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LARGE DEVIATIONS FOR A STOCHASTIC BURGERS' EQUATION

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Abstract. We prove the large deviations principle (LDP) for the law of the solutions to a stochastic Burgers' equation in the presence of an additive noise. Our proof is based on the weak convergence approach.

1. Introduction and Background

The deterministic Burgers' equation is a fundamental partial differential equation (PDE), that arises in many areas of science including, but not limited to, fluid dynamics, aerodynamics, and acoustics. The equation which can describe the motion of hydrodynamic waves, possesses a nonlinearity of quadratic type. The nonlinearity can be treated by means of the Cole-Hopf transformation [7, 19]. This transformation reduces the equation into a linear heat equation, and this allows one to solve an initial value problem (IVP). In this paper, we study a stochastic Burgers' equation in the presence of an additive noise. It should be noted that the stochastic Burgers' equation perturbed by a Gaussian white noise is *equivalent* to the Kardar-Parisi-Zhang (KPZ) equation in that the gradient of the solution to the KPZ equation solves the stochastic Burgers' equation. The stochastic Burgers' equation perturbed by an external noise (of additive or multiplicative type), has been the subject of extensive research in the past two decades (see e.g. [1, 8, 26, 27, 25, 22, 17, 18] and the references therein). In [1], Bertini et al. (1994) consider a Burgers' equation perturbed by a space-time white noise, and prove the existence of solutions by using the stochastic Cole-Hopf transformation. They construct an explicit solution to the linearized equation by means of a generalized Feynman-Kac formula. In [8], Da Prato et al. (1994) consider a stochastic Burgers' equation driven by a Brownian sheet, and prove the existence and uniqueness of global solutions in time, as well as the existence of an invariant measure for the transition semigroup. In [17], Gyöngy proves the existence and uniqueness of the solution to a Burgers' equation on a bounded domain, by establishing an approximation theorem. In [18], his results are further extended to the whole real line. Large deviations properties of a stochastic Burgers' equation perturbed by a space-time white noise were studied by C. Cardon-Weber [5], where the classical approach was adopted. In this work, we prove a large deviation

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principle (LDP) for the law of the solutions to a stochastic Burgers' equation with additive noise where we employ the weak convergence approach. More precisely, we consider the stochastic Burgers' equation studied by Da Prato et al. [8]. The equation reads

$$\frac{\partial X^\epsilon}{\partial t}(t,x) = \frac{\partial^2 X^\epsilon}{\partial x^2}(t,x) + \sqrt{\epsilon}\frac{\partial^2 W}{\partial t \partial x}(t,x) + \frac{1}{2}\frac{\partial}{\partial x}(X^\epsilon(t,x))^2, \tag{1.1}$$

with Dirichlet's boundary conditions $X^\epsilon(t,0) = X^\epsilon(t,1) = 0$ for $t \in [0,T]$, and initial condition $X^\epsilon(0,x) = \xi(x) \in L^2([0,1])$. $W(t,x)$ denotes the Brownian sheet on a filtered probability space, $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$, with definition (2.1).

Since our aim is to prove a large deviations principle for the law of the solutions to Eq. (1.1), we first describe the meaning of this notion. We begin by describing the Friedlin-Wentzell asymptotics [16]. Consider the following m-dimensional stochastic differential equation (SDE), which is driven by finitely many Brownian motions

$$dU^\epsilon(t) = b(U^\epsilon(t))dt + \sqrt{\epsilon}a(U^\epsilon(t))dW(t),$$

supplemented with the following initial condition

$$U^\epsilon(0) = v^\epsilon, \ t \in [0,T].$$

The coefficients a, b are suitably regular, and $W(t)$ is a finite dimensional standard Brownian motion. If $v^\epsilon \to v^0$ as $\epsilon \to 0$, then

$$U^\epsilon \to U^0 \text{ in } C([0,T];\mathbb{R}^m),$$

where U^0 solves

$$\dot{v} = b(v),$$

with

$$v(0) = v^0, \ t \in [0,T].$$

The Freidlin-Wentzell theory describes the asymptotic behavior of probabilities of the large deviations of the solution to the SDE (i.e., U^ϵ), away from its law of large number limit (i.e., U^0), as $\epsilon \to 0$. In this work, we are concerned with the case where the driving Brownian motion is infinite dimensional. In [3], Budhiraja, Dupuis, and Maroulas (2000) use certain variational representations for infinite dimensional Brownian motions [2], and show that, these representations conveniently lay the ground for proving large deviations for a variety of infinite dimensional systems, such as stochastic partial differentials equations. One advantage of their method is that the technical exponential probability estimates (used in the usual proofs based on approximations) are no longer needed; instead, one is required to prove certain qualitative properties of the SPDE under study (such as existence, uniqueness and tightness). Several authors have since adopted the method (see e.g. [23, 24, 28, 29]): In [23], Ren, and Zhang (2005) consider an SDE driven by infinitely many Brownian motions with non-Lipschitz diffusion coefficients. Due the presence of non-Lipschitz coefficients, the problem is not amenable to a small noise LDP analysis based on the standard approximation approach. They reply upon the representation formula for an infinite sequence of real-valued Brownian motions, and prove the Laplace principle by exploiting Theorem 6 of [3]. They further extend their results to multi-dimensional SDEs with non-Lipschitz coefficients in [24]. In [28], Sritharan and Sundar (2006) prove the existence and uniqueness of solutions to a two-dimensional Navier-Stokes equation with multiplicative noise, and then prove the Laplace principle by verifying the assumption of Theorem 5 in [3]. As an application of their method, the authors of [3] prove the Laplace principle (which is equivalent to the large deviations principle for Polish-space-valued random elements) for a class of reaction-diffusion equations. One of their main assumptions is the linear growth condition on the drift coefficients. In this work we prove the Laplace principle for a stochastic Burgers' equation where the linear growth condition is replaced by quadratic growth. The difficulty of the analysis lies in proving the tightness of the nonlinear term.

