Communications on Stochastic Analysis
Vol. 8, No. 2 (2014) 191-203

Serials Publications
www.serialspublications.com

NON-DETECTION PROBABILITY OF DIFFUSING TARGETS IN THE PRESENCE OF A MOVING SEARCHER

PANI W. FERNANDO AND SIVAGURU S. SRITHARAN*

Abstract. In this paper, we compute the non-detection probability of an infinite system of randomly moving independent Brownian targets by a moving searcher which travels according to its own prescribed trajectory.

1. Introduction

Search and Detection theory has developed rapidly in the last few decades within the field of operations research. Its military and civilian applications arise in areas such as anti-submarine warfare (ASW), deep-ocean search for submerged objects, route planning for unmanned vehicles, search and rescue operations, mine field clearing, and fish population management. Bernard Koopman and his colleagues in the ASW research group initiated search and detection theory, during world war II [8]. Their main goal was to find a method to track enemy submarines efficiently. Koopman's work was declassified in 1958.

There are many operational applications in naval warfare where a searcher seeks to detect a moving or stationary object. For example, the searcher might wish to detect a target submarine for attack or closer surveillance. Or the searcher might wish to locate a disabled submarine or unmanned vehicle to assist or recover it. We categorize these applications into three broad cases.

(1) Target is stationary, searcher is moving.
(2) Searcher is stationary, target is moving.
(3) Both the target and searcher are moving.

In [5], J. N. Eagle addressed the second case and obtained two expressions for the non-detection probability of a randomly moving target in the presence of a stationary sensor. First expression was derived from an approximation to the exact solution involved with the diffusion process of the target motion and the second one was obtained from the Monte-Carlo simulation results of the diffusion process associated with the target motion. Marc Mangel [11] focused on the problem involving search a randomly moving target in a 2D bounded domain by a searcher moves in 3D space. He derived the corresponding search equation satisfied by the joint density of the target location and unsuccessful search. In [9], S. N.

Received 2013-9-9; Communicated by P. Sundar.
2010 *Mathematics Subject Classification.* Primary 90B40; Secondary 60H30.
Key words and phrases. Diffusing Brownian targets, non-detection probability, Poisson mapping theorem, Laplace transformation.
* This work has been supported by the Army Research Probability and Statistics Program.

Majumdar and A. J. Bray considered survival probability of a particle moving along a straight line in the presence of diffusing traps in the plane. They obtained an explicit expression for survival probability of the tracer particle for large time t. In addition to the main result, we have also introduced several mathematical theorems to clarify the heuristic results in [9].

In this work, we focus on the third scenario, which is less well developed, but more important in Naval operations research. We allow targets to move according to a diffusion process in the whole plane. The searcher starts its journey from the origin and it follows a deterministic trajectory through the $\mathbb{R}^2$ plane. We have used the ideas in S. N. Majumdar and A. J. Bray [9], to obtain an expression for the probability $\mathbf{P}_{ND}(t)$ that none of the randomly moving targets are detected by searcher up to time t. Our main result exactly coincides with the 2D result in [9] as time t approaches to infinity. Our main result is stated below.

Main Result: Suppose that there are infinitely many Brownian targets(with diffusion constant ϵ) diffuse over whole $\mathbb{R}^2$ plane with density ρ. The searcher starts its journey from the origin and travels along a deterministic path $\mathrm{S}(t)$ with constant speed $\mathbf{v}$(details on $\mathrm{S}(t)$ can be found in section 3). Then the probability $\mathbf{P}_{ND}(t)$ that none of the randomly moving targets detected by searcher up to time t is approximated by:

$$\begin{aligned}\mathbf{P}_{ND}(t) &= \exp\left[-\mathrm{K}(t)\right] \\ &= \exp\left[-2\pi\epsilon(1-e^{-\rho\pi a^2})\left[e^{-\delta\eta}\frac{(t-\delta)\mathrm{E}_1(\delta\eta)+\delta\mathrm{E}_0(\delta\eta)}{\mathrm{E}_1^2(\delta\eta)}-\frac{e^{(\delta-t)\eta}}{\eta}\right]\right],\end{aligned} \tag{1.1}$$

where $\eta = \frac{\mathbf{v}^2}{2\epsilon}$, δ is a small constant such that $\delta < \frac{1}{\eta}$, $\mathrm{E}_1(y) = \int_y^\infty \frac{e^{-z}}{z}\mathrm{d}z$ and $\mathrm{E}_0(y) = \frac{e^{-y}}{y}$ (See [3]). For large time t, $\mathbf{P}_{ND}(t)$ reduce to following simple form

$$\mathbf{P}_{ND}(t) = \exp\left[-\mathrm{K}(t)\right] = \exp\left[\frac{2\pi\epsilon(1-e^{-\rho\pi a^2})t}{\ln(\eta\delta)+\gamma}\right], \tag{1.2}$$

where γ is the Euler-Mascheroni Constant. The article is organized as follows. In the next section, we formulate the target model associated with the search problem that we briefly described in the introduction. In section 3, we obtain an explicit expression for $\mathbf{P}_{ND}(t)$.

2. The Target Model

Let our infinite system of Brownian targets be initially distributed over the $\mathbb{R}^2$ plane according to the spatial Poisson distribution with intensity ρ(See Lemma 2.5 at the end of this section). Then, we allow infinite system of random targets to move independently according to 2D Brownian motion with diffusion constant ϵ. The interesting fact is that we again have the same initial Poisson point distribution with intensity ρ for the system of infinite Brownian targets after each fixed time $t > 0$.

Before we prove this result, we introduce the definition of spatial Point process and two key theorems(Poisson mapping theorem and labeled Poisson point process theorem) which use for the proof of the above result.

Definition 2.1. Let μ be a σ-finite measure on the measurable space $(\mathrm{S}, \mathscr{S})$. Then the collection of random variables $\mathrm{N}(\mathrm{A}, \omega)$, $\mathrm{A} \in \mathscr{S}$, $\omega \in \Omega$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a *Poisson point process* if

(1) The random variables $\mathrm{N}(\mathrm{A}_1), \ldots, \mathrm{N}(\mathrm{A}_n)$ are independent for any finite collection of disjoint measurable subsets $\mathrm{A}_1, \ldots, \mathrm{A}_n \in \mathscr{S}$.
(2) For every $\mathrm{A} \in \mathscr{S}$ the random variable $\mathrm{N}(\mathrm{A})$ has Poisson distribution with mean $\mu(\mathrm{A})$.

