Communications on Stochastic Analysis
Vol. 8, No. 1 (2014) 17-26

OPTIMAL COMBINED DIVIDEND AND PROPORTIONAL REINSURANCE POLICY

ERIYOTI CHIKODZA AND JULIUS N. ESUNGE

Abstract. This paper focuses on a classic optimization problem for an insurance firm in the presence of friction. In particular, we investigate the combined regular-singular control problem in the context of jump diffusions. Of interest to us is the situation in which the company desires to balance its dividend policy and risk exposure. We describe this situation and apply our result to a specific example.

1. Introduction

The problems of combined control of dividends and risk of an insurance firm, as well as portfolio optimization in the prsence of transaction costs have preoccupied many authors, for instance [1, 2, 3, 5, 6]. A more complex challenge for a typical insurance company is that of determining an optimal dividend policy in the presence of friction and simultaneously controlling the level of risk exposure. In such a case the firm is confronted with a mixed stochastic control problem which entails determining the optimal combined dividend and proportional reinsurance policy. Our goal is to investigate this latter problem using the combined regular and singular stochastic control theory.

In particular, we consider the jump diffusion case of the generalized combined regular and singular control problem. Risk control action consists in reinsuring a proportion of the incoming claims the insurance firm is contractually obliged to pay. This practice requires the insurance company to divert a certain percentage of the premiums to the reinsurance company.

We begin the discussion by placing the problem within its proper mathematical framework in Section 2, followed by our main result in Section 3. The final section includes an example on the application of the theory of generalized regular and singular control in the context of jump diffusions. The interested reader will find [4, 7] to be extremely helpful, the former providing a good background, while the latter is the main thrust. This work distinguishes itself from others (for instance [5, 6]), both in the reinsurance strategy chosen and the material in Section 4 which addresses a peculiar scenario.

Received 2014-2-18; Communicated by the editors.
2000 *Mathematics Subject Classification.* 49L20, 45K55, 60H10, 60H30.
Key words and phrases. Combined regular-singular control, jump diffusion, dividend and reinsurance policy.

2. Background and Problem Formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis and $\{B_t\}_{t\geq 0}$ a 1-dimensional standard Brownian motion with respect to $\{\mathcal{F}_t\}$.

Assume that if there are no interventions a stochastic process $\{X_t\}$ evolves according to the following equation [7]:

$$dX_t = \eta(t, X_t)dt + \beta(t, X_t)dB_t + \int_{\mathbb{R}} h(t, X_{t^-}, z)\tilde{N}(dt, dz);$$
$$X_0 = x > 0, \tag{2.1}$$

where η, β and h are Borel measurable real-valued functions. Let $\mathcal{H}$ be the set of all real-valued bounded functions such that Equation (2.1) has a unique strong solution. Here, $\tilde{N}(.,.)$ is a compensated Poisson random measure given by

$$\tilde{N}(dt, dz) = N(dt, dz) - dtv(dz);$$

where $v(.)$ is Lévy measure associated with the Poisson random measure $N(.,.)$.

Now, suppose that we consider harvesting from the system described by Equation (2.1). Then the dynamics of the process $\{X_t\}_{t\geq 0}$ are governed by

$$dX_t = \eta(t, X_t)dt + \beta(t, X_t)dB_t + \int_{\mathbb{R}} h(t, X_{t^-}, z)\tilde{N}(dt, dz)$$
$$-(1+\delta)dL_t; \quad X_0 = x > 0,$$

where L_t represents the total amount of resources harvested from the system up to time t, $0 \leq \delta \leq 1$ and L_t is right continuous, nonnegative, and $\mathcal{F}_t$ adapted.

If $L = 0$, the time-state process $(t, X(t))$ is a jump diffusion whose generator on $C_0^2(\mathbb{R}^2)$ coincides with the integro-differnetial operator $\mathcal{L}$ given by

$$\begin{aligned}\mathcal{L}\phi(s, x) = {} & \frac{\partial \phi}{\partial s}(s, x) + \eta(s, x)\frac{\partial \phi}{\partial x}(s, x) + \frac{1}{2}\beta^2(s, x)\frac{\partial^2 \phi}{\partial x^2} \\ & + \int_{\mathbb{R}} \{\phi(s, x + h(s, x, z)) - \phi(s, x) \\ & - h(s, x, z)\frac{\partial \phi}{\partial x}(s, x)\}v(dz).\end{aligned}$$

Define the terminal time T by

$$T = \inf\{t > s : (t, X_t) \notin S\} \leq \infty,$$

where S is an open and connected subset of $\mathbb{R}$. The terminal time T is the bankruptcy time.