Finally, we note that compared to the proof of C. Cardon Weber [5], the conditions under which we prove the large deviations principle are much weaker, and require less technicalities. For example, the time discretizations required in proving the regularity of the skeleton ([5], p. 60) are completely bypassed, and exponential inequalities for the stochastic integral in Hölder norms ([5], p. 62) are no longer needed. These are most likely the most difficult parts of large deviations analysis based on the standard approximation method. In our alternate proof based on the weak convergence approach, once the well-posedness of the controlled process is established, one only needs to demonstrate the tightness of this process, and its convergence to the limit equation. This leads to a simpler, shorter, and more straightforward proof than that of [5].

Here is the outline of the paper. In Section 2, we introduce the stochastic Burgers' equation under study, and state some properties of the regularizing kernel. In Section 3, we state the variational representation for a Brownian sheet, and subsequently state the large deviations theorem due to Budhiraja, Dupuis and Maroulas ([3, Theorem 7]) which we exploit. In Section 4, we formulate the uniform Laplace principle for the law of the solutions to the stochastic Burgers' equation in the presence of an additive noise. We also lay the ground for proving the Laplace principle by proving the existence and uniqueness of the controlled process. Section 5 is devoted to the proof of the main theorem. As stated in the Introduction, establishing the Laplace principle hinges on proving the tightness and convergence properties of the controlled process. This is carried out in Theorem 5.1. The paper concludes with a brief summary, and some further discussions.

Unless otherwise noted, we adopt the following notation throughout the paper: The notation " $\doteq$ " means "by definition". C denotes a free constant which may take on different values, and depend upon other parameters.

2. Stochastic Burgers Equation

We consider the stochastic Burgers' equation studied by Da Prato et al. [8]. The equation reads

$$\frac{\partial X^\epsilon}{\partial t}(t,x) = \frac{\partial^2 X^\epsilon}{\partial x^2}(t,x) + \sqrt{\epsilon}\frac{\partial^2 W}{\partial t \partial x}(t,x) + \frac{1}{2}\frac{\partial}{\partial x}(X^\epsilon(t,x))^2,$$

with Dirichlet's boundary conditions $X^\epsilon(t,0) = X^\epsilon(t,1) = 0$ for $t \in [0,T]$, and initial condition $X^\epsilon(0,x) = \xi(x) \in L^2([0,1])$. $W(t,x)$ denotes the Brownian sheet on filtered a probability space, $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$, with the following definition.

Definition 2.1 (Brownian Sheet)**.** A family of real valued random variables $\{W(t,x),(t,x)\in[0,T]\times[0,1]\}$ on a filtered probability space is called a *Brownian sheet* if the following hold:
1. If $(t,x)\in[0,T]\times[0,1]$, then $E(W(t,x))=0$.
2. If $0\le s\le t\le T$ and $x\in[0,1]$, then $W(t,x)-W(s,x)$ is independent of $\{\mathcal{F}_s\}$.
3. $\mathrm{Cov}(W(t,x),W(s,y))=\lambda(D_{t,x}\cap D_{s,y})$, where λ is the Lebesgue meaure on $[0,T]\times[0,1]$ and $D_{t,x}\doteq\{(s,y)\in\mathbb{R}_+\times[0,1]:0\le s\le t \text{ and } y_j\le x_j,\ j=1,\dots,d\}$
4. The map $(t,u)\to W(t,u)$ from $[0,T]\times[0,1]$ to $\mathbb{R}$ is continuous with probability one.

By the notion of a solution we mean the following:

Definition 2.2 (Mild Solution)**.** A random field $X^\epsilon\doteq\{X^\epsilon(t,x):t\in[0,T],x\in[0,1]\}$ is called a *mild solution* of (1.1) with initial condition ξ if $(t,x)\to X^\epsilon(t,x)$ is continuous a.s., and $X^\epsilon(t,x)$ is $\{\mathcal{F}_t\}$-measurable for any $t\in[0,T]$, and $x\in[0,1]$, and if

$$X^\epsilon(t,x)=\int_0^1 G_t(x,y)\xi(y)dy+\sqrt{\epsilon}\int_0^t\int_0^1 G_{t-s}(x,y)W(dy,ds)$$
$$-\frac{1}{2}\int_0^t\int_0^1 \partial_y G_{t-s}(x,y)(X^\epsilon(s,y))^2dyds. \tag{2.1}$$

The function $G_t(.,.)$ is the Green kernel associated with the heat operator $\partial/\partial t-\partial^2/\partial^2 x$ with Dirichlet's boundary conditions. We now state some properties of the regularizing kernel.

2.1. Properties of the Regularizing Kernel. There exist positive constants K,a,b,d such that for all $0\le s<t\le T$, and $x,y\in[0,1]$.