Proposition 2.2. *(Poisson mapping theorem) Let* $\mathrm{N}(\mathrm{A})$ *be a Poisson point process on a measure space* $(\mathrm{S}, \mathscr{S})$ *with the measure* μ. *Let* $\mathrm{f} : (\mathrm{S}, \mathscr{S}) \to (\mathrm{G}, \mathscr{G})$ *be a measurable function such that the induced measure* $\tilde{\mu} = \mu \circ \mathrm{f}^{-1}$ *has no atoms. Then* $\mathrm{f}(\mathrm{N}(.))$ *is a Poisson process on the measurable space* $(\mathrm{G}, \mathscr{G})$ *with mean measure* $\tilde{\mu}$.

Proof. See page 18, [7]. □

Definition 2.3. (Labeled Poisson point process) Let $\hat{\mathrm{N}}(\mathrm{A})$ be a Poisson point process on a measure space $(\mathrm{S}, \mathscr{S})$ with the measure μ. Now take a random variable ξ on some measurable space $(\mathrm{F}, \mathscr{F})$. Then $\tilde{\mathrm{N}}(\mathrm{A}, \mathrm{B}) = \{(\mathbf{X}, \mathbf{X}_\xi) \in A \times B\}$ is a *labeled Poisson point process* on the product measurable space $(\mathrm{S} \times \mathrm{F}, \mathscr{S} \otimes \mathscr{F})$ with the mean measure $\mu \times \bar{\mu}_\xi$. Where A, B are arbitrary sets in $\mathscr{B}\left(\mathbb{R}^2\right)$ and $\bar{\mu}_\xi$ is the mean of the labeled random variable ξ.

Proposition 2.4. *Let* $\hat{\mathrm{N}}(\mathrm{A})$ *be a Poisson point process(initial spatial distribution of the targets) over the space* $\left(\mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right)\right)$ *with mean measure* $\rho l(.)$. *Here* $l(.)$ *is the Lebesgue measure on* $\mathbb{R}^2$. *For each* i, $\mathbf{W}_i(t)$ *is the displacement of* i^{th} *Brownian target after time* t. *Then the process,*

$$\tilde{\mathrm{N}}(\mathrm{A}, \mathrm{B}) = \{(\mathbf{W}_i(0), \mathbf{W}_i(t)) \in A \times B : i = 1, 2, 3, \ldots, n\}$$

(n can be finite or infinity), on the product space $\left(\mathbb{R}^2 \times \mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right) \otimes \mathscr{B}\left(\mathbb{R}^2\right)\right)$ *is a labeled Poisson point process with mean measure* $\rho l(.) \times \mu_{\mathbf{W}(t)}$. *Where* A, B *are arbitrary sets in* $\mathscr{B}\left(\mathbb{R}^2\right)$ *and* $l(.)$ *is the Lebesgue measure on* $\mathbb{R}^2$.

Proof. See Theorem 3.11 in page 48, [6]. □

Now we can present the theorem that describe the time stationary property of the target's spatial distribution.

Theorem 2.5. *Let* $\mathrm{N}_0(\mathrm{A})$ *be the Poisson point process(initial spatial distribution of the Brownian targets) over the space* $\left(\mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right)\right)$ *with mean measure* $\rho l(.)$. *Then for any time* $t > 0$, $\mathrm{N}_t(\mathrm{A})$ *is again a Poisson point process(spatial distribution of the Brownian targets at time* t*) with mean measure* $\rho l(.)$. *Where* A *is an arbitrary set in* $\mathscr{B}\left(\mathbb{R}^2\right)$ *and* $l(.)$ *is the Lebesgue measure on* $\mathbb{R}^2$.

Proof. From proposition 2.5, we can conclude that the process

$$\tilde{\mathrm{N}}(\mathrm{A}, \mathrm{B}) = \{(\mathbf{W}_i(0), \mathbf{W}_i(t)) \in A \times B : i = 1, 2, 3, \ldots, n\}$$

on the product space $\left(\mathbb{R}^2 \times \mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right) \otimes \mathscr{B}\left(\mathbb{R}^2\right)\right)$ is a Poisson point process with mean measure $\rho l(.) \times \mu_{\mathbf{W}(t)}$.

Now, define a measurable function from the space $\left(\mathbb{R}^2 \times \mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right) \otimes \mathscr{B}\left(\mathbb{R}^2\right)\right)$ to the space $\left(\mathbb{R}^2, \mathscr{B}\left(\mathbb{R}^2\right)\right)$ such that $\mathrm{f} : (\vec{x}, \vec{y}) \to \vec{x} + \vec{y}$. Then the proposition 2.4

(Poisson mapping theorem) implies that $\mathrm{f}(\tilde{\mathrm{N}}(\mathrm{A},\mathrm{B})) = \mathrm{N}_t(\mathrm{A})$ is a Poisson point process with mean measure $(\rho l(.) \times \mu_{\mathbf{W}(t)}) \circ \mathrm{f}^{-1}$. Now we need to show that $(\rho l(.) \times \mu_{\mathbf{W}(t)}) \circ \mathrm{f}^{-1} = \rho l(.)$. Let $\mathrm{A} \in \mathscr{B}\left(\mathbb{R}^2\right)$. Then,

$$\begin{aligned}
\left((\rho l(.) \times \mu_{\mathbf{W}(t)}) \circ \mathrm{f}^{-1}\right)(\mathrm{A}) &= \int_{\mathbb{R}^2} \mathbf{I}_{\mathrm{A}}(\vec{x}) \mathrm{d}((\rho l(.) \times \mu_{\mathbf{W}(t)}) \circ \mathrm{f}^{-1})(\vec{x}) \\
&= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \mathbf{I}_{\mathrm{A}}(\mathrm{f}((\vec{x},\vec{y}))) \mathrm{d}(\rho l(.) \times \mu_{\mathbf{W}(t)})(\vec{x},\vec{y}) \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{I}_{\mathrm{A}}(\vec{x}+\vec{y}) \mathrm{d}\rho l(\vec{x}) \mathrm{d}\mu_{\mathbf{W}(t)}(\vec{y}) \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathbf{I}_{\mathrm{A}-\vec{y}}(\vec{x}) \mathrm{d}\rho l(\vec{x}) \mathrm{d}\mu_{\mathbf{W}(t)}(\vec{y}) \\
&= \int_{\mathbb{R}^2} \rho l(\mathrm{A}-\vec{y}) \mathrm{d}\mu_{\mathbf{W}(t)}(\vec{y}) \\
&= \rho l(\mathrm{A}) \int_{\mathbb{R}^2} \mathrm{d}\mu_{\mathbf{W}(t)}(\vec{y}) \\
&= \rho l(\mathrm{A}) \mu_{\mathbf{W}(t)}(\mathbb{R}^2) \\
&= \rho l(\mathrm{A}).
\end{aligned} \tag{2.1}$$

□

Remark 2.6. Since all the targets are moving independently over the $\mathbb{R}^2$ plane according to Brownian motion, they don't collide each other as time evolves. The following lemma gives a simple proof of that phenomenon.

