

## MODELLING FINANCIAL INFORMATION BY CONDITIONING

DENNIS IKPE, SURE MATARAMVURA\*, AND RONNIE BECKER

ABSTRACT. In this paper, we model the market information in the information based asset pricing framework of Brody, Hughston and Macrina (BHM) as a conditioned stochastic process (CSP) associated with the conditioning,  $(T, Y, \mu)$  of a functional of a Wiener process. Specifically, the flow of information to the market is generated by the process,  $(I_{tT})_{0 \leq t < T}$  associated with  $(T, Y, \mu)$ . With  $\mu$  set as the *a priori* probability law of a cash flow occurring at time  $T$ , we are able to derive the price process of an asset whose market information is given by the filtration  $(\mathcal{F}_t^I)$  generated by  $I_{tT}$ .

### 1. Introduction

The information based asset pricing theory of Brody, Hughston and Macrina, [4] presents researchers with the important problem of modeling the flow of market information. This problem involves finding a suitable class of models for the information regarding the cash flow of wider class of assets in the market, where issues of tractability and computational complexities are appropriately taken into consideration. In the information based asset pricing framework, an asset is associated with a sequence of random cash flows. The price of the asset is given as the sum of the discounted conditional expectations of the cash flows. The conditional expectation is taken with respect to the filtration generated by the information process. The complexity and tractability of the conditional expectation depends on (i) nature of the cash flow (ii) the law of the information process. As such modelling the information process in the information based asset pricing framework is akin to specifying its law and /or the nature of the cash flow. In this paper, we model the nature of the cash flow by a conditioning on a filtered probability space and define the information process as the unique stochastic process associated with the conditioning. Then we derive the law of the resulting information process using techniques from classical theory of conditioned stochastic differential equations. The information process consist of a signal component and an independent noise which vanishes as the final time approaches. Following the simple Brownian bridge and Gamma bridge models of market information by Brody et al in [4], Hoyle et al in [11] extended the market information model to a wider class of models called Lévy Random Bridges (LRBs). In both [4] and [11], the authors postulate the existence of a market information process and obtain closed form expression for

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prices of European style contracts. In particular they model the signal as the cash flow of a given asset at a specified future time. The information process is then given by the unique stochastic process whose terminal distribution matches the specified *a priori* distribution of the cash flow. However, there is still an ongoing need for more general market information models which are not only attractive due to their tractability but for practical purposes are suitable as market information models for a wider class of assets, different level of investors and broader financial markets. More specifically, there is need to study market information models that are appropriate for asset pricing when the signal is different from the terminal cash flow of an asset. For example the signal may be in the form of realized volatility of an asset (or its derivative) or in the form of the knowledge of the distribution of the first hitting time of the asset to a certain level in a given period. This type of situation is relevant in pricing American style options as well as Asian options. To incorporate situations as described above into the information based asset pricing, we propose in this present work to model the information process as the unique solution of a stochastic differential equation associated with a conditioning. These processes we refer here as Conditioned Stochastic Processes (CSP). We consider CSPs which are semi-martingale and Markovian.

The outline of the paper is as follows; In section 2, we present the precise definition of Conditioned Stochastic Process (CSPs). First we present important results from the theory of conditioned stochastic differential equations (CSDEs)-see [2, 3]. Then we define a Conditioned Stochastic Process as the unique solution to a CSDE. The emphasis is on the properties of CSPs that make them appropriate for the modeling of the dynamics of a market information process. We also verify the bridge and Markov properties of the class of CSPs that we consider here.

In section 3, we describe the information based asset pricing framework of Brody, Hughston and Macrina, [4]. The view in the information based asset pricing framework is that it is unsatisfactory to simply fix the market filtration and assume that asset price processes are adapted to it without indicating the nature of the information, which the background filtration represents. Thus, this framework is based on modeling the flow of market *information*. This information consists of a signal and a perturbing noise component. We present the two existing modeling approaches for the market information process and describe how these approaches differ from our method. In section 4, we model the market information by conditioning. We established that the Brownian bridge information process can be constructed as conditioned stochastic process. Precisely, we re-derive the Brownian bridge information process of [4] as a CSP associated with conditioning of the marginal law of a Brownian motion for a specified initial condition. In conclusion, we derive expression for the price of European option and a binary credit risky bond. Throughout the paper, we fix a probability space  $(\Omega, \mathbb{Q}, \mathcal{F})$  and assume that all processes and filtration under consideration are adapted to it. We consider the time horizon,  $[0, T]$ ,  $T \in (0, \infty]$  and assume that all stochastic processes take value in  $\mathbb{R}$ .