(1) $|G(s,t;x,y)|\le K\frac{1}{|t-s|}\exp\big(-a\frac{|x-y|^2}{t-s}\big)$,
(2) $|\frac{\partial}{\partial x}G(s,t;x,y)|\le K\frac{1}{|t-s|^{3/2}}\exp\big(-b\frac{|x-y|^2}{t-s}\big)$,
(3) $|\frac{\partial}{\partial t}G(s,t;x,y)|\le K\frac{1}{|t-s|^2}\exp\big(-d\frac{|x-y|^2}{t-s}\big)$.

For $\bar\alpha=\frac{\gamma-d}{2\gamma}$ with $\gamma\in(d,\infty)$, and any $\alpha<\bar\alpha$ there exists a constant $\bar K(\alpha)$ such that for all $0<s<t<T$, and all $x,y\in[0,1]$.

(4) $\int_0^T\int_0^1|G_{t-\tau}-G_{s-\tau}|^2d\eta d\tau\le\bar K(\alpha)\rho((t,x),(s,y))^{2\alpha}$
where ρ is the Euclidean distance in $[0,T]\times[0,1]$.

3. Large Deviations for Functionals of a Brownian Sheet

In this section, we visit some of the results presented in [3]. In particular, we state the variational representation for a space-time Brownian sheet, and subsequently state the Laplace principle, or equivalently the large deviations principle (for Polish-space-valued random elements), for a large class of stochastic partial differential equations (SPDEs) driven by a space-time white noise.

Let $(\Omega,\mathcal{F},P,\{\mathcal{F}_t\})$ be the filtered probability space introduced in Section 1, $(L^2([0,1]),\|.\|)$ a normed Hilbert space, and $\phi:\Omega\times[0,T]\to L^2([0,1])$ an $L^2([0,1])$-valued predictable process. Define

$$\mathcal{P}_2 \doteq \left\{ \phi : \int_0^T \|\phi(s)\|_{L^2}^2 ds < \infty \ a.s. \right\}. \tag{3.1}$$

We have the following representation theorem for a Brownian sheet.

Theorem 3.1. *Let $f : \mathcal{C}([0,T]\times[0,1];\mathbb{R}) \to \mathbb{R}$ be a bounded measurable map. Let W be the Brownian sheet defined by Definition 2.1. Then*

$$-\log \mathbb{E}(\exp\{-f(W)\}) = \inf_{u\in\mathcal{P}_2} \mathbb{E}\big(\frac{1}{2}\int_0^T\int_0^1 u^2(s,y)dyds + f(W^u)\big),$$

where $W^u(t,x) \doteq W(t,x) + \int_0^t\int_0^x u(s,y)dyds$.

Let $\mathcal{E}_0$ and $\mathcal{E}$ be Polish spaces, and let the initial condition ξ take values in a compact subspace of $\mathcal{E}_0$. Moreover, for every $\varepsilon > 0$, let $\eta^\varepsilon : \mathcal{E}_0 \times \mathcal{C}([0,T]\times[0,1];\mathbb{R}) \to \mathcal{E}$ be a family of measurable maps. Define $X^{\varepsilon,\xi} \doteq \eta^\varepsilon(\xi, \sqrt{\varepsilon}W)$, and introduce the following:

$$S^N \doteq \left\{ f \in L^2([0,T]\times[0,1]) : \int_{[0,T]\times[0,1]} f^2(s,y)dsdy \le N \right\},\ N \in \mathbb{N}, \tag{3.2}$$

$$\mathcal{P}_2^N \doteq \left\{ u \in \mathcal{P}_2 : u(\omega) \in S^N, P-a.s. \right\}. \tag{3.3}$$

The space S^N is a compact metric space, equipped with the weak topology on $L^2([0,T]\times[0,1])$. For $u \in L^2([0,T]\times[0,1])$, define

$$\mathrm{Int}(u)(t,x) \doteq \int_0^t\int_0^x u(s,y)dsdy. \tag{3.4}$$

The following is the standing Assumption of Theorem 3.2.

ASSUPMTION 1: There exists a measurable map $\eta^0 : \mathcal{E}_0 \times \mathcal{C}([0,T]\times[0,1];\mathbb{R}) \to \mathcal{E}$ such that

1. For every $M < \infty$ and compact set $K \subset \mathcal{E}_0$, the set

$$\Gamma_{M,K} \doteq \{\eta^0(\xi, \mathrm{Int}(u)) : u \in S^M, \xi \in K\},$$

is a compact subset of $\mathcal{E}$.

2. Consider $M < \infty$ and the families $\{u^\epsilon\} \subset \mathcal{P}_2^M$, and $\{\xi^\epsilon\} \subset \mathcal{E}_0$ such that $u^\epsilon \to u$, and $\xi^\epsilon \to \xi$ in distribution, as $\varepsilon \to 0$. Then

$$\eta^\varepsilon\big(\xi^\epsilon, \sqrt{\varepsilon}W + \mathrm{Int}(u^\varepsilon)\big) \to \eta^0\big(\xi, \mathrm{Int}(u)\big),$$

in distribution as $\epsilon \to 0$.

For $h \in \mathcal{E}$, and $\xi \in \mathcal{E}_0$. Define

$$I_\xi(h) \doteq \inf_{\{u\in L^2([0,T]\times[0,1]): h \doteq \eta^0(\xi, Int(u))\}} \left\{ \frac{1}{2}\int_0^T\int_0^1 u^2(y,s)dyds \right\}. \tag{3.5}$$

The following theorem which we are going to exploit is due to Budhiraja, Dupuis and Maroulas ([3], Theorem 7), and states the uniform Laplace principle for the family $\{X^{\varepsilon,\xi}\}$.