Lemma 2.7. *Let $\{\mathbf{W}_i(t) : i = 1, 2, 3, \ldots\}$ be the set of independent Brownian motions with same diffusion constant ϵ which prescribed the random motions of all targets in $\mathbb{R}^2$ plane. Then the targets do not collide with each other for any time $t > 0$ except possibly at the location $(0, 0)$.*

Proof. Consider arbitrary two targets which are governed by Brownian motions $\mathbf{W}_i(t) = (\mathbf{w}_i^x(t), \mathbf{w}_i^y(t))$ and $\mathbf{W}_i(t) = (\mathbf{w}_j^x(t), \mathbf{w}_j^y(t))$. Now suppose that the two targets collide each other at time $t > 0$ and the location (x, y). Since $\mathbf{W}_i(t)$ and $\mathbf{W}_j(t)$ are pairwise independent Brownian motions, we have

$$\begin{aligned}
\mathrm{E}\left(\mathbf{W}_i(t) \cdot \mathbf{W}_j(t)\right) &= \mathrm{E}\left(\mathbf{w}_i^x(t)\mathbf{w}_j^x(t) + \mathbf{w}_i^y(t)\mathbf{w}_j^y(t)\right) \\
&= \mathrm{E}\left(\mathbf{w}_i^x(t)\right)\mathrm{E}\left(\mathbf{w}_j^x(t)\right) + \mathrm{E}\left(\mathbf{w}_i^y(t)\right)\mathrm{E}\left(\mathbf{w}_j^y(t)\right) = 0.
\end{aligned} \tag{2.2}$$

On the other hand

$$\mathrm{E}\left(\mathbf{W}_i(t) \cdot \mathbf{W}_j(t)\right) = \mathrm{E}\left(\mathbf{w}_i^x(t)\mathbf{w}_j^x(t) + \mathbf{w}_i^y(t)\mathbf{w}_j^y(t)\right) = x^2 + y^2 = 0. \tag{2.3}$$

The tesult (2.3) implies that the targets do not collide with each other for any time $t > 0$ except possibly at the location $(0, 0)$. □

In the next lemma, we describe how the infinite system of Brownian targets initially distributed over the $\mathbb{R}^2$ plane.

Lemma 2.8. *We consider a bounded set* $\mathrm{A} \in \mathscr{B}\left(\mathbb{R}^2\right)$ *containing* N *number of independently moving targets according to Brownian motion with same diffusive constant* ϵ*. We assume that the* N *number of targets are uniformly distributed over the set* A*. Let* $l(.)$ *be the Lebesgue measure over the* $\mathbb{R}^2$ *plane. Then the targets are initially distributed according to the spatial Poisson point distribution with a rate* $\rho = \frac{\mathrm{N}}{l(\mathrm{A})}$ *over the* $\mathbb{R}^2$ *plane.*

Proof. Let $\mathrm{B} \in \mathscr{B}\left(\mathbb{R}^2\right)$ such that $\mathrm{B} \subseteq \mathrm{A}$. Now let $\mathbb{N}(\mathrm{B})$ be a random variable which represents the number of targets located in the set B. Then $\mathbf{P}(\mathbb{N}(\mathrm{B}) = n|\mathrm{N}$ number of targets in A) is the probability that n number of targets initially lies inside the set B out of N targets in the set A. Then

$$\begin{aligned}\mathbf{P}(\mathbb{N}(\mathrm{B}) = n|\mathrm{N} \text{ number of targets in A}) &= \mathbf{C}_{\mathrm{n}}^{\mathrm{N}} \left(\frac{l(\mathrm{B})}{l(\mathrm{A})}\right)^{\mathrm{n}} \left(1 - \frac{l(\mathrm{B})}{l(\mathrm{A})}\right)^{\mathrm{N-n}} \\ &= \frac{\mathrm{N}!}{\mathrm{n}!(\mathrm{N}-\mathrm{n})!} \left(\frac{\rho l(\mathrm{B})}{\mathrm{N}}\right)^{\mathrm{n}} \left(1 - \frac{\rho l(\mathrm{B})}{\mathrm{N}}\right)^{\mathrm{N-n}}.\end{aligned} \tag{2.4}$$

When $\mathrm{N}, l(\mathrm{A}) \to \infty$ while keeping density $\rho = \frac{\mathrm{N}}{l(\mathrm{A})}$ fixed, we can easily argue that $\frac{\mathrm{N}!}{\mathrm{N}^{\mathrm{n}}(\mathrm{N}-\mathrm{n})!}\left(1 - \frac{\rho l(\mathrm{B})}{\mathrm{N}}\right)^{\mathrm{n}} \to 1$. Then we get

$$\mathbf{P}(\mathbb{N}(\mathrm{B}) = n) = \frac{(\rho l(\mathrm{B}))^{\mathrm{n}} \exp[-\rho l(\mathrm{B})]}{\mathrm{n}!}, \tag{2.5}$$

as $\mathrm{N}, l(\mathrm{A}) \to \infty$ with density $\rho = \frac{\mathrm{N}}{l(\mathrm{A})}$ fixed. This implies that the initial distribution of the targets is spatially Poisson with intensity parameter ρ. □

3. Non-detection Probability $\mathbf{P}_{ND}(t)$

We introduce the searcher to the system at time $t = 0$ at the origin and it follows its own deterministic path $\mathrm{S}(t)$. Then one can show that the number of targets encountered by the searcher up to time t over the $\mathbb{R}^2$ plane is Poissonly distributed with mean parameter $\mathrm{K}(t) = \rho \int_{\mathbb{R}^2} \hat{\mathbf{P}}(\vec{b}, t)\mathrm{d}x\mathrm{d}y$ as follows, where $\vec{b} = (x, y) \in \mathbb{R}^2$.