Let $\mathcal{A}$ be the set of all mixed regular-singular controls of the form (η, β, h, L) such that Equation (2.1) has a unique strong solution. We call $\mathcal{A}$ the set of admissible controls. Define the performance functional, $J^{(\eta,\beta,h,L)}$ by

$$J^{(\eta,\beta,h,L)}(s, x) = E^{s,x}\left[\int_s^T u(s + t, X(t))dt + \int_s^T \pi(s + t, X(t))dL_t\right]$$

where $E^{s,x}$ is the expectation with respect to the probability law P.

In this case we impose the requirement that the utility rate $u : \mathbb{R}^2 \to \mathbb{R}$ be continuous, non-decreasing and concave.

The problem is to find the function Φ, and the optimal mixed control

$$(\eta^*, \beta^*, h^*, L^*) \in \mathcal{A}$$

such that

$$\Phi(s,x) = \sup_{(\eta,\beta,h,L)\in\mathcal{A}} J^{(\eta,\beta,h,L)}(s,x) = J^{(\eta^*,\beta^*,h^*,L^*)}(s,x). \tag{2.2}$$

3. Main Result

The central piece of this paper is the following theorem, and its proof.

Theorem 3.1. *Assume that $\pi(t,\xi)$ is decreasing with respect to ξ and for all t.*

(1) Suppose there exists a non-negative function $\phi \in C_0^2(S) \cap C(\bar{S})$ such that:
  (i) $\pi(t,x) - (1+\delta)\frac{\partial\phi}{\partial x} \le 0$ for all $(t,x) \in S$;
  (ii) $\mathcal{L}\phi(t,x) \le 0$ for all $(t,x) \in S$ and for all controls $\eta, \beta, h \in \mathcal{H}$;
  *(iii) $E^{s,x}\Big[\int_0^T \{|\beta(t,X(t)) \bigtriangledown \phi(t,X(t))|^2 + \int_{\mathcal{R}} |\phi(t,x+h)$
  $-\phi(t,x)|^2 v(dz)\}dt\Big] < \infty$; and*
  (iv) $\{\phi(t,X(t))\}_{t\le T}$ is uniformly integrable for all $(t,x) \in S$.
  Then

$$\phi(s,x) \ge \Phi(s,x) \text{ for all } (s,x) \in S. \tag{3.1}$$

(2) Define the non-intervention region D by

$$D = \left\{(t,x) \in S : \max\{\pi(t,x) - (1-\delta)\frac{\partial\phi}{\partial x}, L\phi(t,x)\} \le 0\right\}.$$

Suppose also that for all $(t,x) \in \bar{D}$:
  (v) $\mathcal{L}\phi(t,x) = 0$
  Moreover, suppose there exists a mixed control $\hat{v} = (\hat{\eta}, \hat{\beta}, \hat{h}, \hat{L}) \in \mathcal{A}$ such that
  (vi) $(t, X^{\hat{v}}(t)) \in \bar{D}$ for all $t > s$
  (vii) $\left(\frac{\partial\phi}{\partial x}(t, X^{\hat{v}}(t)) - \pi(t,x)\right) d\hat{L}^{(c)}(t) = 0$ where $i = 1, 2, ..., n$
  (viii) $\Delta\phi(t_j, X^{\hat{v}}(t_j)) := \phi(t_j, X^{\hat{v}}(t_j)) - \phi(t_j, X^{\hat{v}}(t_j^-))$ at all jumping times $t_j \ge s$; and
  (ix) $\lim_{R\to\infty} E^{s,x}[\phi(T_R, X^{\hat{v}}(T_{R^-})] = 0$, where

$$T_R = T \wedge R \wedge \inf\{t > s : |X^v(t)| \ge R\} \text{ for } R > 0.$$

Then

$$\phi(s,x) = \Phi(s,x) \text{ for all } (s,x) \in S \tag{3.2}$$

and

$v^* := \hat{v}$ is an optimal combined regular-singular control.