Theorem 3.2. *Let $\eta^0 : \mathcal{E}_0 \times \mathcal{C}([0,T]\times[0,1];\mathbb{R}) \to \mathcal{E}$ be a measurable map satisfying Assumption 1. Suppose that for all $h \in \mathcal{E}$, $\xi \to I_\xi(h)$ is a lower semi-continuous map from $\mathcal{E}_0$ to $[0,\infty]$. Then for every $\xi \in \mathcal{E}_0$, $I_\xi(h) : \mathcal{E} \to [0,\infty]$, is a rate function on $\mathcal{E}$ and the family $\{I_\xi, \xi \in \mathcal{E}\}$ of rate functions has compact level sets on compacts. Furthermore, the family $\{X^{\varepsilon,\xi}\}$ satisfies the the uniform Laplace principle on $\mathcal{E}$ with rate function I_ξ, uniformly in ξ on compact subset of $\mathcal{E}_0$.*

Note that the controlled equation (i.e. the equation under the change of measure), reads as follows:

$$Y_\xi^{\epsilon,u}(t,x) = \int_0^1 G_t(x,y)\xi(y)dy + \sqrt{\epsilon}\int_0^t\int_0^1 G_{t-s}(x,y)W(dy,ds)$$
$$+\int_0^t\int_0^1 G_{t-s}(x,y)u(y,s)dyds - \frac{1}{2}\int_0^t\int_0^1 \partial_y G_{t-s}(x,y)(Y_\xi^{\epsilon,u}(s,y))^2 dyds. \quad (3.6)$$

4. Large Deviations for the Stochastic Burgers' Equation

In this section, we formulate the Laplace principle for the law of the solutions to Eq. (1.1), and lay the ground for its proof.

In ([17, Theorem 2.1]), Gyöngy proves for any $\epsilon > 0$, the existence and uniqueness of the solution process, which admits a modification in $C([0,T], L^p[0,1])$, $p \geq 2$. As shown by Gyongy ([17, Proposition 3.5]) the mild form of the solution, i.e.

$$X^\epsilon(t,x) = \int_0^1 G_t(x,y)\xi(y)dy + \sqrt{\epsilon}\int_0^t\int_0^1 G_{t-s}(x,y)W(dy,ds)$$
$$-\frac{1}{2}\int_0^t\int_0^1 \partial_y G_{t-s}(x,y)(X^\epsilon(s,y))^2 dyds, \quad (4.1)$$

is equivalent to the weak form. Recall that the function $G_t(.,.)$ is the Green kernel associated with the heat operator $\partial/\partial t - \partial^2/\partial^2 x$ with Dirichlet's boundary conditions.

Note that the controlled version of (4.1) is (3.6). The controlled process corresponds to the following controlled equation

$$\frac{\partial Y_\xi^{\epsilon,u}}{\partial t}(t,x) = \frac{\partial^2 Y_\xi^{\epsilon,u}}{\partial x^2}(t,x) + \sqrt{\epsilon}\frac{\partial^W}{\partial t\partial x}(t,x) + \frac{1}{2}\frac{\partial}{\partial x}(Y_\xi^{\epsilon,u}(t,x))^2 + u(t,x). \quad (4.2)$$

Since establishing an LDP for the law of the solution process depends on the qualitative properties of the controlled process, we first need to establish the existence and uniqueness of Eq. (3.6).

4.1. Existence and Uniqueness Results.

We reply upon the following existence and uniqueness result of Gyöngy ([17, Theorem 2.1]).

Theorem 4.1 (Existence and Uniqueness of Solution Mapping)**.** *For any filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$, with a Brownian sheet defined as before, and $\xi \in L^p[0,1]$, $p \geq 2$ there exists a measurable function*

$$\eta^\epsilon : L^2([0,1]) \times C([0,T]\times[0,1];\mathbb{R}) \to C\big([0,T]; L^p([0,1])\big),$$

such that $X^\epsilon \doteq \eta^\epsilon(\xi, \sqrt{\epsilon}W)$*, (with* ξ *denoting the initial condition) is the unique, mild solution of (1.1).*

For $h \in C\big([0,T]; L^2([0,1])\big)$, we define the following action functional

$$I_\xi(h) \doteq \inf_u \int_0^T \int_0^1 u^2(s,y)dyds, \tag{4.3}$$

where the infimum is taken over all $u \in L^2([0,T]\times[0,1])$ such that

$$\begin{aligned} h(t,x) = &\int_0^1 G_t(x,y)\xi(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y)u(s,y)dyds \\ &- \frac{1}{2}\int_0^t \int_0^1 \partial_y G_{t-s}(x,y)h^2(s,y)dyds. \end{aligned} \tag{4.4}$$

We now announce the main theorem of this paper.

Theorem 4.2 (Main Theorem)**.** *The processes* $\{X^\epsilon(t) : t \in [0,T]\}$ *satisfy the uniform Laplace principle on* $C\big([0,T]; L^2([0,1])\big)$ *with rate function* I_ξ *given by (4.3).*

As mentioned before, the proof of Theorem 4.2 hinges on the existence and uniqueness of the controlled process (3.6). We have the following theorem whose proof is mainly based on Girsanov's theorem.