## 2. Conditioned Stochastic Process

**2.1. General framework.** In this section, we give the general set-up for information based asset pricing where the market information is modelled as conditioned stochastic process in the Wiener space.

We consider a constant horizon time,  $T \in (0, \infty]$  and the space

$$(C_\infty, \{\mathcal{F}_t\}_{t \geq 0}, \{X_t\}_{t \geq 0}, \mathbb{P}),$$

where  $C_\infty$  is the space of continuous functions  $\mathbb{R}_+ \rightarrow \mathbb{R}$  (for  $T > 0$ ,  $C_T$  will denote the space of continuous functions  $[0, T] \rightarrow \mathbb{R}$ ),  $\{X_t\}_{t \geq 0}$  is the coordinate process defined by  $X_t(g) = g(t)$ , and  $\mathbb{P}$  is the Wiener measure. We want to construct the information process,  $I_{tT} = F(H_T, \eta_{tT})$ , which generates the filtration  $\{\mathcal{F}_t\}_{t \geq 0}$ .  $H_T$  is an  $\mathcal{F}_T$ -measurable random variable with *a priori* law,  $\mu$  and  $\eta_{tT}$  is the noise.  $I_{tT}$  is defined as the conditioned stochastic process associated with the conditioning,  $(T, Y, \mu)$ , where a certain functional,  $Y$  of the process  $\{X_t\}$  is forced to assume the law,  $\mu$  at a fixed future time,  $T \in (0, \infty]$ .

**Definition 2.1.** A *conditioning* on a probability space is a triplet  $(T, Y, \mu)$  with the following properties;

- (1) there exists a jointly measurable process  $\Lambda_t^y$ ,  $0 \leq t < T$ ;  $y \in \mathbb{R}$  such that for any random variable  $Z$  bounded and  $\mathcal{F}_t$ -measurable,  $t < T$ , and  $\mathbb{P}_Y$  (where  $\mathbb{P}_Y$  is the law of  $Y$  under  $\mathbb{P}$ ) -a.s.  $y \in \mathbb{R}$

$$\mathbb{E}(Z|Y = y) = \mathbb{E}(\Lambda_t^y Z) \quad (2.1)$$

- (2)  $\text{Supp } \mu \subset \text{Supp } \mathbb{P}_Y$  and  $L^1(\mathbb{R}, \mathbb{P}_Y) \subset L^1(\mathbb{R}, \mu)$ .

$T \in (0, \infty]$  corresponds to the period of time on which the conditioning is made.  $Y$  is a  $\mathcal{F}_T$ -measurable random variable with values in  $\mathbb{R}$  and differentiable in Malliavin sense. It represents a functional of the trajectory being conditioned.  $\mu$  is the probability measure on the Borel  $\sigma$ -algebra,  $\mathcal{B}(\mathbb{R})$  corresponding to the actual conditioning.

**Theorem 2.2.** *There exists a unique probability measure,  $\mathbb{P}^\mu$  such that*

- *if  $Z : (C_T; \mathcal{F}_T) \rightarrow (\mathbb{R}; \mathcal{B}(\mathbb{R}))$  is a bounded random variable then*

$$\mathbb{E}_\mu(Z|Y) = \mathbb{E}(Z|Y) \quad (2.2)$$

- *the law of  $Y$  under  $\mathbb{P}^\mu$  is precisely  $\mu$ .*

$\mathbb{P}^\mu$  is given by the formula: For  $A \in \mathcal{F}_T$ ;

$$\mathbb{P}^\mu(A) = \int_{\mathbb{R}} \mathbb{P}(A|Y = y) \mu(dy). \quad (2.3)$$

The proof of the above theorem can be found in ([2], p. 119).

Given a conditioning,  $(T, Y, \mu)$  on a Wiener space, we define the associated conditioned stochastic process as follows;

**Definition 2.3.** Let a standard Brownian motion  $\{w_t\}_{0 \leq t < T}$  be defined on a filtered probability space  $(\Omega, \{\mathcal{H}_t\}_{0 \leq t < T}, \mathbb{Q})$ , the stochastic process

$$I_{tT} = \int_0^t \left[ \frac{\int_{\mathbb{R}} \alpha_s^y \Lambda_s^y \mu(dy)}{\int_{\mathbb{R}} \Lambda_s^y \mu(dy)} \right]_{s \leq t} ds + w_t, \quad t < T \quad (2.4)$$

is called a *conditioned stochastic process* (CSP) associated with the conditioning  $(T, Y, \mu)$ .