Proof. In view of the generalized Itô's formula for semi-martingales (see for example [8], page 74, Theorem 33), and mindful of Equation (2.1) we obtain the following:

$$\begin{aligned}
&\phi(s+T_R, X(T_R)) - \phi(s, X(0)) \\
&= \int_0^{T_R} \left(\frac{\partial\phi}{\partial s}(s,x) + \eta(s,x) + \frac{\partial\phi}{\partial x}(s,x) + \frac{1}{2}\beta^2(s,x)\frac{\partial^2\phi}{\partial x^2} \right. \\
&\quad + \left. \int_R \{\phi(s, x+h(s,x,z)) - \phi(s,x) - h(s,x,z)\frac{\partial\phi}{\partial x}(s,x)\}v(dz) \right) dt \\
&\quad \int_0^{T_R} \beta(s,x)\frac{\partial\phi}{\partial x} dB(t) - (1+\delta)\int_0^{T_R} \frac{\partial\phi}{\partial x} dL(t) \\
&\quad \int_0^{T_R}\int_R \{\phi(s, x+h(s,x,z)) - \phi(s,x)\}\tilde{N}(ds,dz) \\
&\quad + \sum_{0<t_j<T_R} \{\phi(t_j, X_{t_j}) - \phi(t_j, X_{t_j^-}) - \Delta X_{t_j}(s+,x,z)\frac{\partial\phi}{\partial x}(s,x)\}, \qquad (3.3)
\end{aligned}$$

where the sum is taken over all the jumping times $t_j \in [0, T_R]$ and

$$\Delta X_{t_j} = X_{t_j} - X_{t_j^-}$$

represents jumps at t_j, and similarly for $\Delta\phi$. Applying Dynkin's formula (see for example [7], page 12, Theorem 1.24) to Equation (3.3), we get:

$$\begin{aligned}
&E^{s,x}[\phi(s+T_R, X(T_R))] \\
&= \phi(s,x) + E^{s,x}\left[\int_0^{T_R} \mathcal{L}\phi(t,x)dt - \int_0^{T_R} u dt - \int_0^{T_R} \frac{\partial\phi}{\partial x} dL(t)\right] \\
&\quad + E^{s,x}\left[\sum_{0<t_j<T_R} \{\phi(t_j, X_{t_j}) - \phi(t_j, X_{t_j^-}) - \Delta X_{t_j}(s+,x,z)\frac{\partial\phi}{\partial x}(s,x)\}\right]. \qquad (3.4)
\end{aligned}$$

From condition $1(ii)$ we obtain

$$\begin{aligned}
&E^{s,x}[\phi(s+T_R, X(T_R))] \\
&\le \phi(s,x) - E^{s,x}\left[\int_0^{T_R} u dt + \int_0^{T_R} \frac{\partial\phi}{\partial x} dL(t)\right] \\
&\quad + E^{s,x}\left[\sum_{0<t_j<T_R} \{\phi(t_j, X_{t_j}) - \phi(t_j, X_{t_j^-}) - \Delta L(t_j)\frac{\partial\phi}{\partial x}(s,x)\}\right],
\end{aligned}$$

where $\Delta L(t_j) = L(t_j) - L(t_j^-)$. Consequently,

$$\begin{aligned}
&E^{s,x}[\phi(s+T_R, X(T_R))] \\
&\le \phi(s,x) - E^{s,x}\left[\int_0^{T_R} u dt + \int_0^{T_R} \frac{\partial\phi}{\partial x} dL^c(t)\right] \\
&\quad + E^{s,x}\left[\sum_{0<t_j<T_R} \{\phi(t_j, X_{t_j}) - \phi(t_j, X_{t_j^-})\}\right] \qquad (3.5)
\end{aligned}$$

where $L^c(t)$ is the continuous part of $L(t)$ defined by

$$L^c(t) = L(t) - \sum_{0<t_j<t} \Delta L(t_j).$$

Applying the mean value theorem on the two terms under the summation in (3.5) we obtain

$$\Delta\phi(t_j, X_{t_j}) = -\frac{\partial\phi}{\partial x}(s+t_j, \hat{X}_{t_j}).\Delta L(t_j) \tag{3.6}$$

for some point $\hat{X}_{t_j}$ joining the two points $X_{t_j^-}$ and X_{t_j}.