Theorem 4.3 (Existence and Uniqueness of Controlled Process)**.** *Let* η^ϵ *denote the solution mapping, and let* $u \in \mathcal{P}_2^N$ *for some* $N \in \mathbb{N}$*. For* $\epsilon > 0$ *and* $\xi \in L^2([0,1])$ *define*

$$Y_\xi^{\epsilon,u} \doteq \eta^\epsilon\big(\xi, \sqrt{\epsilon}W + Int(u)\big),$$

then $Y_\xi^{\epsilon,u}$ *is the unique solution of equation (4.2).*

Proof. For a fixed $u \in \mathcal{P}_2^N$, define

$$\frac{dQ^{u,\epsilon}}{dP} \doteq \exp\left\{-\frac{1}{\sqrt{\epsilon}}\int_0^T \int_0^1 u(s,y)W(dyds) - \frac{1}{2\epsilon}\int_0^T \int_0^1 u^2(s,y)dyds\right\}.$$

Since

$$\exp\left\{-\frac{1}{\sqrt{\epsilon}}\int_0^T \int_0^1 u(s,y)W(dyds) - \frac{1}{2\epsilon}\int_0^T \int_0^1 u^2(s,y)dyds\right\},$$

is an exponential martingale, we have that $Q^{u,\epsilon}$ is a probability measure on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$. Clearly, $Q^{u,\epsilon}$ is equivalent to P. By Girsanov's theorem ([10], Theorem 10.14), we can conclude that $\tilde{W} \doteq W + \epsilon^{-1/2}\mathrm{Int}(u)$ is a Brownian sheet under $Q^{u,\epsilon}$. By Theorem 4.1, $Y_\xi^{\epsilon,u}$ is the unique solution of (4.1) with $\tilde{W}$ replaced by W under the measure $Q^{u,\epsilon}$. Note that this is precisely equation (3.6) (i.e., the controlled process) on $(\Omega, \mathcal{F}, Q^{\epsilon,u}, \{\mathcal{F}_t\})$. By the equivalence of the measures we conclude that $Y_\xi^{\epsilon,u}$ is the unique solution of Eq. (4.2) under the measure P, and the proof is complete. □

The next Theorem is a uniqueness result regarding the limit equation which we will use in the proof of Theorem 5.1.

Theorem 4.4 (Uniqueness of Limit Equation)**.** *Fix $\xi \in L^2([0,1])$ and $u \in L^2([0,T] \times [0,1])$. Then there exists a unique function h in $\mathcal{C}\big([0,T]; L^2([0,1])\big)$ which satisfies equation (4.4).*

The proof of this theorem is almost verbatim to that of Theorem 4.1, and thus omitted. We now state two theorems and two lemmas which we are going to use in the proof of the main theorem. The next theorem, is a multiparameter extension of the Burkholder-Davis-Gundy (BDG) Inequality ([21, Theorem 4.2.1]).

Theorem 4.5 (The Burkholder-Davis-Gundy Inequality)**.** *Suppose X is continuous and adapted, and for all $t \in \mathbb{R}_+^N$, we have $E\big[\int_{[0,t]} X_s^2 ds\big] < \infty$. Then for all t, and all $p \geq 1$, we have that,*

$$E\left[\sup_{s\in[0,t]} \Big\{ \int_{[0,s]} X_r dB_r \Big\}^{2p}\right] \leq c(p) 4^{pN} E\left[\Big\{ \int_{[0,t]} X_r^2 dr \Big\}^p\right]$$

The following theorem ([21, Theorem 2.5.2]) shows that under enough smoothness, X which denotes a stochastic process has a Hölder continuous modification.

Theorem 4.6. *Let $(X = X_t; t \in \mathbb{R}_+^N)$ denote a stochastic process that satisfies the following for some $C, p > 0$ and $r > N$:*

$$E\big[|X_s - X_t|^p\big] \leq C|s-t|^r, \;\; s,t \in \mathbb{R}_+^N$$

then, there exists a modification $(Y = Y_t; t \in \mathbb{R}_+^N)$ of X that is Hölder continuous of any order $q \in [0, p^{-1}(r-N)[$

The following lemma([21, Exercise 2.5.1]) is used in proving the tightness of the stochastic integral.

Lemma 4.7. *In the setting of Theorem 4.6, we have, for all $\tau \in \mathbb{R}_+^N$, $0 < q < p$ and all $Q \in]0, p^{-1}(r-N)[$ that, there exists a finite constant C (which depends on p, q, Q, r and τ) such that for all $\delta \in [0,1[$,*

$$E\left[\sup_{s,t\in[0,\tau]:|s-t|\leq\delta} |X_s - X_t|^q\right] \leq C\delta^{Qp}$$

The next lemma ([17, Lemma 3.3]) is used in proving the tightness of the term entailing the derivative of the kernel in (5.1).

Lemma 4.8. *Let $\rho \in [1,\infty)$, and $q \in [1,\rho)$. Moreover, let $\zeta_n(t,y)$ be a sequence of random fields on $[0,T] \times [0,1]$ such that $\sup_{t\leq T} |\zeta_n(t,.)|_q \leq \theta_n$, where θ_n is a finite random variable for every n. Assume that the sequence θ_n is bounded in probability, i.e.*

$$\lim_{c\to\infty} \sup_n P(\theta_n \geq C) = 0.$$

Then the sequence $J(\zeta_n) \doteq \int_0^t \int_0^1 \partial_y G(r,t;x,y)\zeta_n(r,y) dy dr$, $t \in [0,T]$, $x \in [0,1]$ is uniformly tight in $C\big([0,T]; L^\rho([0,1])\big)$.