Theorem 3.1. *Let* $\mathrm{A} \in \mathscr{B}\left(\mathbb{R}^2\right)$*. Suppose that* A *containing* N *number of independently moving targets according to Brownian motion with same diffusive constant* ϵ*. Denote* $\hat{\mathbf{P}}(\vec{b}_i, t)$ *the probability that* i^{th} *target starting from the point* $\vec{b}_i = (x_i, y_i) \in \mathrm{A}$ *and detected by the searcher before time* t*. Then the probability that* n *number of targets encountered by searcher up to time* t *is given by* $\mathbf{P}(\mathrm{n}, t) = \frac{(\mathrm{K}(t))^{\mathrm{n}} \exp[-\mathrm{K}(t)]}{\mathrm{n}!}$ *with the mean parameter* $\mathrm{K}(t) = \rho \int_{\mathbb{R}^2} \hat{\mathbf{P}}(\vec{b}, t)dxdy$*.*

Proof. Since the infinite system of Brownian targets is Poissonly distributed over $\mathbb{R}^2$ plane, the N number of targets are uniformly distributed over the set A, the probability that a target starting from set A and detected by searcher before time t is given by $\frac{1}{l(\mathrm{A})} \int_{\mathrm{A}} \hat{\mathbf{P}}(\vec{b}, t)\mathrm{d}x\mathrm{d}y$.

Therefore, the probability that n number of targets detected by searcher out of N number of targets starting from set A before time t is

$$\begin{aligned}\mathbf{P}^{\mathrm{N}}(\mathrm{n},t) &= \mathbf{C}_{\mathrm{n}}^{\mathrm{N}}\left(\frac{1}{l(\mathrm{A})}\int_{\mathrm{A}}\hat{\mathbf{P}}(\vec{b},t)\mathrm{d}x\mathrm{d}y\right)^{\mathrm{n}}\left(1-\frac{1}{l(\mathrm{A})}\int_{\mathrm{A}}\hat{\mathbf{P}}(\vec{b},t)\mathrm{d}x\mathrm{d}y\right)^{\mathrm{N}-\mathrm{n}}\\ &= \frac{\mathrm{N}!}{\mathrm{n}!(\mathrm{N}-\mathrm{n})!}\left(\frac{\rho}{\mathrm{N}}\int_{\mathrm{A}}\hat{\mathbf{P}}(\vec{b},t)\mathrm{d}x\mathrm{d}y\right)^{\mathrm{n}}\left(1-\frac{\rho}{\mathrm{N}}\int_{\mathrm{A}}\hat{\mathbf{P}}(\vec{b},t)\mathrm{d}x\mathrm{d}y\right)^{\mathrm{N}-\mathrm{n}}.\end{aligned} \tag{3.1}$$

As we have argued in Lemma 2.5, we can easily obtain that

$$\mathbf{P}^{\mathrm{N}}(\mathrm{n},t) \to \mathbf{P}(\mathrm{n},t) = \frac{(\mathrm{K}(t))^{\mathrm{n}}\exp\left[-\mathrm{K}(t)\right]}{\mathrm{n}!}, \tag{3.2}$$

as $\mathrm{N}, l(\mathrm{A}) \to \infty$ while keeping density $\rho = \frac{\mathrm{N}}{l(\mathrm{A})}$ fixed. Here $\mathrm{K}(t) = \rho\int_{\mathbb{R}^2}\hat{\mathbf{P}}(\vec{b},t)\mathrm{d}x\mathrm{d}y$. Hence we have conclude the Theorem 2.2. □

By setting $n = 0$ in (3.2), we get

$$\mathbf{P}_{ND}(t) = \mathbf{P}(0,t) = \exp\left[-\mathrm{K}(t)\right]. \tag{3.3}$$

Now our goal is to compute $\mathrm{K}(t)$ to obtain the non-detection probability $\mathbf{P}_{ND}(t)$. In [2], a useful method is introduced to compute $\mathrm{K}(t)$ associated with any general path $\mathrm{S}(t)$. They have formulated an implicit integral equation for $\mathrm{K}(t)$ associated with Gaussian transition density and target's spatial intensity ρ. We modify it according to our model as follows.

$$1 - e^{-\rho\pi a^2} = \int_0^t \dot{\mathrm{K}}(\tilde{t})\mathbf{P}\left(\mathrm{S}(t),t|\mathrm{S}(\tilde{t}),\tilde{t}\right)\mathrm{d}\tilde{t}, \tag{3.4}$$

where a is the radius of the searcher's sensor, $\dot{\mathrm{K}} = \frac{\mathrm{dK}}{\mathrm{d}t}$, $\mathbf{P}\left(\mathrm{S}(t),t|\mathrm{S}(\tilde{t}),\tilde{t}\right) = [2\pi\epsilon(t-\tilde{t}+\delta)]^{-1}\exp\left\{-[\mathrm{S}(t)-\mathrm{S}(\tilde{t})]^2/2\epsilon(t-\tilde{t}+\delta)\right\}$ is the de-singularized 2D transition probability density associated with Brownian motion and $\mathrm{S}(t)$ is the deterministic path of the searcher.

For the sake of completeness, we sketch the heuristic derivation of the integral equation (3.4). Note that there are two ways to compute the probability density that a target meets the searcher at the location $\mathrm{S}(t)$ at time t. Since the targets are spatially distributed according to homogeneous Poisson point process with intensity ρ(see theorem 2.6), the probability that a target detected by the searcher at the location $\mathrm{S}(t)$ at time t is $1 - \mathbf{P}\left(\mathbb{N}(\mathbb{B}_{\mathrm{S}(t)}(a)) = 0\right) = 1 - e^{-\rho l(\mathbb{B}_{\mathrm{S}(t)}(a))} = 1 - e^{-\rho\pi a^2}$. Here $\mathbb{B}_{\mathrm{S}(t)}(a)$ represents the searcher's coverage area which is a disk with radius a and center $\mathrm{S}(t)$. Secondly, a target meets the searcher for the first time during time interval $(\tilde{t},\tilde{t}+\mathrm{d}\tilde{t})$ is $\dot{\mathrm{K}}(\tilde{t})\mathrm{d}\tilde{t}$. Therefore the probability density of the above target again meets the searcher at the location $\mathrm{S}(t)$ at time t which is given by transition probability density $\mathbf{P}\left(\mathrm{S}(t),t|\mathrm{S}(\tilde{t}),\tilde{t}\right)$. By equating the results of these two methods gives the equation (3.4).

Remark 3.2. In general, the transition probability density function associated with Brownian motion doesn't have a δ term in the denominator and inside of the exponential function. But in 2D case of the integral equation, we have to introduce very small value $\delta > 0$ to make the right hand side of integral equation (3.4) to be well defined.