From equation 3.6 and condition (i) of Theorem (3.1) we see that

$$\phi(s,x) \geq E^{s,x}\left[\int_0^{T_R} udt + \int_0^{T_R} \pi(s+t)dL(t) + \phi(s+T_R, X^L(T_R))\right]. \tag{3.7}$$

Letting $R \to \infty$ in (3.7) we obtain

$$\phi(s,x) \geq \lim_{R\to\infty} \sup E^{s,x}\left[\int_0^{T_R} udt + \int_0^{T_R} \pi(s+t)dL(t)\right] \geq J^L(s,x). \tag{3.8}$$

This proves the requirement (3.1) for an arbitrarily chosen $L(t)$.

Equality in Equations (3.6), (3.7), and (3.8) follows in view of conditions (iv), (v) and (vi) and upon replacing $L(t)$ by $L^*(t)$. Consequently,

$$\phi(s,x) = \lim_{R\to\infty} E^{s,x}\left[\int_0^{T_R} udt + \int_0^{T_R} \pi(s+t)dL(t)\right]. \tag{3.9}$$

Taken together, Equations (3.1) and (3.9) yield the condition in Equation (3.2). In particular, we see that

v^* is an optimal combined regular singular control.

This completes the proof of Theorem 3.1. □

4. Application.

In this final section, we present a concrete situation in which the results from the preceding section are relevant.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space satisfying the usual conditions. Let $B(t)$ be a 1-dimensional standard Brownian motion with respect to $\{\mathcal{F}_t\}$.

Assume that the management of an insurance company is allowed to control the risk exposure of the firm by applying proportional reinsurance policy. Let $a(t)$ be the retention level, which is the fraction of all incoming claims that the firm will insure by itself. In addition, the company also applies a dividend policy. Suppose that if a dividend of magnitude ξ is paid out to share holders, a transaction cost $\delta\xi$ is incurred by the company as they process the dividends. Here $\delta \in (0,1)$ is a constant of proportionality. Thus, if the total dividend paid out up to time t is $L(t)$, the total transaction cost incurred is $\delta L(t)$.

Now suppose that as a result of applying the two dimensional policy, $c(t)$, given by

$$c(t) = (a(t), L(t)),$$

the liquid reserves, $X(t)$ at time t, of the insurance company evolve according to the following stochastic differential equation

$$dX(t) = \mu a(t)dt + \sigma a(t)dB(t) + \gamma a(t)\int_{\mathcal{R}} z\tilde{N}(dt,dz) - (1+\delta)dL(t),\ X(0) = x > 0, \tag{4.1}$$

where $L(t)$ is the cumulative dividend paid out up to time t and $0 < \delta < 1$ is as explained above. Note that μ and σ are positive constants and $\tilde{N}(.,.) = N(dt,dz) - v(dz)dt$ is a compensated Poisson random measure with Lévy measure v. It is also assumed that $\gamma z < 0$.

Define the performance functional J, by

$$J^c(x) := E\left[\int_0^\tau e^{-\rho t} X^\alpha(t) dL(t)\right],$$

where $\{X(t)\}$ is a solution of 4.1, τ is a stopping time given by

$$\tau = \inf\{t : X(t) \leq 0\} \text{ (time to bankruptcy)},$$

$\rho > 0$ is a discount factor, $0 < \alpha \leq 1$ and $E[.]$ denotes expectation with respect to probability law P.

Problem. *We seek the optimal combined control $c^* := (a^*(t), L^*(t))$ and the function $V(x)$ such that*

$$V(x) = \sup_{c\in\mathcal{A}} J^c(x) = J^{c^*}(x)$$

where $\mathcal{A}$ is the space of all admissible policies $c(t)$.

Solution. *We apply Theorem (3.1) to solve the above problem.*

In view of the problem represented by (2.2) we observe that

$$\eta(t,X_t) = \mu a(t), \quad \beta(t,X_t) = \sigma a(t), \quad h(t,X_{t^-},z) = \gamma a(t) z$$

where μ,σ,γ are given constants and $0 \leq a(t) \leq 1$.

Next, let us find the function V and the optimal control c^*. The following fact will be instrumental.

