that the system is Markovian, i.e. that the coefficients are deterministic functions of t and $X(t)$. This is a limitation of the dynamic programming approach to solving stochastic control problems.

In Section 5 we shall see that there is an alternative approach to stochastic control, called *the maximum principle*, which does not require that the system is Markovian.

4. Risk Minimization

4.1. Introduction. Let $p \in [1,\infty]$. A *convex risk measure* is a map $\rho : L^p(\mathcal{F}_T) \to \mathbb{R}$ with the following properties:

(i) (Convexity): $\rho(\lambda F + (1-\lambda)G) \le \lambda\rho(F) + (1-\lambda)\rho(G)$; for all $F, G \in L^p(\mathcal{F}_T)$,
i.e. diversification reduces the risk.

(ii) (Monotonicity): $F \le G \Rightarrow \rho(F) \ge \rho(G)$; for all $F, G \in L^p(\mathcal{F}_T)$,
i.e. smaller wealth has bigger risk.

(iii) (Translation invariance): $\rho(F+\alpha) = \rho(F) - \alpha$ if $a \in \mathbb{R}$; for all $F \in L^p(\mathcal{F}_T)$,
i.e. adding a constant to F reduces the risk accordingly.

Remark 4.1. We may regard $\rho(F)$ as the amount we need to add to the position F in order to make it "acceptable", i.e. $\rho(F + \rho(F)) = 0$. (F is acceptable if $\rho(F) \le 0$).

One can prove that basically any risk convex measure ρ can be represented as follows:

$$\rho(F) = \sup_{Q \in \wp} \{\mathbb{E}_Q(-F) - \zeta(Q)\} \tag{4.1}$$

for some family $\wp$ of measures $Q \ll \mathbb{P}$ and for some convex penalty function $\zeta : \wp \to \mathbb{R}$. We refer to [2] for more information about risk measures.

Returning to the financial market in Example 3.1, suppose we want to *minimize the risk* of the terminal wealth, rather than maximize the expected utility. Then the problem is to minimize $\rho(X_\pi(T))$ over all possible admissible portfolios $\pi \in \mathcal{A}$.

Hence we want to solve the problem

$$\inf_{\pi \in \mathcal{A}} (\sup_{Q \in \wp} \{\mathbb{E}_Q[-X_\pi(T)] - \zeta(Q)\}). \tag{4.2}$$

This is an example of a *stochastic differential game* (of zero-sum type). Heuristically, this can be interpreted as the problem to find the best possible π under the worst possible scenario Q.

The game above is a special case of the following general *zero-sum stochastic differential game*:

We have 2 players and 2 types of controls, u_1 and u_2, and we put $u = (u_1, u_2)$. We assume that player number i controls u_i, for $i = 1, 2$. Suppose the state $Y(t) = Y_u(t)$ has the form

$$\begin{aligned} \mathrm{d}Y(t) &= b(Y(t), u(t))\mathrm{d}t + \sigma(Y(t), u(t))\mathrm{d}B(t) \\ &\quad + \int_{\mathbb{R}} \gamma(Y(t), u(t), \zeta)\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta)\,;\, Y(0) = y. \end{aligned} \tag{4.3}$$

We define the *performance functional* as follows:

$$J_{u_1,u_2}(y) = \mathbb{E}^y[\int_0^{\tau_{\mathcal{S}}} f(Y(t),u_1(t),u_2(t))\mathrm{d}t + g(Y(\tau_{\mathcal{S}}))\mathbf{1}_{\tau_{\mathcal{S}}<\infty}]. \tag{4.4}$$

Problem: Find $\Phi(y)$ and $u_1^* \in \mathcal{A}_1$, $u_2^* \in \mathcal{A}_2$ such that

$$\Phi(y) := \inf_{u_2\in\mathcal{A}_2}(\sup_{u_1\in\mathcal{A}_1} J_{u_1,u_2}(y)) = J_{u_1^*,u_2^*}(y). \tag{4.5}$$

4.2. The HJBI equation for stochastic differential games. Here we need a new tool, namely the *Hamilton-Jacobi-Bellman-Isaacs (HJBI)* equation, which in this setting goes as follows:

Theorem 4.2. *(The HJBI equation for zero-sum games ([3])) Suppose we can find a function* $\varphi \in \mathbb{C}^2(\mathcal{S})\bigcap\mathbb{C}(\bar{\mathcal{S}})$ *(continuous up to the boundary of* $\mathcal{S}$*) and a Markov control pair* $(\hat{u_1}(y),\hat{u_2}(y))$ *such that*