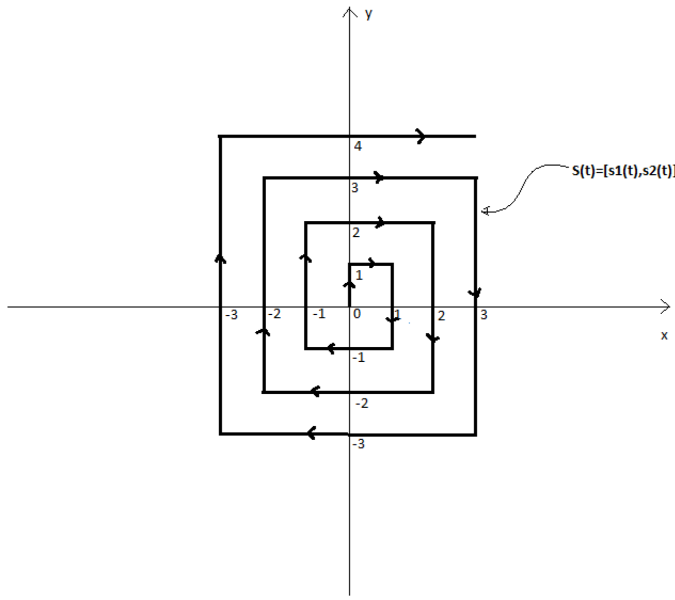

FIGURE 1. Trajectory of the Searcher

Let us discuss about the function $\mathrm{S}(t)$ governed by the deterministic trajectory of the searcher starting at origin. We are interested to analyze this problem with the searcher which starts at time $t = 0$, at the origin and travel through a spiral shape trajectory.

The function $\mathrm{S}(t)$ of the above trajectory can be expressed as follows.

$$\mathrm{S}(t) = (\mathrm{s}_1, \mathrm{s}_2) = \begin{cases} (\mathbf{v}t - n(4n-3), n\mathbf{v}) \text{ if } & \dfrac{4n(n-1)+1}{\mathbf{v}} \leq t \leq \dfrac{4n(n-2)}{\mathbf{v}} \\ (n\mathbf{v}, \mathbf{v}t - n(4n-1)) \text{ if } & \dfrac{n(4n-2)}{\mathbf{v}} \leq t \leq \dfrac{4n^2}{\mathbf{v}} \\ (\mathbf{v}t - n(4n+1), -n\mathbf{v}) \text{ if } & \dfrac{4n^2}{\mathbf{v}} \leq t \leq \dfrac{n(4n+2)}{\mathbf{v}} \\ (-n\mathbf{v}, \mathbf{v}t - n(4n+3)) \text{ if } & \dfrac{n(4n+2)}{\mathbf{v}} \leq t \leq \dfrac{4n^2+4n+1}{\mathbf{v}}, \end{cases}$$

for $n = 1, 2, 3, \ldots$ and

$$\mathrm{S}(t) = (\mathrm{s}_1(t), \mathrm{s}_2(t)) = (0, \mathbf{v}t) \quad \text{if } 0 \leq t \leq \frac{1}{\mathbf{v}}, \tag{3.5}$$

for $n = 0$, where $\mathbf{v}$ is the speed of the searcher.

Now we would like to derive a simple expression for transition probability density function $\mathbf{P}\left(\mathrm{S}(t), t | \mathrm{S}(\tilde{t}), \tilde{t}\right)$ for the function $\mathrm{S}(t)$ given by (3.5). For any $n \geq 1$, assume that $\tilde{t}, t \in \left[\frac{4n(n-1)+1}{\mathbf{v}}, \frac{4n(n-2)}{\mathbf{v}}\right]$, then

$$\begin{aligned} \mathbf{P}\left(\mathrm{S}(t), t|\mathrm{S}(\tilde{t}), \tilde{t}\right) &= \frac{1}{2\pi\epsilon(t-\tilde{t}+\delta)} \exp\left[\frac{|\mathrm{S}(t)-\mathrm{S}(\tilde{t})|^2}{2\epsilon(t-\tilde{t}+\delta)}\right] \\ &= \frac{1}{2\pi\epsilon(t-\tilde{t}+\delta)} \exp\left[\frac{-\mathbf{v}^2(t-\tilde{t})^2}{2\epsilon(t-\tilde{t}+\delta)}\right] \\ &\approx \frac{1}{2\pi\epsilon(t-\tilde{t}+\delta)} \exp\left[\frac{-\mathbf{v}^2(t-\tilde{t})}{2\epsilon}\right] \\ &= \frac{1}{2\pi\epsilon(t-\tilde{t}+\delta)} \exp\left[-\eta(t-\tilde{t})\right], \end{aligned} \tag{3.6}$$

where $\eta = \frac{\mathbf{v}^2}{2\epsilon}$. Second approximation of (3.6) holds due to fact that δ is a very small positive value. Similarly, we can show that $\mathbf{P}\left(\mathrm{S}(t), t|\mathrm{S}(\tilde{t}), \tilde{t}\right)$ has the same expression for other three cases when $n \geq 1$ and the case $n = 0$ of the trajectory function $\mathrm{S}(t)$.

Therefore, the above analysis implies that for any $t > 0$, the integral equation (3.4) involved with the searcher trajectory function $\mathrm{S}(t)$ is reduced to following simple form.

$$1 - e^{-\rho\pi a^2} = \frac{1}{2\pi\epsilon} \int_0^t \frac{\exp[-\eta(t-\tilde{t})]}{(t-\tilde{t}+\delta)} \dot{\mathrm{K}}(\tilde{t}) \mathrm{d}\tilde{t}, \tag{3.7}$$

We use Laplace transform method to solve implicit integral equation (3.7) to solve $\mathrm{K}(t)$. Let $\tilde{\mathrm{K}}(s) = \int_0^\infty \mathrm{K}(t) e^{-st} \mathrm{d}t$. Apply Laplace transform to (3.7) to get

$$(1 - e^{-\rho\pi a^2}) \int_0^\infty e^{-st} \mathrm{d}t = \frac{1}{2\pi\epsilon} \int_0^\infty e^{-st} \int_0^t \frac{\exp[-\eta(t-\tilde{t})]}{(t-\tilde{t}+\delta)} \dot{\mathrm{K}}(\tilde{t}) \mathrm{d}\tilde{t} \mathrm{d}t. \tag{3.8}$$

By applying change of order of integration to (3.8), one can obtain,

$$\frac{2\pi\epsilon(1 - e^{-\rho\pi a^2})}{s} = \int_0^\infty e^{-s\tilde{t}} \dot{\mathrm{K}}(\tilde{t}) \mathrm{d}\tilde{t} \int_0^\infty \frac{\exp[-(s+\eta)\tau]}{\tau+\delta} \mathrm{d}\tau. \tag{3.9}$$