Lemma 4.1. *If $dL(t) = 0$, the function V satisfies the following Hamilton-Jacobi-Bellman equation*

$$\max_{a\in[0,1]}\left[\frac{\partial V}{\partial s}(s,x) + \mu a\frac{\partial V}{\partial x}(s,x) + \frac{1}{2}\sigma^2 a^2 \frac{\partial^2 V}{\partial x^2}(s,x) + \int_{\mathcal{R}}\{V(s,x+\gamma az) - V(s,x) - \gamma az\frac{\partial V}{\partial x}(s,x)\}v(dz)\right] = 0 \tag{4.2}$$

with initial condition

$$V(0) = 0. \tag{4.3}$$

In view of the above, we propose a candidate value function of the form

$$V(s,x) = e^{-\rho s} f(x).$$

In this case, Equation (4.2) can be expressed as

$$\max_{a\in[0,1]}[-\rho f(x)+\mu a f^{'}(x)+\frac{1}{2}\sigma^2a^2f^{''}(x) + \int_{\mathbb{R}}\{f(x+\gamma az)-f(x)-\gamma azf^{'}(x)\}v(dz)\Bigg]=0. \tag{4.4}$$

From Equation (4.4), we observe that the first order condition for the optimal control $a:=\hat{a}$ is given by

$$\mu f^{'}(x)+\sigma^2\hat{a}f^{''}(x) + \int_{\mathbb{R}}\{f^{'}(x+\gamma\hat{a}z)-f^{'}(x)\}\gamma zv(dz)=0. \tag{4.5}$$

Now let us choose $f(x)=e^{rx}$ for some constant $r\in\mathbb{R}$. Putting $a=\hat{a}$ and $V(s,x)=e^{-\rho s}e^{rx}$ into (4.2) we get

$$g(r):=-\rho+\mu\hat{a}r+\frac{1}{2}\sigma^2\hat{a}^2r^2+\int_{\mathbb{R}}\{e^{r\gamma\hat{a}z}-1-r\gamma\hat{a}z\}v(dz)=0.$$

Since $g(0)=-\rho<0$ and

$$\lim_{r\to\infty}g(r)=\lim_{r\to-\infty}g(r)=\infty,$$

we observe that the equation $g(r)=0$ has two distinct real solutions $r_1=r_1(a), r_2=r_2(a)$ such that

$$r_2(a)<0<r_1(a).$$

It follows that the solution to (4.2)-(4.3) is

$$C(e^{r_1x}-e^{r_2x}),$$

where C is an arbitrary constant. As such, setting

$$f(x)=C(e^{r_1(a)x}-e^{r_2(a)x})$$

in (4.2) we conclude that the optimal proportional reinsurance policy $a:=\hat{a}$ is the solution, namely

$$\begin{aligned}&\mu[r_1(\hat{a})e^{r_1(\hat{a})x}-r_2(\hat{a})e^{r_2(\hat{a})x}]+\sigma^2\hat{a}[r_1^2(\hat{a})e^{r_1(\hat{a})x}-r_2^2(\hat{a})e^{r_2(\hat{a})x}]\\ &\quad+\int_{\mathbb{R}}\{r_1(\hat{a})e^{r_1(\hat{a})(x+\gamma\hat{a}z)}-r_2(\hat{a})e^{r_2(\hat{a})(x+\gamma\hat{a}z)}\\ &\quad-r_1(\hat{a})e^{r_1(\hat{a})x}-r_2(\hat{a})e^{r_2(\hat{a})x}\}\gamma zv(dz)=0.\end{aligned} \tag{4.6}$$

Next, let us consider the dividend control. To this end, for $\alpha=-\frac{1}{2}$, the performance functional is given by

$$J^{(u)}(s,x):=E\left[\int_s^\tau e^{-\rho(s+t)}(X(t))^{-\frac{1}{2}}dL(t)\right].$$

Now, we observe that if we apply the "*take the money and run*"-strategy, $\dot{u}$, then all the resources are taken out immediately, and no reinsurance policy is applied, that is $a^*(t)=0$. Such a strategy is described by

$$\dot{u}(s)=\dot{L}(s)=(1-\delta)x.$$

The value function obtained from this strategy is

$$\Phi(s,x) = e^{-\rho s}x^{-\frac{1}{2}}(1-\delta)x = e^{-\rho s}(1-\delta)\sqrt{x};\ x > 0.$$

This strategy is not likely to be optimal since it does not take into account the impact of transaction costs on total discounted gains, neither does it cater for the price increases as the resources diminish. The strategy also does not consider the benefits that accrue from reinsuring a proportion of the incoming claims.