Here,  $\alpha_t^y$  is a measurable process such that:

- (1) For  $\mathbb{P}_Y - a.s.$ ,  $y \in \mathbb{R}$ ,  $\alpha_t^y$  is  $\mathcal{F}$ -predictable
- (2) For  $\mathbb{P}_Y - a.s.$ ,  $y \in \mathbb{R}$  and for  $0 \leq t < T$

$$\mathbb{P} \left( \int_0^t \alpha_s^y ds < +\infty \right) = 1.$$

- (3) For  $\mathbb{P}_Y - a.s.$ ,  $y \in \mathbb{R}$  and for  $0 \leq t < T$

$$\langle \Lambda^y, X \rangle_t = \int_0^t \alpha_s^y \Lambda_s^y ds.$$

**Lemma 2.4.** *The probability law of the conditioned stochastic process (CSP),  $\{I_{tT}\}_{0 \leq t < T}$ , associated with the conditioning,  $(T, Y, \mu)$  is given by*

$$\mathbb{P}^\mu(I_{tT} \in dx) = \int_{y=-\infty}^{y=\infty} \mathbb{P}(I_{tT}|Y = y)\mu(dy)dx. \quad (2.5)$$

*Proof.* This is a direct consequence of theorem (2.2).  $\square$

**2.2. Markov property of conditioned stochastic processes.** It is desirable in the information based asset pricing framework for the model of market information to possess the Markov property. In this section, we show that CSPs are Markov processes. This simplifies subsequent calculations in the determination of asset price dynamics. As we shall see, the Markov property of the CSPs follows from the independent increments of the driving Wiener process. The independent increment property of Lévy processes makes the extension of this approach to a general Lévy models a viable endeavor.

**Proposition 2.5.** *Given a conditioning,  $(T, Y, \mu)$ , the associated Conditioned Stochastic Process,  $\{I_{tT}\}_{0 \leq t < T}$  is a Markov process.*

*Proof.* Let  $\mathcal{F}_t^I$  denote the filtration generated by the process,  $\{I_{tT}\}_{0 \leq t < T}$ . Then we need to verify that

$$\mathbb{E}[h(I_{tT})|\mathcal{F}_s^I] = \mathbb{E}[h(I_{tT})|I_{sT}] \quad (2.6)$$

for any bounded measurable function  $h(x)$  and for all  $s, t$  such that  $0 \leq s \leq t < T$ . It suffices to show that

$$\mathbb{E}[h(I_{tT})|I_{sT}, I_{s_1T}, I_{s_2T}, \dots, I_{s_nT}] = \mathbb{E}[h(I_{tT})|I_{sT}] \quad (2.7)$$

for any collection of times  $t, s, s_1, s_2, \dots, s_n$  such that  $0 \leq s_n \leq \dots \leq s_2 \leq s_1 \leq s \leq t < T$ . First we note that the conditioned process,  $\{I_{tT}\}_{0 \leq t < T}$  can be expressed in the form

$$I_{tT} = \int_0^t H(s, (X_u)_{u \leq s}) ds + \omega_t \quad (2.8)$$

where  $H$  is a predictable function such that for all  $t < T$

$$\frac{\int_{\mathbb{R}} \alpha_s^y \Lambda_s^y \mu(dy)}{\int_{\mathbb{R}} \Lambda_s^y \mu(dy)} = H(t, (X_s)_{s \leq t}). \quad (2.9)$$

We know that for a standard Brownian motion, the increments,  $\omega_s - \omega_{s_1}, \omega_{s_1} - \omega_{s_2}, \dots, \omega_{s_{n-1}} - \omega_{s_n}$  are independent for  $s_n \leq s_{n-1} \leq \dots \leq s_2 \leq s_1 \leq s$ . Then it follows that

$$\begin{aligned} \mathbb{E}[h(I_{tT})|I_{sT}, I_{s_1T}, I_{s_2T}, \dots, I_{s_nT}] &= \mathbb{E}[h(I_{tT})|I_{sT}, I_{sT} - I_{s_1T}, \dots, I_{s_{n-1}T} - I_{s_nT}] \\ &= \mathbb{E}[h(I_{tT})|\omega_s, \omega_s - \omega_{s_1}, \dots, \omega_{s_{n-1}} - \omega_{s_n}] \\ &= \mathbb{E}[h(I_{tT})|I_{sT}], \end{aligned}$$

since  $I_{tT}$  and  $I_{sT}$  are independent of  $\omega_s - \omega_{s_1}, \omega_{s_1} - \omega_{s_2}, \dots, \omega_{s_{n-1}} - \omega_{s_n}$ .  $\square$