(i) $A_{u_1,\hat{u_2}(y)}\varphi(y) + f(y,u_1,\hat{u_2}(y)) \le 0$; $\forall u_1 \in \mathcal{A}_1$ *and* $\forall y \in \mathcal{S}$
(ii) $A_{\hat{u_1}(y),u_2}\varphi(y) + f(y,\hat{u_1}(y),u_2) \ge 0$; $\forall u_2 \in \mathcal{A}_2$ *and* $\forall y \in \mathcal{S}$
(iii) $A_{\hat{u_1}(y),\hat{u_2}(y)}\varphi(y) + f(y,\hat{u_1}(y),\hat{u_2}(y)) = 0$; $\forall y \in \mathcal{S}$
(iv) $\lim_{t\to\tau_{\mathcal{S}}} \varphi(Y_u(t)) = g(Y_u(\tau_{\mathcal{S}}))\mathbf{1}_{\tau_{\mathcal{S}}<\infty}$ *for all* u
(v) *"growth conditions".*

Then

$$\begin{aligned}\varphi(y) = \Phi(y) &= \inf_{u_2}(\sup_{u_1} J_{u_1,u_2}(y)) = \sup_{u_1}(\inf_{u_2} J_{u_1,u_2}(y)) \\ &= \inf_{u_2} J_{\hat{u_1},u_2}(y) = \sup_{u_1} J_{u_1,\hat{u_2}}(y) \\ &= J_{\hat{u_1},\hat{u_2}}(y).\end{aligned}$$

Proof. The proof is similar to the proof of the HJB equation. □

Remark 4.3. For the sake of the simplicity of the presentation, in (v) above and also in (iv) of Theorem 4.5 we choose not to specify the rather technical "growth conditions"; we just mention that they are analogous to the conditions $(iii) - (iv)$ in Theorem 3.2. We refer to [3] for details. For a specification of the growth conditions in Theorem 5.1 we refer to Theorem 2.1 in [8].

To apply this to our risk minimization problems, we parametrize the family $\wp$ of measures $Q \ll P$ as follows:

For given predictable processes $\theta_0(t), \theta_1(t,\zeta)$ we put $\theta := (\theta_1, \theta_2)$ and define the process $Z_\theta(t)$ as follows:

$$\mathrm{d}Z_\theta(t) = Z_\theta(t^-)[\theta_0(t)\mathrm{d}B(t) + \int_{\mathbb{R}} \theta_1(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta)]; \quad Z_\theta(0) > 0,\ \theta_1 > -1$$

i.e.

$$\begin{aligned}Z_\theta(t) = Z_\theta(0)\exp\Big[&\int_0^t \theta_0(s)\mathrm{d}B(s) - \frac{1}{2}\int_0^t \theta_0^2(s)ds + \int_0^t\int_{\mathbb{R}} \ln(1+\theta_1(s,\zeta)) \\ &\tilde{N}(\mathrm{d}s,\mathrm{d}\zeta) + \int_0^t\int_{\mathbb{R}}\{\ln(1+\theta_1(s,\zeta)) - \theta_1(s,\zeta)\}\nu(\mathrm{d}\zeta)\mathrm{d}s\,\Big].\end{aligned} \tag{4.6}$$

Define a probability measure $Q_\theta \ll \mathbb{P}$ on $\mathcal{F}_T$ by putting $\frac{dQ_\theta}{dP} = Z_\theta(T)$. Then $Z_\theta(t) = \frac{d(Q_\theta|\mathcal{F}_t)}{d(P|\mathcal{F}_t)}$ and $Z_\theta(t) = \mathbb{E}[Z_\theta(T)|\mathcal{F}_t]$ for all $t \leq T$. If we restrict ourselves to this family $\wp$ of measures $Q = Q_\theta$ for $\theta \in \Theta$ the risk minimization problem gets the form:

$$\inf_{\pi\in\Pi}(\sup_{\theta\in\Theta}\{\mathbb{E}_{Q_\theta}[-X_\pi(T)] - \zeta(Q_\theta)\}) = \inf_{\pi\in\Pi}(\sup_{\theta\in\Theta}\{\mathbb{E}[-Z_\theta(T)X_\pi(T)] - \zeta(Q_\theta)\})$$

For example, if $\zeta(Q_\theta) = \int_0^{\tau_s} \lambda\,(Y(s), \theta(s))\,ds$, then this problem is a special case of the zero-sum stochastic differential game.