Our choice of a candidate value function is inspired by a kind of "chattering strategy". To this end, we propose

$$\phi(s,x) := \frac{2e^{-\rho s}}{1+\delta}\sqrt{x} \tag{4.7}$$

as a possible value function.

We note that the function given in Equation (4.7) satisfies the conditions of Theorem 3.1.

Using the second-order integro-partial-differential operator

$$\begin{aligned}\mathcal{L}\phi(s,x) &= \frac{\partial\phi}{\partial s} + \mu a\frac{\partial\phi}{\partial x} + \frac{1}{2}\sigma^2 a^2\frac{\partial^2\phi}{\partial x^2}\\ &\quad + \int_{\mathcal{R}}\left\{\phi(s,x+\gamma az) - \phi(s,x) - \gamma az\frac{\partial\phi}{\partial x}\right\}v(dz),\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{L}\phi(s,x) &= \frac{e^{-\rho s}}{1+\delta}\left[-2\rho x^{\frac{1}{2}} + \mu a x^{-\frac{1}{2}} - \frac{\sigma^2a^2}{4}x^{-\frac{3}{2}}\right.\\ &\quad \left. + \int_{\mathcal{R}}\{2\sqrt{x+\gamma az} - 2x^{\frac{1}{2}} - \gamma azx^{-\frac{1}{2}}\}v(dz)\right]\\ &\le \frac{e^{-\rho s}}{1+\delta}\left[-2\rho x^{\frac{1}{2}} + \mu a x^{-\frac{1}{2}} - \frac{\sigma^2a^2}{4}x^{-\frac{3}{2}}\right.\\ &\quad \left. + \int_{\mathcal{R}}\{2\sqrt{x} - 2x^{\frac{1}{2}} - \gamma azx^{-\frac{1}{2}}\}v(dz)\right]\\ &= \frac{e^{-\rho s}}{1+\delta}\left[-2\rho x^{\frac{1}{2}} + \mu a x^{-\frac{1}{2}} - \frac{\sigma^2a^2}{4}x^{-\frac{3}{2}}\right.\\ &\quad \left. - \int_{\mathcal{R}}\gamma zax^{-\frac{1}{2}}v(dz)\right].\end{aligned}$$

We have applied the fact that $\gamma z \le 0$ and $a \in [0,1]$. Thus

$$\begin{aligned}\mathcal{L}\phi(s,x) &\le \frac{-2\rho e^{-\rho s}}{1+\delta}x^{-\frac{3}{2}}\left[x^2 - \frac{\mu a}{2\rho}x + \frac{\sigma^2}{8\rho} + x\int_{\mathcal{R}}\frac{\gamma az}{2\rho}v(dz)\right]\\ &= \frac{-2\rho e^{-\rho s}}{1+\delta}x^{-\frac{3}{2}}\left[x^2 + \left(\int_{mathcalR}\frac{\gamma az}{2\rho}v(dz) - \frac{\mu a}{2\rho}\right)x + \frac{\sigma^2a^2}{8\rho}\right].\end{aligned}$$

From the above, if

$$\left(\int_{\mathcal{R}}\gamma zv(dz) - \mu\right)^2 \le 2\rho\sigma^2$$

then$\phi(x) = \Phi(x)$.

Putting

$$f(x) = \frac{2\sqrt{x}}{1+\delta}$$

in (4.2) gives

$$\frac{\mu}{\sqrt{x}} - \frac{\sigma^2 \hat{a}}{2\sqrt{x^3}} + \int_{\mathcal{R}} \left\{ \frac{1}{\sqrt{x+\gamma\hat{a}z}} - \frac{1}{\sqrt{x}} \right\} \gamma z v(dz) = 0. \tag{4.8}$$

In summary, the following is true:

Theorem 4.2. *Suppose $X^{(u)}(t)$ is given by Equation (4.1).*

(1) Assume that

$$\left(\int_{\mathbb{R}} \gamma z v(dz) - \mu \right)^2 \leq 2\rho\sigma^2.$$

Then

$$\Phi(s,x) := \frac{2e^{-\rho s}}{1+\delta}\sqrt{x}$$

where σ and ρ are defined as before. This value is achieved in the limit if we apply the strategy $\tilde{u}^{(m,\eta)}$ described above with $\eta \to 0$ and $m \to \infty$, that is, by applying the policy of immediate chattering to 0.