**2.3. Conditional terminal law.** In this section, we want to show that  $\mathcal{F}_s^I$ -conditional law (density) of the terminal value  $I_{TT} = \lim_{t \rightarrow T} I_{tT}$  exists. Given a conditioning  $(T, Y, \mu)$ , we write the law of  $Y$  under  $\mathbb{P}$  as  $\mathbb{P}_Y(Y \in dy|\mathcal{F}_t^I) = p_t(x, y)dy$ . Let  $\mu_s$  denote the  $\mathcal{F}_s^I$ -conditional law of  $I_{TT}$ . We have  $\mu_0(A) = \mu(A)$ . Then for  $s > 0$ , it follows from theorem 2.4 and equation (2.1) that

$$\mu_s(z; dy) = \frac{\Lambda_s^y \mu(dy)}{\int_{-\infty}^{\infty} \Lambda_s^y \mu(dy)}. \quad (2.10)$$

The *a priori*  $k$ th moment of the terminal value,  $I_{TT}$  is given by

$$\int_{-\infty}^{\infty} |y|^k \mu_t(x; dy).$$

If

$$\int_{-\infty}^{\infty} |y|^k \mu(dy) < \infty, \quad (2.11)$$

then the  $\mathcal{F}_t^I$ -conditional  $k$ th moment of  $I_{TT}$  is finite.

**Proposition 2.6.** *Let  $M_k(I_{tT}) = \int_{-\infty}^{\infty} |y|^k \mu_t(x; dy)$ . If (2.11) holds for  $k \in \mathbb{Z}$ , then  $M_k(I_{tT})$  is a martingale with respect to  $\mathcal{F}_t^I$ .*

*Proof.* We want to show that under the probability  $\mathbb{P}^\mu$ ,

$$\mathbb{E}(M_k(I_{tT})|\mathcal{F}_s) = M_k(I_{sT}).$$

Note that for  $s \leq t$ ,  $\mathcal{F}_s \subseteq \mathcal{F}_t$ . Then using the tower property of conditional expectation we have

$$\begin{aligned} \mathbb{E}(M_k(I_{tT})|\mathcal{F}_s) &= \mathbb{E}(\mathbb{E}(|y|^k|\mathcal{F}_t) |\mathcal{F}_s) \\ &= \mathbb{E}(|y|^k|\mathcal{F}_s) \text{ ( by the tower property)} \\ &= \mathbb{E}(|y|^k|I_{sT}) \\ &= M_k(I_{sT}) \end{aligned}$$

as required.  $\square$

### 3. Information Based Asset Pricing

In this section we give a brief description of the information based asset pricing theory of Brody, Hughston and Macrina, [4, 5]. The objective is to outline the important ideas behind the framework so that the reader will easily follow the applications of conditioned stochastic processes to the theory.

We fix a time horizon,  $[0, T]$ ,  $T > 0$  and a probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$ . We consider a deterministic interest rate  $\{r_t > 0\}$  then the time sprice of a zero-coupon bond that pays 1 unit of currency at maturity  $t$ , is given by

$$P_{st} = \exp\left(-\int_s^t r_u du\right), \quad (s \leq t). \quad (3.1)$$

For  $t < T$  the time  $t$  price of a cash flow,  $H_T$  at time  $T$  is given by

$$H_{tT} = P_{tT}\mathbb{E}(H_T|\mathcal{F}_t),$$

where  $\{\mathcal{F}_t\}$  is the *market information*. This represents the information available to the market participants. We assume that there exists an information process  $\{I_{tT}\}_{0 \leq t < T}$  (possibly multi-dimensional) which generates the market filtration (i.e.  $\mathcal{F}_t = \sigma(\{I_{sT}\}_{0 \leq s \leq t})$ ). The challenge to the analyst is to construct an appropriate model for  $\{I_{tT}\}_{0 \leq t < T}$  for an asset with cash flow  $H_T = h(X_T)$  for some function  $h(x)$  and market factor  $X_T$ , called the  $X$ -factor. In their initial paper, Brody et al. [4] considered a finite time horizon,  $T < \infty$  and used a *heuristic* approach to construct a model of the market information process. The Brownian bridge and Gamma bridge information processes of Brody, Hughston and Macrina, [4] were first obtained by this approach. Brownian bridge information process is modelled explicitly as

$$I_{tT} = \alpha t H_T + \beta_{tT}, \quad I_{TT} = H_T,$$

where the process  $\{\beta_{tT}\}_{0 \leq t \leq T}$  is a standard Brownian bridge and the constant,  $\alpha$  denotes the rate of information arrival. Similarly, Gamma bridge information process is modelled explicitly as