By further simplifying,

$$\tilde{\mathrm{K}}(s) = \frac{2\pi\epsilon(1 - e^{-\rho\pi a^2})}{s^2 \tilde{\mathrm{g}}(s)}, \tag{3.10}$$

where

$$\tilde{\mathrm{g}}(s) = \int_0^\infty \frac{\exp[-(\eta+s)\tau]}{\tau+\delta} \mathrm{d}\tau. \tag{3.11}$$

Now we apply inverse Laplace transform (Bromwich contour integral approach) for $\tilde{\mathrm{K}}(s)$ to obtain $\mathrm{K}(t)$,

$$\mathrm{K}(t) = \frac{C}{2\pi i} \lim_{b\to\infty} \int_{a-ib}^{a+ib} \frac{e^{st}}{s^2 \tilde{\mathrm{g}}(s)} \mathrm{d}s, \tag{3.12}$$

where $C = 2\pi\epsilon(1 - e^{-\rho\pi a^2})$. Simple calculation leads the $\tilde{\mathrm{g}}(s)$ to the following form,

$$\tilde{\mathrm{g}}(s) = e^{\delta(s+\eta)} \int_{\delta(s+\eta)}^\infty \frac{e^{-z}}{z} \mathrm{d}z = e^{\delta(s+\eta)} \mathrm{E}_1(\delta(s+\eta)), \tag{3.13}$$

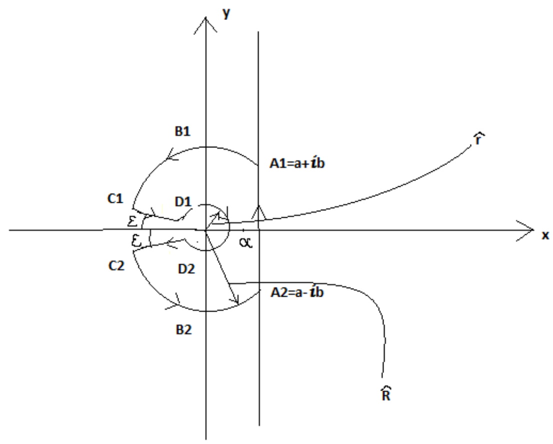

FIGURE 2. The Bromwich contour

where $\mathrm{E}_\nu(y) = \int_y \frac{e^{-z}}{z^\nu}\mathrm{d}z$ is the exponential integral function. Then,

$$\mathrm{K}(t) = \frac{C}{2\pi i}\lim_{b\to\infty}\int_{a-ib}^{a+ib}\frac{e^{st}}{s^2 e^{\delta(s+\eta)}\mathrm{E}_1(\delta(s+\eta))}\mathrm{d}s. \tag{3.14}$$

By observing (3.14), we can see that $\mathrm{H}(s) = \frac{e^{st}}{s^2 e^{\delta(s+\eta)}\mathrm{E}_1(\delta(s+\eta))}$ has a pole at $s = 0$ with order 2 and branch point at $s = -\eta$. This implies that $\mathrm{H}(s)$ is analytic except on the set $(-\infty, -\eta] \cup \{0\}$. According to the residue theorem, the integral of $\mathrm{H}(s)$ through the circular path $A1B1C1D1D2C2B2A2A1$ is equal to $2\pi i Res(\mathrm{H}(s), 0)$. That is,

$$\mathbf{I}_{A2A1} + \mathbf{I}_{A1C1} + \mathbf{I}_{C1D1} + \mathbf{I}_{D1D2} + \mathbf{I}_{D2C2} + \mathbf{I}_{C2A2} = 2\pi i Res(\mathrm{H}(s), 0). \tag{3.15}$$

Therefore,

$$\int_{a-ib}^{a+ib}\mathrm{H}(s)\mathrm{d}s = 2\pi i Res(\mathrm{H}(s), 0) - (\mathbf{I}_{A1C1} + \mathbf{I}_{C1D1} + \mathbf{I}_{D1D2} + \mathbf{I}_{D2C2} + \mathbf{I}_{C2A2}). \tag{3.16}$$

In the above figure, $\hat{R}$ and $\hat{r}$ represent the radii of large circular arc and small circular arc respectively. It can be noticed that $b \to \infty$ when $\hat{R}$ approaches to ∞ since $\hat{R}^2 = a^2 + b^2$. Let ϵ denote the angle between $C1D1$ or $C2D2$ and negative x-axis.

Let $s + \eta = \hat{R}e^{i\theta}$, then $\mathrm{d}s = i\hat{R}e^{i\theta}\mathrm{d}\theta$ and the integrand

$$\mathrm{H}(s) = \frac{e^{(\hat{R}e^{i\theta}-\eta)t}}{(\hat{R}e^{i\theta}-\eta)^2 e^{\delta\hat{R}e^{i\theta}}\mathrm{E}_1(\delta\hat{R}e^{i\theta})}$$

along the paths $A1C1$ and $C2A2$. Therefore, for any $\epsilon > 0$, the integrand $\mathrm{H}(s)$ is a continuous function of θ over $A1C1$ and $C2A2$. This gives us the existence of $\lim_{\epsilon\to 0}\mathbf{I}_{A1C1}$ and $\lim_{\epsilon\to 0}\mathbf{I}_{C2A2}$ for any fixed $\hat{R} > 0$. Similarly, we can argue that

$\lim_{\epsilon\to 0}\mathbf{I}_{D1D2}$ exists for any fixed $\hat{r} > 0$. $\mathbf{J}$ denotes the limit of $\mathbf{I}$ as $\epsilon \to 0$. For example $\mathbf{J}_{D1D2} = \lim_{\epsilon\to 0}\mathbf{I}_{D1D2}$.

Along the arc $D1D2$, $s + \eta = \hat{r}e^{i\theta}$ with $-\pi + \epsilon \le \theta \le \pi - \epsilon$. Then,

$$\mathbf{J}_{D1D2} = \int_{-\pi}^{\pi} \frac{e^{(\hat{r}e^{i\theta}-\eta)t} i\hat{r}e^{i\theta}}{(\hat{r}e^{i\theta}-\eta)^2 e^{\delta\hat{r}e^{i\theta}} \mathrm{E}_1(\delta\hat{r}e^{i\theta})} \mathrm{d}\theta. \tag{3.17}$$

The result $\lim_{\hat{r}\to 0}\mathrm{E}_1(\delta\hat{r}e^{i\theta}) = \infty$ and the fact that $\hat{r}$ appears in the numerator of the integrand of (3.17) gives us $\lim_{\hat{r}\to 0}\mathbf{J}_{D1D2} = 0$.