Extension of HJBI to non-zero sum games. In this case we have two performance functionals, one for each player:

$$J^{(i)}_{u_1,u_2}(y) = \mathbb{E}^y\left[\int_0^{\tau_s} f_i(Y(t), u_1(t), u_2(t))dt + g_i(Y(\tau_s))\mathbf{1}_{\tau_s<\infty}\right] \quad ; \quad i = 1, 2 \tag{4.7}$$

(In the zero-sum game we have $J^{(2)} = -J^{(1)}$). The pair $(\hat{u_1}, \hat{u_2})$ is called a *Nash equilibrium* if

(i) $J^{(1)}_{u_1,\hat{u_2}}(y) \leq J^{(1)}_{\hat{u_1},\hat{u_2}}(y)$ for all u_1
(ii) $J^{(2)}_{\hat{u_1},u_2}(y) \leq J^{(2)}_{\hat{u_1},\hat{u_2}}(y)$ for all u_2

Remark 4.4. Note that this is not a very strong equilibrium: One can sometimes obtain a better result for both players at points which are not Nash equilibria.

The next result is an extension of the HJBI equation to the non-zero sum games:

Theorem 4.5. *(The HJBI equation for non-zero stochastic differential games* [3]*)*

Suppose $\exists\ \varphi_i \in \mathcal{C}^2(\mathcal{S})$, *and a Markovian control* $(\hat{\theta}, \hat{\pi})$ such that*:*

(i) $A_{u_1,\hat{u_2}(y)}\varphi_1(y) + f_1(y, u_1, \hat{u_2}(y)) \leq A_{\hat{u_1}(y),\hat{u_2}(y)}\varphi_1(y) + f_1(y, \hat{u_1}(y), \hat{u_2}(y)) = 0$; *for all* u_1
(ii) $A_{\hat{u_1}(y),u_2}\varphi_2(y) + f_2(y, \hat{u_1}(y), u_2) \leq A_{\hat{u_1}(y),\hat{u_2}(y)}\varphi_2(y) + f_2(y, \hat{u_1}(y), \hat{u_2}(y)) = 0$; *for all* u_2*.*
(iii) $\lim_{t\to\tau_s^-} \varphi_i(Y_{u_1,u_2}(t)) = g_i(Y_{u_1,u_2}(\tau_s))\mathbf{1}_{\tau_s<\infty}$ *for* $i = 1, 2$ *and for all* u_1, u_2
(iv) *"growth conditions".*

Then $(\hat{u_1}, \hat{u_2})$ *is a Nash equilibrium and*

$$\varphi_1(y) = \sup_{u_1\in\mathcal{A}} J_1^{u_1,\hat{u_2}}(y) = J_1^{\hat{u_1},\hat{u_2}}(y) \tag{4.8}$$

$$\varphi_2(y) = \sup_{u_2\in\mathcal{A}_2} J_2^{\hat{u_1},u_2}(y) = J_2^{\hat{u_1},\hat{u_2}}(y). \tag{4.9}$$

5. Stochastic Control (2): The Maximum Principle Approach

We have mentioned that the dynamic programming approach to stochastic control only works if the system is Markovian. However, for non-Markovian systems the *maximum principle* approach still works. In this section we describe this method.

Consider a controlled Itô-Lévy process of the form

$$\begin{aligned} \mathrm{d}X(t) &= b(X(t), u(t), \omega)\mathrm{d}t + \sigma(t, X(t), u(t), \omega)\mathrm{d}B(t) \\ &+ \int_{\mathbb{R}} \gamma(t, X(t), u(t), \zeta, \omega)\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta) \end{aligned} \tag{5.1}$$

Here $b(t, x, u, \omega)$ is a given $\mathcal{F}_t$-adapted process, for each x and u and similarly with σ and γ. So this system is not necessarily Markovian.

The *performance functional* has the form:

$$J(u) = \mathbb{E}[\int_0^T f(t, X(t), u(t), \omega)\mathrm{d}t + g(X(T), \omega)]$$

where $T > 0$ is a fixed constant.

Problem: Find $u^* \in \mathcal{A}$ so that $\sup_{u \in \mathcal{A}} J(u) = J(u^*)$.

5.1. The Maximum Principle Approach. Define the *Hamiltonian* as follows:

$$H(t, x, u, p, q, r(\cdot)) = f(t, x, u) + b(t, x, u)p + \sigma(t, x, u)q + \int_{\mathbb{R}} \gamma(t, x, u, \zeta)r(\zeta)\nu(\mathrm{d}\zeta). \tag{5.2}$$

Here $r(\cdot)$ is a real function on $\mathbb{R}$.