$$I_{tT} = H_T \gamma_{tT}, \quad (0 \leq t \leq T),$$

where  $\{\gamma_{tT}\}_{0 \leq t \leq T}$  is a Gamma bridge process starting at 0 and ending at 1. See [4, 5] for more explanations on this approach. More recently, Hoyle et al. [11] used a different approach to construct market information process, which we refer to as the *probabilistic* approach. The view in Hoyle's paper is that the task of modeling the evolution of market information can be reduced to that of specifying the law of the information process,  $I_{tT}$ . In this approach, the authors consider a finite time horizon and  $I_{tT}$  is modelled as a *Lévy Random Bridge* (LRB). The LRBs are obtained by the specification of the law,  $LRB_{\mathcal{C}}([0, T], \{f_t\}, \nu)$  or  $LRB_{\mathcal{D}}([0, T], \mathcal{Q}_t, \mathcal{P})$  for continuous and discrete cases respectively. For example, the Brownian bridge information process corresponds to the case where  $\{f_t\}$  is the density process of a standard Brownian motion and  $\nu$  is the marginal law of a Brownian motion. In this paper we use a different approach to model the market information. Specifically, we model the market information process by a conditioning on the Wiener space. Our approach allows for infinite time horizon and to price assets whose cash flows are determined at random times. The literature on this topic has so far

avoided incorporation of random times in the framework. For example, “default” of credit risky bonds is redefined in the framework to simply imply a failure of the bond issuer to meet debt obligation at the maturity date of the bond. Several extensions of the framework till date has maintained this view (See,[4, 11]). The key idea in modeling the information process as a conditioned stochastic process is that the signal component can be generalized to include other functionals of the market factor than its value at a fixed future date. For instance, we can consider a case where the signal is in the form of the knowledge of the distribution of first hitting time of the market factor to a certain level in a given period. In a subsequent paper, we shall use this approach to construct price processes for more general class of assets. It was mentioned in [11] that the Brownian bridge information process is identical in law to a conditioned Brownian motion. In the next section we define the conditioned stochastic information process and construct the Brownian bridge information process explicitly as a conditioned stochastic process using the results from previous sections. The construction of the Gamma bridge information process using our approach would require the development of the theory of conditioned stochastic processes for general Lévy subordinators and we have omitted it in this present paper.

#### 4. Market Information by Conditioning

**4.1. Conditioned Brownian motion.** Perhaps the best way to illustrate the intuition that market information process as described in the information based asset pricing framework can be modelled by a conditioning in the Wiener space is by way of example. Thus the objective in this section is to construct the Brownian bridge information process of Brody et al. [4] as a conditioned stochastic process. Specifically, we obtain the explicit form of the process as a solution of a conditioned stochastic differential equation associated with the conditioning of the marginal law of a standard Brownian motion. The intention is that this simple application of the theory of conditioned stochastic differential equations will motivate further research in the theory of CSDEs associated with the conditioning of the marginal law of other forms of Lévy processes (e.g. Gamma and Stable-half processes) which in turn will pave way for further application of the theory to the construction of wider class of market information process. In [4], the Brownian bridge information process is given by

$$Y_{tT} = \alpha t X_T + \beta_{tT}, \quad Y_{TT} = X_T, \tag{4.1}$$

where  $\alpha$  is a constant, denoting rate of information flow to market participants,  $X_T$  is the payoff of the asset at the terminal point,  $T$  and  $\beta_{tT}$  is the Brownian bridge process on the interval  $[0, T]$ , with  $\beta_{0,T} = \beta_{TT} = 0$ .

*Remark 4.1.* Recall that  $\beta_{tT}$  has mean zero and the covariance of  $\beta_{sT}$  and  $\beta_{tT}$  is  $\frac{s(T-t)}{T}$ .

We consider the filtered probability space  $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})$  on which a standard  $(\mathcal{F}_t, \mathbb{Q})$  Brownian motion  $\{\omega_t\}_{0 \leq t \leq T}$  is defined. The expression for the CSDE associated with the conditioning of marginal law of a standard Brownian motion is given by (2.4).