5. Proof of the Main Theorem

In light of Theorem 3.2, it suffices to verify Assumption 1. The following key convergence theorem leads to the proof.

Theorem 5.1 (Convergence of the Controlled Process)**.** *Let $M < \infty$, and suppose that $\xi^\epsilon \to \xi$ and $u^\epsilon \to u$ in distribution as $\epsilon \to 0$ with $\{u^\epsilon\} \subset \mathcal{P}_2^M$. Then $Y^{\epsilon,u^\epsilon}_{\xi^\epsilon} \to Y^{0,u}_{\xi}$ in distribution.*

Proof. For the convenience of the reader, we divide the proof into two steps.
Step I: Tightness of $Y^{\epsilon,u^\epsilon}_{\xi^\epsilon}$ in $C\big([0,T];L^2([0,1])\big)$
Recall

$$\begin{aligned} Y^{\epsilon,u^\epsilon}_{\xi^\epsilon}(t,x) &= \int_0^1 G_t(x,y)\xi^\epsilon(y)dy + \sqrt{\epsilon}\int_0^t\int_0^1 G_{t-s}(x,y)W(dy,ds) \\ &+ \int_0^t\int_0^1 G_{t-s}(x,y)u^\epsilon(y,s)dyds \\ &- \frac{1}{2}\int_0^t\int_0^1 \partial_y G_{t-s}(x,y)(Y^{\epsilon,u^\epsilon}_{\xi^\epsilon}(s,y))^2 dyds \\ &\doteq I_1^\epsilon + I_2^\epsilon + I_3^\epsilon + I_4^\epsilon. \end{aligned} \tag{5.1}$$

We show tightness of I_i^ϵ for $i = 1,2,3,4$ in $C\big([0,T];L^2([0,1])\big)$, and therefore assert the claim. Since $\xi^\epsilon \in L^2([0,1])$, the tightness of I_1^ϵ follows by the following lemma.

Lemma 5.2. *Let $\psi \in L^2([0,1])$. Then $(t \to G_t\psi)$ belongs to $C\big([0,T];L^2([0,1])\big)$, and*

$$\psi \to \{t \to G_t\psi\},$$

is a continuous map in ψ.

To show the tightness of I_2^ϵ, it suffices to show that ([20, Theorem 4.10])

$$\lim_{\lambda\to\infty}\sup_\epsilon P[|I_2^\epsilon(t,x))| > \lambda] = 0 \qquad \text{for any fixed } (t,x) \in [0,T]\times[0,1], \tag{5.2}$$

$$\lim_{\delta\to 0}\sup_\epsilon P\left[\sup_{\rho((t,x),(s,y))\le\delta} |I_2^\epsilon(s,x) - I_2^\epsilon(t,y)| > \zeta\right] = 0,\ \forall\zeta > 0. \tag{5.3}$$

To prove (5.2) which we call the space criterion it suffices to show that

$$\sup_\epsilon E\big[|I_2^\epsilon(t,x)|^2\big] < \infty.$$

By Ito's isometry, and the properties of the regularizing kernel we have

$$\sup_\epsilon E\big[|I_2^\epsilon(t,x)|^2\big] = \sup_\epsilon \epsilon\int_0^1\int_0^t G_{t-s}^2 dyds < \infty,$$

which by Chebyshev's inequality leads to

$$\lim_{\lambda\to\infty}\sup_\epsilon P[|I_2^\epsilon(t,x))| > \lambda] \le \lim_{\lambda\to\infty}\frac{1}{\lambda^2}\sup_\epsilon E\big[|I_2^\epsilon(t,x)|^2\big] = 0.$$

This concludes the proof of the space criterion. To prove (5.3) which we call the time criterion, we have

$$\begin{aligned} E\Big[\Big| \int_0^T \int_0^1 (G_{t-\tau} - G_{s-\tau}) W(d\eta, d\tau) \Big| \Big]^{2p} &\le C \Big(\int_0^T \int_0^1 |G_{t-\tau} - G_{s-\tau}|^2 d\eta d\tau \Big)^p \\ &\le C \bar{K}(\alpha) \rho((t,x),(s,y))^{2\alpha p}, \end{aligned} \tag{5.4}$$

where the BDG inequality, and the properties of the regularizing Kernel have been used. In (4.5) let p, and $0 < \alpha < \bar{\alpha} < 1/2$, be such that $\alpha p > 2$. Then the assumption of Theorem 4.6 is satisfied, and therefore by lemma 4.7 we have

$$E\Big[\sup_{\rho((t,x),(s,y)) \le \delta} |I_2^\epsilon(t,x) - I_2^\epsilon(s,y)|^q \Big] \le C \delta^{Qp}.$$

Finally, by Chebyshev's inequality

$$\begin{aligned} &\lim_{\delta \to 0} \sup_\epsilon P\Big[\sup_{\rho((t,x),(s,y)) \le \delta} |I_2^\epsilon(s,x) - I_2^\epsilon(t,y)| > \zeta \Big] \\ &\le \lim_{\delta \to 0} \frac{1}{\zeta^q} E\Big[\sup_{\rho((t,x),(s,y)) \le \delta} |I_2^\epsilon(t,x) - I_2^\epsilon(s,y)|^q \Big] \le \frac{C}{\zeta^q} \lim_{\delta \to 0} \delta^{Qp} = 0, \ \forall \zeta > 0. \end{aligned}$$

This concludes the proof of the time criterion, and the tightness of I_2^ϵ is established.