When s lies on the line $C1D1$, let $s + \eta = re^{i(\pi-\epsilon)}$. Then

$$\mathrm{H}(s) = \frac{e^{(re^{i(\pi-\epsilon)}-\eta)t}}{(re^{i(\pi-\epsilon)}-\eta)^2 e^{\delta re^{i(\pi-\epsilon)}} \mathrm{E}_1(\delta re^{i(\pi-\epsilon)})} \tag{3.18}$$

where $\hat{r} \le r \le \hat{R}$. By using the results (5), (9) in [13] with the identity $\mathrm{E}_i(-x) = -\mathrm{E}_1(x)$ for $x > 0$(See [3]), we get

$$\lim_{\epsilon\to 0}\mathrm{E}_1(\delta re^{i(\pi-\epsilon)}) = -\mathrm{E}_i(\delta r) - i\pi. \tag{3.19}$$

Then,

$$\mathbf{J}_{C1D1} = e^{-t\eta}\int_{\hat{R}}^{\hat{r}} \frac{e^{r(\delta-t)}}{(r+\eta)^2\left[\mathrm{E}_i(\delta r) + i\pi\right]}\mathrm{d}r. \tag{3.20}$$

Similarly, we can show that

$$\mathbf{J}_{D2C2} = e^{-t\eta}\int_{\hat{r}}^{\hat{R}} \frac{e^{r(\delta-t)}}{(r+\eta)^2\left[\mathrm{E}_i(\delta r) - i\pi\right]}\mathrm{d}r. \tag{3.21}$$

Then,

$$\begin{aligned}
\lim_{\hat{R}\to\infty}\lim_{\hat{r}\to 0}(\mathbf{J}_{C1D1} + \mathbf{J}_{D2C2}) &= 2\pi i e^{-t\eta}\int_0^{\infty} \frac{e^{r(\delta-t)}}{(r+\eta)^2\left[\mathrm{E}_i^2(\delta r) + \pi^2\right]}\mathrm{d}r \\
&= 2\pi i e^{\delta\eta}\int_{\eta}^{\infty} \frac{e^{\alpha(\delta-t)}}{\alpha^2\left[\mathrm{E}_i^2(\delta\alpha - \delta\eta) + \pi^2\right]}\mathrm{d}\alpha \\
&= 2\pi i e^{\delta\eta}\int_{\eta}^{\infty} \frac{e^{-\alpha t}\mathrm{d}\mathrm{E}_i(\delta\alpha)}{\alpha\left[\mathrm{E}_i^2(\delta\alpha - \delta\eta) + \pi^2\right]} \\
&\approx 2\pi i \frac{e^{(\delta-t)\eta}}{\eta}\int_{\eta}^{\infty} \frac{\mathrm{d}\mathrm{E}_i(\delta\alpha)}{\left[\mathrm{E}_i^2(\delta\alpha) + \pi^2\right]} \\
&= 2\pi i \frac{e^{(\delta-t)\eta}}{\pi\eta}\left[\tan^{-1}\left(\frac{\mathrm{E}_i(\delta\alpha)}{\pi}\right)\right]_{\alpha=\eta}^{\alpha=\infty} \\
&\approx 2\pi i \frac{e^{(\delta-t)\eta}}{\eta}.
\end{aligned} \tag{3.22}$$

In the result (3.22), the first approximation is obtained by using the fact that $\delta\eta < 1$ is a very small value and by plugging lower limit η for $e^{-t\alpha}$. The second approximation is due to the fact that $\delta\eta < 1$ is a very small value and the results

$\mathrm{E}_i(0) = -\infty$ and $\mathrm{E}_i(-\infty) = +\infty$. Notice that, on the arcs $A1C1$ and $C2A2$, when $|s| \to \infty$,

$$\begin{aligned}\mathrm{H}(s) &= \frac{e^{st}}{s^2 e^{\delta(s+\eta)}\mathrm{E}_1(\delta(s+\eta))} \\ &= \frac{\delta(s+\eta)e^{st}}{s^2 e^{\delta(s+\eta)}e^{-\delta(s+\eta)}} \\ &= \frac{\delta(s+\eta)e^{st}}{s^2}.\end{aligned} \tag{3.23}$$

The above result hold since $\mathrm{E}_1(s) = \frac{e^{-s}}{s}\left(1 + \mathrm{O}(\frac{1}{s})\right)$ as $|s| \to \infty$ (See pages 2-4 in [1]). Let $s + \eta = \hat{R}e^{i\theta}$. Therefore as $|s| \to \infty$,

$$|\mathrm{H}(s)| \leq \frac{\delta \hat{R} e^{(\hat{R}\cos\theta - \eta)t}}{|\hat{R}e^{i\theta} - \eta|^2}. \tag{3.24}$$

Notice that, when s lies on arcs $A1B1$ and $B2A2$, $|\mathrm{H}(s)| \leq \frac{\delta \hat{R} e^{at}}{|\hat{R}-\eta|^2}$ as $\hat{R} \to \infty$. By using the fact that the arc lengths of $A1B1$ and $B2A2$ are approaching to a as $\hat{R} \to \infty$ together with above estimate, we can argue that $\mathbf{J}_{A1B1}$ and $\mathbf{J}_{B2A2}$ tend to zero as $\hat{R} \to \infty$.

Now, let $s + \eta = \hat{R}e^{i\theta}$ on arcs $B1C1$ and $C2B2$. From inequality (3.23)

$$\begin{aligned}|\mathbf{J}_{B1C1}| &\leq \frac{\delta \hat{R} e^{-\eta t}}{(\hat{R}-\eta)^2} \int_\beta^\pi e^{t\hat{R}\cos\theta}\mathrm{d}\theta \\ &\leq \frac{\delta \hat{R} e^{-\eta t}}{(\hat{R}-\eta)^2} \left[e^{t\hat{R}\cos\beta}(\frac{\pi}{2} - \beta) + \int_0^{\frac{\pi}{2}} e^{-t\hat{R}\sin\theta}\mathrm{d}\theta \right] \\ &\leq \frac{\delta \hat{R} e^{-\eta t}}{(\hat{R}-\eta)^2} \left[e^{\eta t}(\frac{\pi}{2} - \beta) + \frac{\pi}{2\hat{R}} \right],\end{aligned} \tag{3.25}$$

where $\cos\beta = \frac{\eta}{\hat{R}}$. The last inequality holds due to Jordan's inequality. From the estimate (3.25), we can see that $\mathbf{J}_{B1C1}$ approaches to zero as $\hat{R} \to \infty$. In similar manner, we can show that, $\mathbf{J}_{C2B2}$ approaches to zero as $\hat{R} \to \infty$.