**Proposition 4.2.** *The market information process defined by (4.1) is a Conditioned Stochastic Process associated with the conditioning of the marginal law of a standard Brownian motion, with the initial condition  $Y_{0T} = 0$ .*

*Proof.* Let  $(T, \omega_T, \nu)$  denote the conditioning of the marginal law of a standard Brownian motion.  $W_T$  is the value at  $T$  of a standard Brownian motion.  $\nu$  is a probability measure such that for  $x \in \mathbb{R}$ ,

$$\int_{\mathbb{R}} x^2 \nu(dx) < \infty$$

and the law of  $W_T$  in  $\mathbb{Q}$  is  $\nu$ . Let  $\alpha_t^y = \frac{y-x}{T-t}$ , then from (2.8) we obtain;

$$dI_{tT} = \frac{\int_{-\infty}^{+\infty} \left( \frac{x-I_{tT}}{T-t} \right) e^{\frac{x^2}{2T} - \frac{(x-I_{tT})^2}{2(T-t)}} \nu(dx)}{\int_{-\infty}^{+\infty} e^{\frac{x^2}{2T} - \frac{(x-I_{tT})^2}{2(T-t)}} \nu(dx)} dt + d\omega_t, \quad t < T, \quad I_{0T} = 0. \quad (4.2)$$

Now, let us choose  $\nu(dx) = \frac{e^{-(x-mT)^2/2T}}{\sqrt{2\pi T}} dx$ , then Eq. (4.2) becomes

$$dI_{tT} = m dt + d\omega_t.$$

The corresponding solution associated with the initial value,  $I_{0T} = 0$  is

$$I_{tT} = mt + \omega_t, \quad t < T. \quad (4.3)$$

Recall that the Brownian bridge,  $\beta_{tT}$  can be transformed into a Brownian motion with drift as

$$\beta_{tT} = \omega_t - \frac{t}{T} \omega_T.$$

Hence, Eq. (4.3) can be re-written as

$$I_{tT} = \left( m + \frac{\omega_T}{T} \right) t + \beta_{tT}. \quad (4.4)$$

Now, setting  $m + \frac{\omega_T}{T} = \alpha X_T$  (where  $\alpha$  is constant and  $X_T$  a random variable) gives the first expression for the information process in (4.1). To conclude, we show that

$$I_{TT} = \lim_{t \rightarrow T} I_{tT} = X_T.$$

Indeed, observe that for  $\alpha = \frac{1}{T}$  as in [4], and from (4.4),

$$I_{TT} = \lim_{t \rightarrow T} I_{tT} = mT + \omega_T = X_T$$

as required.  $\square$

The interpretation of the above proposition is that the Brownian bridge market information process can be constructed explicitly by the specification of an appropriate conditioning,  $(T, Y, \mu)$  on the Wiener space. This corresponds to the situation where the functional is the law of a coordinate process and the conditioning is the specification of the *a priori* law of the cash flow such that the associated CSP provides the required information about the *a posteriori* law of the Cash flow. In some real life applications, it may be easier to specify a conditioning on other functionals of the coordinate process than its law. For example, its quadratic variation or first hitting time of a level. This is particularly desirable when the payoff (cash flow) is in the form of such a functional. For instance, when the *a priori* law



of the cash flow is more difficult to determine compared to that of its quadratic variation or its first hitting time of a level. In this case, the market information regarding a contingent claim on the asset at a future time can be obtained by an appropriate conditioning on the functional. In the following sections we price European option on a credit risky discount bond.

**4.2. European options.** In this section we focus on a market with a single factor<sup>1</sup> denoted by  $X_T$ . We work in continuous time. The asset is modelled by a random cash flow  $H_T = h(X_T)$  occurring at time  $T$ . For simplicity we consider contingent cash flow of the form  $h(x) = x$ . We assume that  $X_T$  is an integrable random variable with *a priori* probability law  $\mu$ . The market information regarding  $X_T$  is provided by the process  $\{I_{tT}\}_{0 \leq t < T}$ . The information process  $I_{tT}$  is the unique conditioned stochastic process associated with the conditioning  $(T, Y, \mu)$ .  $Y$  is a functional of a coordinate process on the Wiener space. Using the Markov property of the information process, the time  $t < T$  price of the cash flow is given by

$$H_{tT} = P_{tT} \mathbb{E}_\mu[X_T | I_{tT}], \quad (4.5)$$

where  $P_{tT}$  is the discount factor defined in (3.1). The  $\mathcal{F}_{tT}^I$ -conditional law of  $X_T$  is given by (2.10) as

$$\mu_t(x; dy) = \frac{\Lambda_t^y \mu(dy)}{\int_{-\infty}^{\infty} \Lambda_t^y \mu(dy)}. \quad (4.6)$$

Then we obtain

$$H_{tT} = P_{tT} \int_{-\infty}^{\infty} y \mu_t(x; dy). \quad (4.7)$$

In the case of conditioning of the marginal law of a standard Brownian motion we have

$$\Lambda_t^y = \sqrt{\frac{T}{T-t}} \exp \left[ \frac{y^2}{2T} - \frac{(y-x)^2}{2(T-t)} \right]. \quad (4.8)$$