(2) If

$$\left(\int_{\mathbb{R}} \gamma z v(dz) - \mu \right)^2 > 2\rho\sigma^2$$

then the value function takes the form

$$\Phi(s,x) = \begin{cases} e^{-\rho s} A(e^{r_1 x} - e^{r_2 x}), & \text{for } 0 \leq x < x^* \\ e^{-\rho s}\left(\frac{2}{1+\delta}\sqrt{x} - \frac{2}{1+\delta}\sqrt{x^*} + B\right), & \text{for } x^* \leq x \end{cases}$$

for some constants $A > 0, B > 0$ and $x^* > 0$ where r_1 and r_2 are the solutions of the equation

$$-\rho + \mu a r + \frac{1}{2}\sigma^2 a^2 r^2 + \int_{\mathcal{R}} \{ e^{r\gamma a z} - 1 - r\gamma a z \} v(dz) = 0$$

with $r_2 < 0 < r_1$ and $|r_2| > r_1$.

In conclusion, we have:

Remark 4.3. In both cases 1. and 2. of the above Theorem, the corresponding optimal dividend policy is the following:

(a) If $x > x^*$ it is optimal to apply immediate chattering from x down to x^*.
(b) If $0 < x < x^*$ it is optimal to apply the harvesting equal to the local time of the downward reflected process $\bar{X}(t)$ at x^*.

Constants A, B and x^* can be determined using the principle of smooth fit at x^*. The optimal reinsurance policy $a^* := \hat{a}$ is determined by the solution of

(c) Equation (4) if $x > x^*$.

(d)

$$\mu[r_1(\hat{a})e^{r_1(\hat{a})x} - r_2(\hat{a})e^{r_2(\hat{a})x}] + \sigma^2\hat{a}[r_1^2(\hat{a})e^{r_1(\hat{a})x} - r_2^2(\hat{a})e^{r_2(\hat{a})x}]$$
$$+ \int_{\mathcal{R}} \{r_1(\hat{a})e^{r_1(\hat{a})(x+\gamma\hat{a}z)} - r_2(\hat{a})e^{r_2(\hat{a})(x+\gamma\hat{a}z)}$$
$$- r_1(\hat{a})e^{r_1(\hat{a})x} - r_2(\hat{a})e^{r_2(\hat{a})x}\}\gamma z v(dz) = 0$$

if $0 < x < x^*$.

Acknowledgment. Our gratitude goes to the referees who provided very helpful comments.

References

1. Akian, M., Menaldi, J. L., and Sulem, A.: On an investment-consumption model with transaction costs, *SIAM Journal of Control and Optimization* **34** (1996) no.1 329–364.
2. Davis, M. H. A. and Norman, A. R.: Portfolio selection with transaction costs, *Mathematics of Operations Research* **15** (1990), no. 4, 676–713.
3. Framstad, N. C., Øksendal, B., and Sulem, A.: Optimal consumption and portfolio in a jump diffusion market with proportional transaction costs, *Journal of Mathematical Economics* **35** (2001) 233–257.
4. Hanson, F.: *Applied Stochastic Processes and Control for Jump-Diffusions: Modeling, Analysis, and Computation (Advances in Design and Control)*, SIAM (2007).
5. He, L. and Liang, Z.: Optimal financing and dividend control of the insurance company with proportional reinsurance policy, *Insurance, Mathematics and Economics* **42** (2008) 976–983.
6. He, L., Liang, Z. and Wu, J.: Optimal dividend and reinsurance strategy of a property insurance company under catastrophe risk, arXiv:1009.1269 (2010).
7. Øksendal, B. and Sulem, A.: *Applied Stochastic Control of Jump Diffusions,* Springer-Verlag (2005).
8. Protter, P.: *Stochastic Integration and Differential Equations,* Springer-Verlag (2004).

Eriyoti Chikodza: Department of Mathematics and Computer Science, Great Zimbabwe University, P.O. Box 1235, Masvingo, Zimbabwe
E-mail address: egchikodza@yahoo.co.uk

Julius N. Esunge: Department of Mathematics, University of Mary Washington, Fredericksburg, VA 22401, USA
E-mail address: jesunge@umw.edu