The *backward stochastic differential equation (BSDE)* in the adjoint processes $p(t), q(t), r(t, \zeta)$ is defined as follows:

$$\begin{cases} \mathrm{d}p(t) &= -\frac{\partial H}{\partial x}(t, X(t), u(t), p(t), q(t), r(t, \cdot))\mathrm{d}t + q(t)\mathrm{d}B(t) \\ &\quad + \int_{\mathbb{R}} r(t, \zeta)\tilde{N}(\mathrm{d}t, \mathrm{d}\zeta); \quad 0 \le t \le T \\ p(T) &= g'(X(T)). \end{cases} \tag{5.3}$$

This equation is called *backward* because we are given the terminal value $p(T)$, not the initial value $p(0)$. One can prove in general that under certain conditions on the drift term there exists a unique solution (p, q, r) of such equations. Note that this particular BSDE is linear in p,q and r and hence easy to solve (if we know X and u). See [9], [10] and [11] for more information about BSDEs.

Theorem 5.1. *(The Mangasarian (sufficient) maximum principle)*
Suppose $\hat{u} \in \mathcal{A}$, with corresponding $\hat{X}(t) = X_{\hat{u}}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)$. Suppose the functions $x \to g(x)$ and $(x, u) \to H(t, x, u, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot))$ are concave for each t and ω and that

$$\max_{v \in \mathcal{V}} H(t, \hat{X}(t), v, \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)) = H(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, \cdot)), \tag{5.4}$$

for all t, where $\mathcal{V}$ is the set of all possible control values. Moreover, suppose that some growth conditions are satisfied. Then $\hat{u}$ is an optimal control.

Let us apply this to the optimal portfolio problem of Example 3.1. We want to maximize $\mathbb{E}[U(X_u(T))]$ over all admissible portfolios u, where $u(t)$ represents the amount invested in the risky asset at time t. The wealth process $X_u(t)$ generated by u is given by

$$dX(t) = u(t)[\alpha(t,\omega)\mathrm{d}t + \beta(t,\omega)\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma_0(t,\zeta,\omega)\tilde{N}(dt,d\zeta)]. \tag{5.5}$$

In this case the Hamiltonian is

$$H = u\alpha(t)p + u\beta(t)q + u\int_{\mathbb{R}} \gamma_0(t,\zeta)r(\zeta)\nu(\mathrm{d}\zeta) \tag{5.6}$$

$$b(t,x,u) = u\alpha(t), \quad \sigma(t,x,u) = u\beta(t), \quad \gamma(t,x,u,\zeta) = u\gamma_0(t,\zeta). \tag{5.7}$$

The BSDE (5.3) becomes

$$\begin{cases} \mathrm{d}p(t) &= q(t)\mathrm{d}B(t) + \int_{\mathbb{R}} r(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta); \quad 0 \le t \le T \\ p(T) &= U'(X_u(T)). \end{cases} \tag{5.8}$$

Note that u appears linearly in H. Therefore we guess that the coefficient of u must be 0. Otherwise one could make H arbitrary big by choosing u suitably.

Hence we obtain the following two equations that must be satisfied for an optimal triple $(p(t), q(t), r(t,\cdot))$:

$$\alpha(t)p(t) + \beta(t)q(t) + \int_{\mathbb{R}} \gamma(t,z)r(t,\zeta)\nu(\mathrm{d}\zeta) = 0 \tag{5.9}$$

$$\begin{cases} \mathrm{d}p(t) &= q(t)\mathrm{d}B(t) + \int_{\mathbb{R}} \gamma(t,\zeta)r(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta) \\ p(T) &= U'(X_u(T)). \end{cases} \tag{5.10}$$

By using a necessary version of the maximum principle we can prove that these two conditions are both necessary and sufficient for a control u to be optimal. We formulate this as follows:

Theorem 5.2. *A control u is optimal for the utility maximization problem in Example 3.1 if and only if the solution* $(p(t), q(t), r(t,\cdot))$ *of the BSDE* (5.10) *satisfies the equation* (5.9)*.*

This result can be used to find the optimal portfolio in some cases. To illustrate this, we proceed as follows: Using Malliavin calculus we get:

$$\begin{cases} p(t) &= \mathbb{E}[R|\mathcal{F}_t] \\ q(t) &= \mathbb{E}[D_t R|\mathcal{F}_t] \qquad \text{where } R = U'(X_u(T)) \\ r(t,\zeta) &= \mathbb{E}[D_{t,\zeta} R|\mathcal{F}_t] \end{cases} \tag{5.11}$$

Substituting this back into (5.9) we get:

$$\alpha(t)\mathbb{E}[R|\mathcal{F}_t] + \beta(t)\mathbb{E}[D_t R|\mathcal{F}_t] + \int_{\mathbb{R}} \gamma(t,\zeta)\mathbb{E}[D_{t,\zeta} R|\mathcal{F}_t]\nu(\mathrm{d}\zeta) = 0.$$

This is a *Malliavin-type differential equation* in the unknown random variable R.