As for the tightness of I_3^ϵ, we have

$$\begin{aligned} \sup_{\epsilon \in (0,1)} I_3^\epsilon &\doteq \sup_{\epsilon \in (0,1)} \int_0^t \int_0^1 G_{t-s}(x,y) u^\epsilon(y,s) dy ds \\ &\le \Big(\int_0^t \int_0^1 G_{t-s}^2(x,y) dy ds \Big)^{1/2} \sup_{\epsilon \in (0,1)} \Big(\int_0^1 \int_0^t (u^\epsilon)^2 dy ds \Big)^{1/2} \le C(T), \end{aligned} \tag{5.5}$$

where Hölder's inequality, properties of the regularizing kernel, and boundedness of the controls in $L^2([0,T] \times [0,1])$ have been used. This establishes the tightness of I_3^ϵ.

As for the tightness of I_4^ϵ, we mainly use Lemma 4.8. Letting $q = 1$, and $\rho = 2$ in that lemma, it suffices to show that

$$\sup_{t \le T} |Y_{\xi^\epsilon}^{\epsilon, u^\epsilon}(t, .)|_{L^2},$$

is bounded in probability, i.e.

$$\lim_{c \to \infty} \sup_{\epsilon \in (0,1)} P\big(\sup_{t \le T} |Y_{\xi^\epsilon}^{\epsilon, u^\epsilon}(t, .)|_{L^2} \ge C \big) = 0. \tag{5.6}$$

In [17], Gyöngy considers the following class of semi-linear stochastic partial differential equations which includes the stochastic Burgers' equation as a special case.

$$\frac{\partial}{\partial t}v(t,x) = \frac{\partial^2}{\partial x^2}v(t,x) + f(t,x,v(t,x)) + \frac{\partial}{\partial x}g(t,x,v(t,x)) + \sigma(t,x,v(t,x))\frac{\partial^2}{\partial t\partial x}W(t,x). \tag{5.7}$$

Let $g(t,x,r) \doteq \frac{1}{2}r^2$, $f(t,x,r) \doteq u^\epsilon(t,x)$, and $\sigma(t,x,r) \doteq \sqrt{\epsilon}$ so that we recover the controlled equation (4.2). In [17], Gyöngy proves the existence and uniqueness of the solutions to Eq. (5.7), by an approximation procedure. Let $f_n(t,x,r)$, and $g_n(t,x,r)$ be sequences of bounded measurable functions such that they are globally Lipschitz in $r \in \mathbb{R}$, and $f_n \doteq f$, $g_n \doteq g$ for $|r| \leq n$, $f_n = g_n \doteq 0$ for $|r| \geq n+1$. Note that f_n, and g_n satisfy the same growth conditions as f, and g. We have, by ([17, Proposition 4.7]), that there exists a unique solution, say $Y^{\epsilon,u^\epsilon}_{\xi^\epsilon,n}$, to the semi-linear equation (5.7) with f and g replaced by f_n and g_n. That is, $Y^{\epsilon,u^\epsilon}_{\xi^\epsilon,n}$, is the unique solution to the truncated equation. Moreover, $Y^{\epsilon,u^\epsilon}_{\xi^\epsilon,n}$ converges to $Y^{\epsilon,u^\epsilon}_{\xi^\epsilon}$ in $C\big([0,T];L^2([0,1])\big)$ in probability (uniformly in ϵ) as n approaches infinity. It has been demonstrated in [17] that, for every $n \geq 1$

$$\lim_{c\to\infty}\sup_{\epsilon\in(0,1)} P\big(\sup_{t\leq T}|Y^{\epsilon,u^\epsilon}_{\xi^\epsilon,n}(t,.)|_{L^2} \geq C\big) = 0. \tag{5.8}$$

Observe that

$$\begin{aligned}\sup_{\epsilon\in(0,1)} & P\big(\sup_{t\leq T}|Y^{\epsilon,u^\epsilon}_{\xi^\epsilon}|_{L^2} \geq C\big)\\ &\leq \sup_{\epsilon\in(0,1)} P\big(\sup_{t\leq T}|Y^{\epsilon,u^\epsilon}_{\xi^\epsilon} - Y^{\epsilon,u^\epsilon}_{\xi^\epsilon,n}|_{L^2} + \sup_{t\leq T}|Y^{\epsilon,u^\epsilon}_{\xi^\epsilon,n}|_{L^2} \geq C\big)\\ &\leq \sup_{\epsilon\in(0,1)} P\big(\sup_{t\leq T}|Y^{\epsilon,u^\epsilon}_{\xi^\epsilon} - Y^{\epsilon,u^\epsilon}_{\xi^\epsilon,n}|_{L^2} \geq \frac{C}{2}\big) + \sup_{\epsilon\in(0,1)} P\big(\sup_{t\leq T}|Y^{\epsilon,u^\epsilon}_{\xi^\epsilon,n}|_{L^2} \geq \frac{C}{2}\big).\end{aligned} \tag{5.9}$$