In addition, if  $\mu(dy)$  admits a density,  $\rho(y)$  then the  $\mathcal{F}_t^I$ -conditional law of the cash flow becomes

$$\mu_t(x; dy) = \frac{\Lambda_t^y \rho(y) dy}{\int_{-\infty}^{\infty} \Lambda_t^y \rho(y) dy}. \quad (4.9)$$

Now, we want to determine the price of European option on the price  $H_{tT}$  at time  $t$ . The time  $s$ , ( $0 \leq s \leq t$ ) price of a  $t$ -maturity put option on  $H_{tT}$  with strike  $K$  is given by

$$C_{st} = P_{st} \mathbb{E}[(K - H_{tT})^+ | I_{sT}]. \quad (4.10)$$

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<sup>1</sup>The case of multiple cash flow follows analogously in which case  $N$  cash flows,  $H_{T_1}, H_{T_2}, \dots, H_{T_N}$ , are to be received at dates,  $T_1 \leq T_2 \leq \dots \leq T_N$  respectively. For each date  $T_j$  and each  $X$ -factor,  $X_{T_j}^i$ , we have the conditioning  $(T_j, Y_j, \mu_j)$  where  $Y_j$  is a  $\mathbb{R}^j \rightarrow \mathbb{R}$  random functional,  $\mu_j$  is  $\mathbb{R}^j \rightarrow \mathbb{R}^*$  measure and  $\mathbb{R}^*$  is some space of probability measures.

Using results from the previous sections we have

$$\begin{aligned}
C_{st} &= P_{st} \mathbb{E}_\mu \left[ (K - P_{tT} \mathbb{E}_\mu [H_T | I_{tT}])^+ | I_{sT} \right] \\
&= P_{st} \mathbb{E}_\mu \left[ \left( \int_{-\infty}^{\infty} (K - P_{tT} y) \mu_t(x; dy) \right)^+ | I_{sT} \right] \\
&= P_{st} \mathbb{E}_\mu \left[ \frac{1}{\int_{-\infty}^{\infty} \Lambda_t^y \mu(dy)} \left( \int_{-\infty}^{\infty} (K - P_{tT} y) \Lambda_t^y \mu(dy) \right)^+ | I_{sT} \right].
\end{aligned} \tag{4.11}$$

The process  $D_t$  defined by

$$D_t = \int_{-\infty}^{\infty} \Lambda_t^y \mu(dy) \tag{4.12}$$

is a density process and under the Wiener measure  $\mathbb{P}$ ,  $\frac{1}{D_t}$  is a martingale. Then the expression for the option price becomes

$$C_{st} = \frac{P_{st}}{D_s} \mathbb{E}_\mathbb{P} \left[ \left( \int_{-\infty}^{\infty} (K - P_{tT} y) \Lambda_t^y \mu(dy) \right)^+ | I_{sT} \right] \tag{4.13}$$

$$= \frac{P_{st}}{D_s} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} (K - P_{tT} y) \Lambda_t^y \mu(dy) \right)^+ \mu_s(z; dy). \tag{4.14}$$

Let the set  $A_t$  be defined by

$$A_t = \left\{ x \in (-\infty, \infty) : \int_{-\infty}^{\infty} (K - P_{tT} y) \Lambda_t^y \mu(dy) > 0 \right\}^2 \tag{4.15}$$

the expression for the put option price becomes

$$C_{st} = P_{st} \int_{-\infty}^{\infty} \int_{x \in A_t} (K - P_{tT} y) \mu_s(z; dy) \mu_{st}(dx; y) \tag{4.16}$$

where  $\mu_{st}(x; y) = \frac{\mu_t(x; y)}{D_s}$ .

Notice that  $H_{tT}$  can be written as

$$H_{tT} = \zeta(t, I_{tT}) \tag{4.17}$$

then the set  $A_t$  becomes

$$A_t = \{I \in (-\infty, \infty) : \zeta(t, I) < K\}.$$

The break even market information  $I_{tT}^*$  is such that  $A_t = \{x : x \in (-\infty, I_{tT}^*)\}$ , where  $\zeta$  is some deterministic function.