And also, one can derive $Res(\mathrm{H}(s), 0)$ as follows,

$$\begin{aligned}Res(\mathrm{H}(s), 0) &= \frac{\mathrm{d}}{\mathrm{d}s}\left[s^2\mathrm{H}(s)\right]_{s=0} \\ &= e^{-\delta\eta}\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{e^{(t-\delta)s}}{\mathrm{E}_1(\delta(s+\eta))}\right]_{s=0} \\ &= e^{-\delta\eta}\left[\frac{e^{(t-\delta)s}(t-\delta)\mathrm{E}_1(\delta(s+\eta)) - e^{(t-\delta)s}\frac{\mathrm{d}\mathrm{E}_1(\delta(s+\eta))}{\mathrm{d}s}}{\mathrm{E}_1^2(\delta(s+\eta))}\right]_{s=0} \\ &= e^{-\delta\eta}\left[e^{(t-\delta)s}\frac{(t-\delta)\mathrm{E}_1(\delta(s+\eta)) + \delta\mathrm{E}_0(\delta(s+\eta))}{\mathrm{E}_1^2(\delta(s+\eta))}\right]_{s=0} \\ &= e^{-\delta\eta}\frac{(t-\delta)\mathrm{E}_1(\delta\eta) + \delta\mathrm{E}_0(\delta\eta)}{\mathrm{E}_1^2(\delta\eta)}.\end{aligned} \tag{3.26}$$

Fourth equality of the above result holds due to $\frac{d\mathrm{E}_1(x)}{dx} = -\mathrm{E}_0(x)$. Finally, by combining above results, we can get a expression for $\mathbf{P}_{ND}(t)$ as follows.

$$\begin{aligned}\mathbf{P}_{ND}(t) &= \exp\left[-\mathrm{K}(t)\right] \\ &= \exp\left[-2\pi\epsilon(1-e^{-\rho\pi a^2})\left[e^{-\delta\eta}\frac{(t-\delta)\mathrm{E}_1(\delta\eta)+\delta\mathrm{E}_0(\delta\eta)}{\mathrm{E}_1^2(\delta\eta)} - \frac{e^{(\delta-t)\eta}}{\eta}\right]\right]. \end{aligned} \tag{3.27}$$

For very large time t, one easily argue that $\mathbf{P}_{ND}(t)$ reduce to following form

$$\mathbf{P}_{ND}(t) = \exp\left[-\mathrm{K}(t)\right] = \exp\left[\frac{2\pi\epsilon(1-e^{-\rho\pi a^2})}{\ln(\eta\delta)+\gamma}\right]t. \tag{3.28}$$

Here $\mathrm{E}_1(\delta\eta) \approx -\ln(\eta\delta) - \gamma$ since $\eta\delta << 1$ and

$$\lim_{x\to 0}\mathrm{E}_1(x) = -\gamma - \ln(x),$$

where γ is the Euler-Mascheroni Constant. Therefore we have conclude the main result. Now let us briefly discuss about the asymptotic behavior of the non-detection probability $\mathbf{P}_{ND}(t)$ (3.28) by varying the speed $\mathbf{v}$ of the moving searcher. We infer that when the searcher moves very slowly comparing to the targets(i.e. $\eta = \frac{\mathbf{v}^2}{2\epsilon}$ is a very small value which is closer to zero but not equal to zero), the decay rate of the non-detection probability $\mathbf{P}_{ND}(t)$ (3.28) is very small. On the other hand, it is evident that when searcher travels very fast comparing to motion of the targets(while preserving $\delta < \frac{1}{\eta}$), the non-detection probability $\mathbf{P}_{ND}(t)$ (3.28) decays very fast.

Acknowledgment: Authors would like to thank Prof. V. S. Mandrekar, Prof. J. Eagle, Prof. A. Washburn, Prof. P. Jacobs, Prof. D. Gaver, Prof. J. Royset and Prof. P. Sundar for stimulating discussions regarding on this project. And also authors would like to thank the anonymous referee for many valuable comments.

References

1. Bleistein, N. and Handelsman, R. A.: *Asymptotic Expansions of Integrals*, Dover Publications, New York, 1986.
2. Bray, A. J., Majumdar, S. N., and Blythe, R. A.: Formal Solution of a Class of Reaction-Diffusion Model: Reduction to a Single Particle Problem, *The American Physical Society, Physical Review E* **67**(2003), 060102(R).
3. Chandrasekhar, S.: *Radiative Transfer*, Dover Publications, New York, 1960.
4. Churchill, R. V.: *Operational Mathematics*, McGraw-Hill Book Company, 1958.
5. Eagle, J. N.: Estimating the Probability of a Diffusing Target Encountering a Stationary Sensor, *Naval Research Logistics***34**(1987), 43–51.
6. Geiger, J.: Applied Stochastic Processes, *in preprint*.
7. Kingman, J. F. C.: *Poisson Processes*, Clarendon Press, Oxford, 1993.
8. Koopman, B. O.: *Search and Screening*, Military Operations Research Society, Alexandria, Virginia, 1980.
9. Majumdar, S. N. and Bray, A. J.: Survival Probability of a Ballistic Tracer Particle in the Presence of Diffusing Traps, *The American Physical Society, Physical Review E*, **68**(2003), 045101(R).
10. Mangel, M.: Optimal Search for and Mining of Under Water Mineral Resources, *Siam J. Appl. Math.*, **43**,1(1981), 99–106.

11. Mangel, M.: Search for a Randomly Moving Object, *Siam J. Appl. Math.*, **40**,2(1981), 327–338.
12. Mangel, M.: Search Theory: A Differential Equations Approach, *Lecture Notes in Pure and Appl.Math.*(1988), 55–103.
13. Pegoraro, V. and Slusallek, P.: On the Evaluation of the Complex-Valued Exponential Integral, *Journal of Graphics Tools* **15**,3(2011), 183–198.
14. Stone, L. D.: *Theory of Optimal Search*, Academic Press, New York, 1975.
15. Van Den Berg, J., Meester, R. and White, D. G.: Dynamic Boolean Models, *Stochastic Processes and their Applications*, *69*(1997), 247–257.
16. Wagner, D.: *Naval Operations Analysis*, U. S. Naval Institute, Annapolis, Maryland, 1968.
17. Washburn, A. R.: *Search and Detection*, Military Applications Section, Operations Research Society of America, 1981.

Pani W. Fernando: Center for Decision, Risk, Controls and Signals Intelligence (DRCSI), Naval Postgraduate School, Monterey, CA 93943, USA
E-mail address: panif71@gmail.com

Sivaguru S. Sritharan: Center for Decision, Risk, Controls and Signals Intelligence (DRCSI), Naval Postgraduate School, Monterey, CA 93943, USA
E-mail address: sssritha@nps.edu