By letting C approach infinity, and exploiting the boundedness in probability of $|Y^{\epsilon,u^\epsilon}_{\xi^\epsilon,n}|_{L^2}$, we get

$$\lim_{c\to\infty}\sup_{\epsilon\in(0,1)} P\big(\sup_{t\leq T}|Y^{\epsilon,u^\epsilon}_{\xi^\epsilon}|_{L^2} \geq C\big) \leq \lim_{c\to\infty}\sup_{\epsilon\in(0,1)} P\big(\sup_{t\leq T}|Y^{\epsilon,u^\epsilon}_{\xi^\epsilon} - Y^{\epsilon,u^\epsilon}_{\xi^\epsilon,n}|_{L^2} \geq \frac{C}{2}\big).$$

Now by letting n tend to infinity (due the convergence in probability of $Y^{\epsilon,u^\epsilon}_{\xi^\epsilon,n}$ to $Y^{\epsilon,u^\epsilon}_{\xi^\epsilon}$) we conclude that

$$\lim_{c\to\infty}\sup_{\epsilon\in(0,1)} P\big(\sup_{t\leq T}|Y^{\epsilon,u^\epsilon}_{\xi^\epsilon}(t,.)|_{L^2} \geq C\big) = 0.$$

This verifies the assumption of Lemma 4.8, and the tightness of I^ϵ_4 is established. This concludes the tightness of $Y^{\epsilon,u^\epsilon}_{\xi^\epsilon}$ in $C\big([0,T];L^2([0,1])\big)$.

Step II: Convergence to the Limit Equation

Having the tightness of I_i^ϵ for $i = 1, 2, 3, 4$ at hand, by Prohorov's theorem, we can extract a subsequence along which each of the aforementioned processes and $Y_{\xi^\epsilon}^{\epsilon,u^\epsilon}$ converge in distribution to I_i^0 and $Y_\xi^{0,u}(t,x)$ in $C\big([0,T];L^2([0,1])\big)$. We aim to show that the respective limits are as follows:

$$\begin{aligned}
I_1^0 &= \int_0^1 G_t(s,y)\xi(y)dy,\\
I_2^0 &= 0,\\
I_3^0 &= \int_0^t\int_0^1 G_{t-s}(x,y)u(y,s)dyds,\\
I_4^0 &= -\frac{1}{2}\int_0^t\int_0^1 \partial_y G_{t-s}(x,y)(Y_\xi^{0,u}(s,y))^2dyds.
\end{aligned}$$

The case $i = 1$ follows from lemma (5.2). The case $i = 2$ follows from Lemma 3 in [3]. Note that convergence in probability in $C([0,T]\times[0,1])$ implies the same in $C\big([0,T];L^2([0,1])\big)$. As for $i = 3$, we invoke the Skorokhod Representation Theorem [14], and thus assume almost sure convergence (on a larger, common probability space). Denote the right-hand-side of I_3^0 by $\tilde{I}_3^0$. We have

$$|I_3^\epsilon - \tilde{I}_3^0| \le \int_0^t\int_0^1 |G_{t-s}||u^\epsilon - u|dyds \le C(T)\sup_{x,t}|u^\epsilon - u|,$$

and thus converges to zero as $\epsilon \to 0$ since $u^\epsilon \to u$, and

$$\int_0^t\int_0^1 |G_{t-s}|dyds \le C(T).$$

By the fact that the limit is unique, and that $\tilde{I}_3^0$ is a continuous random field (by Theorem 4.4) we conclude that $I_3^0 = \tilde{I}_3^0$. For $i = 4$, we invoke the Skorokhod Representation Theorem [14] again. Denote the right-hand-side of I_4^0 by $\tilde{I}_4^0$. We have

$$\begin{aligned}
|I_4^\epsilon - \tilde{I}_4^0| &\le \int_0^t\int_0^1 |\partial_y G_{t-s}||(Y_{\xi^\epsilon}^{\epsilon,u^\epsilon})^2 - (Y_\xi^{0,u})^2|dyds\\
&\le C(T)\sup_{x,t}|Y_{\xi^\epsilon}^{\epsilon,u^\epsilon} - Y_\xi^{0,u}|\ \sup_{x,t}|Y_{\xi^\epsilon}^{\epsilon,u^\epsilon} + Y_\xi^{0,u}|,
\end{aligned}$$

and thus converges to zero as $\epsilon \to 0$ since $Y_{\xi^\epsilon}^{\epsilon,u^\epsilon} \to Y_\xi^{0,u}$, and

$$\int_0^t\int_0^1 |\partial_y G_{t-s}|dyds \le C(T).$$

By the same exact reasoning as the third case, we conclude that $I_4^0 = \tilde{I}_4^0$. We have proven that along a subsequence, the controlled process converges to the limit equation. This concludes the proof of Theorem (4.2). ☐

6. Conclusions

We studied the large deviations properties of a stochastic Burgers' equation in the presence of an additive noise by employing the weak convergence approach. Compared to previous works, our method had the advantage of avoiding time discretizations as well as technical exponential tail estimates, which to our belief, are the most difficult parts of large deviations analysis for infinite dimensional models. Our proof hinged on three existence and uniqueness results: that of the solution mapping, the controlled process, and the limit equation. Furthermore, we showed that, along a subsequence the controlled process converges to the limit equation. We thus provided a short, and simple proof of the large deviations principle for the law of the solutions to the stochastic Burgers' equation by adopting the weak convergence approach.

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