This type of Malliavin differential equation is discussed in [5]. The general solution of this equation is $R = R_{c,\theta}(T)$, where

$$\begin{aligned} R_{c,\theta}(t) = c\exp\Bigg[\int_0^t \theta_0(s)\mathrm{d}B(s) - \frac{1}{2}\int_0^t \theta_0^2(s)\mathrm{d}s \\ + \int_0^t\int_{\mathbb{R}} \ln(1+\theta_1(s,\zeta))\tilde{N}(\mathrm{d}s,\mathrm{d}\zeta) \\ + \int_0^t\int_{\mathbb{R}} \{\ln(1+\theta_1(s,\zeta)) - \theta_1(s,\zeta)\}\nu(\mathrm{d}\zeta)\mathrm{d}s\Bigg] \end{aligned} \tag{5.12}$$

i.e. $\mathrm{d}R_{c,\theta}(t) = R_{c,\theta}(t^-)\left[\theta_0(t)\mathrm{d}B(t) + \int_{\mathbb{R}}\theta_1(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta)\right]$
for arbitrary constant $c \in \mathbb{R}$, and any $\theta_0(t), \theta_1(t,z)$ satisfying the equation:

$$\alpha(t) + \beta(t)\theta_0(t) + \int_{\mathbb{R}} \gamma_0(t,z)\theta_1(t,z)\nu(\mathrm{d}z) = 0 \tag{5.13}$$

Note that if $c = 1$, then by the Girsanow theorem for Lévy processes, $R(t)$ is the density process (Radon-Nikodym derivative process) of an *equivalent martingale measure* Q, i.e., a measure Q equivalent to $\mathbb{P}$ such that the risky asset price given by

$$\mathrm{d}S(t) = S(t^-)\left[\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t) + \int_{\mathbb{R}}\gamma_0(t,\zeta)\tilde{N}(\mathrm{d}t,\mathrm{d}\zeta)\right] \tag{5.14}$$

is a martingale under Q.

For simplicity, assume that $\nu = 0$ from now on (i.e., that there are no jumps). Then (5.13) becomes: $\alpha(t) + \beta(t)\theta_0(t) = 0$ i.e.

$$\theta_0(t) = -\frac{\alpha(t)}{\beta(t)}.$$

Since $R_{c,\theta}(T) = U'(X(T))$ we have $X(T) = I(R_{c,\theta}(T))$, where $I := (U')^{-1}$.
Now that θ_0 is known, what about c?

Recall the equation for $X(t) = X_u(t)$:

$$\begin{cases} \mathrm{d}X(t) &= u(t)\left[\alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B(t)\right] \\ X(T) &= I(R_{c,\theta}(T)); \quad \theta = \theta_0 = -\frac{\alpha(t)}{\beta(t)}. \end{cases} \tag{5.15}$$

If we define $Z(t) = u(t)\beta(t)$, then we see that $X(t)$ satisfies the BSDE

$$\begin{cases} \mathrm{d}X(t) = \frac{\alpha(t)}{\beta(t)}Z(t)\mathrm{d}t + Z(t)\mathrm{d}B(t) \\ X(T) = I(R_{c,\theta}(T)). \end{cases} \tag{5.16}$$

The solution of this linear BSDE is

$$X(t) = \frac{1}{\Gamma(t)}\mathbb{E}\left[I(R_{c,\theta}(T))\Gamma(T)|\mathcal{F}_t\right] \tag{5.17}$$

where $\mathrm{d}\Gamma(t) = -\Gamma(t)\frac{\alpha(t)}{\beta(t)}\mathrm{d}B(t); \quad \Gamma(0) = 1.$

Now put $t = 0$ and take expectation to get

$$X(0) = x = \mathbb{E}\left[I(R_{c,\theta}(T))\Gamma(T)\right].$$

This determines the constant c and hence the optimal terminal wealth $X_u(T)$. Then, when the optimal terminal wealth $X_u(T)$ is known, one can find the corresponding optimal portfolio u by solving the BSDE above for $Z(t)$ and using that $Z(t) = u(t)\beta(t)$. We omit the details.

Remark 5.3. The advantage of this approach is that it applies to a general non-Markovian setting, which is inaccessible for dynamic programming. Moreover, this approach can be extended to case when the agent has only *partial information* to her disposal, which means that her decisions must be based on an information flow which is a subfiltration of $\mathcal{F}$. More information can be found in the references below.

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