If the information is generated by the conditioned stochastic process associated with the conditioning of the marginal law of a Brownian motion  $\{B_t\}$ , then  $\Lambda_t^y$  is given by (4.8) and the functional,  $Y$  is defined by  $Y = B_T$ . In this case,  $\mu_{st}(x; y)$  is normal random variable with mean  $\bar{Z}(y) = \frac{T-t}{T-s} I_{sT} + \frac{t-s}{T-s} y$  and variance  $\sigma^2 = \frac{t-s}{T-s} (T-t)$ . Then the inner integral in (4.16) can be written as

$$\int_{-\infty}^{I_t^*} \mu_{st}(dx; y) = \mathcal{N} \left( \frac{I_t^* - \bar{Z}(y)}{\sigma} \right),$$

<sup>2</sup>Note that  $\Lambda_t^y$  is a function of  $x$ . See (4.8).

where  $\mathcal{N}(x)$  is the standard normal cumulative distribution function.

The option price becomes

$$C_{st} = \int_{-\infty}^{\infty} \left[ (P_{st}K - P_{sT}y) \mathcal{N} \left( \frac{I_t^* - \bar{Z}(y)}{\sigma} \right) \mu_s(z; dy) \right]. \quad (4.18)$$

**4.3. Binary credit risky bond.** We consider a credit risky bond that pays  $H_T \in [h_0, h_1]$ ,  $h_0 < h_1$  at maturity time  $T$ . The bond pays a principle of  $h_1$  when there is no default but a recovery amount of  $h_0$  in the case of a partial default. The bond price,  $H_{tT}$  at time  $t$  is give by

$$H_{tT} = P_{tT} \mathbb{E}_\mu(H_T | I_{tT}).$$

$I_{tT}$  is the conditioned stochastic process associated with the conditioning  $(T, Y, \mu)$ . From (4.7) the expression for the bond price becomes

$$\begin{aligned} H_{tT} &= P_{tT} \int_{-\infty}^{\infty} y \mu_t(x; dy) \\ &= P_{tT} \left[ \sum_{i=0}^1 h_i \mu_t(x; h_i) \right] \\ &= P_{tT} \left[ \frac{h_0 \Lambda_t^{h_0} \mu(h_0) + h_1 \Lambda_t^{h_1} \mu(h_1)}{\Lambda_t^{h_0} \mu(h_0) + \Lambda_t^{h_1} \mu(h_1)} \right]. \end{aligned}$$

Assume that *a priori*,  $\mu(h_0) = \rho_0$  and  $\mu(h_1) = \rho_1 = (1 - \rho_0)$ . then we have

$$H_{tT} = P_{tT} \left[ \frac{h_0 \Lambda_t^{h_0} \rho_0 + h_1 \Lambda_t^{h_1} \rho_1}{\Lambda_t^{h_0} \rho_0 + \Lambda_t^{h_1} \rho_1} \right]. \quad (4.19)$$

In addition, when the information is generated by the conditioned stochastic process associated with the conditioning of the marginal law of a Brownian motion  $\{B_t\}$ , then  $\Lambda_t^y$  is given by (4.8) and the functional,  $Y$  is defined by  $Y = B_T$ . Particularly, the function  $\zeta(t, I_t)$  is given by

$$\zeta(t, I_t) = P_{tT} \left[ \frac{h_0 \Lambda^{h_0}(I_t) \rho_0 + h_1 \Lambda^{h_1}(I_t) \rho_1}{\Lambda^{h_0}(I_t) \rho_0 + \Lambda^{h_1}(I_t) \rho_1} \right].$$

Then the equation  $\zeta(t, x) = K$  can be solved explicitly for  $x$  using the appropriate initial condition.

The price of a put option on  $H_{tT}$  is then given by

$$C_{st} = \sum_{i=0}^1 \left[ (P_{st}K - P_{sT}h_i) \mathcal{N} \left( \frac{I_t^* - \bar{Z}(h_i)}{\sigma} \right) \mu_s(z; h_i) \right]. \quad (4.20)$$

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DENNIS IKPE: MATHEMATICAL SCIENCE, UNIVERSITY OF SOUTH AFRICA, PRETORIA, 0003 SOUTH AFRICA AND, MATHEMATICS AND APPLIED MATHEMATICS, UNIVERSITY OF CAPE TOWN, CAPE TOWN, 1700, SOUTH AFRICA  
*E-mail address:* `ikpecd@unisa.ac.za`

SURE MATARAMVURA: ACTUARIAL SCIENCE, UNIVERSITY OF CAPE TOWN, CAPE TOWN, 1700, SOUTH AFRICA  
*E-mail address:* `sure.mataramvura@uct.ac.za`

RONNIE BECKER: AFRICAN INSTITUTE FOR MATHEMATICAL SCIENCES, MUIZENBERG 7945, SOUTH AFRICA  
*E-mail address:* `ronnie@aims.ac.za`