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STABILITY FOR SOME LINEAR STOCHASTIC FRACTIONAL SYSTEMS

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Abstract. We obtain a closed expression for the solution of a linear Volterra integral equation with an additive Hölder continuous noise, which is a fractional Young integral, and with a function as initial condition. This solution is given in terms of the Mittag-Leffler function. Then we study the stability of the solution via the fractional calculus. As an application we analyze the stability in the mean of some stochastic fractional integral equations with a functional of the fractional Brownian motion as an additive noise.

1. Introduction

There is a wide range of applications of fractional calculus to different areas of human knowledge, as in engineering, physics, chemistry, etc. (see, for instance, [10], [26] and [6]). Among these we can mention applications on viscoelasticity [3], analysis of electrode processes [11] and Lorenz systems [8]. Therefore, currently fractional systems (i.e., equations involving fractional derivatives and/or integrals) have been studied by several authors, among them we can mention Hilfer [10], Kilbas et al. [13], Miller and Ross [20], Podlubny [26], Samko et al. [30], etc. In particular, the stability of fractional systems with constant initial condition has been considered.

In the deterministic case, the stability of fractional linear systems has been analyzed by Matignon [19] and the stability of non-linear fractional systems has been studied by several authors via Lyapunov method (see for example Li et al. [15] and references therein). In particular, non-linear fractional systems with a function as initial condition using also the Lyapunov technique have been considered in the Ph.D.Thesis of Martínez-Martínez [18]. Moreover, Junsheng et al. [12] give the form of solution for a linear fractional equation with a constant initial condition in terms of the Mittag-Leffler function by means of the Adomian decomposition method. We observe that Lemma 3.1 below is an extension of the results established in [12].

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The stability of stochastic systems driven by a Brownian motion has been examined too. For instance, Applebay and Freeman [2] have gotten the equivalence between almost sure exponential convergence and the p-th mean exponential convergence to zero for integrodifferential equations with an Itô integral noise. To do so, the solution is given in terms of the principal matrix or fundamental solution of the system. This method is similar to Adomian one. The Lyapunov function techniques have been used to deal with stability in probability for Itô-Volterra integral equation (see Li et al. [16]), with some stochastic type stability criteria for stochastic integrodifferential equations with infinite delay (see Zhang and Li [35]) and with conditional stability of Skorohod Volterra type equations with anticipative kernel (see Zhang and Zhang [36]). The mean square stability for Volterra-Itô equations with a function as initial condition has been established by Bao [4] by means of Gronwall lemma.

On the other hand, the fractional Brownian motion (fBm) $B = \{B_t\}_{t\geq 0}$ is a Gaussian process with stationary increments and is the only one which is self-similar (with index $H \in (0,1)$). Its covariance function is

$$\mathbf{E}(B_t B_s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \quad t, s \geq 0.$$

Since B is not a semimartingale for $H \neq 1/2$, we are not able to apply the classical Itô calculus. Thus, we need another approach to deal with it, such as Young integration (see Nualart and Răşcanu [25], and León and Tindel [14]), or Malliavin calculus (see Nualart [23]). The fractional Brownian motion has many applications due to its properties; for instance it has long range memory for $H > 1/2$ and intermitency for $H < 1/2$ (see Nualart [23]), B has Hölder continuous paths for any exponent less than H because of Kolmogorov continuity theorem (see Decreusefond and Üstünel [7]). Hence, the analysis of stochastic differential equations driven by fBm is considered by several authors these days for different interpretations of stochastic integrals (see, e.g. Lyons [17], Quer-Sardanyons and Tindel [27], León and Tindel [14], Nualart [23], and Nualart and Răşcanu [25]). In this paper we use the Young and Skorohod integrals (see Young [32] and Nualart [24]).

Zeng et al. [34] have stated the stability in probability and moment exponential stability for stochastic differential equation driven by fractional Brownian motion with parameter $H > 1/2$ utilizing the Lyapunov function techniques. Also, Nguyen [22] has established the exponential stability for linear stochastic differential equations with time-varying delays and with an additive noise of the form $\int_0^\cdot \sigma(s) dW_s^H$. Here $H > 1/2$,

$$W_t^H = \int_0^t (t-s)^{H-1/2} dW(s),$$

W is a Brownian motion and σ is a deterministic function such that

$$\int_0^\infty \sigma^2(s) e^{2\lambda s} ds < \infty,$$

for some $\lambda > 0$. Towards this end, the author uses that the solution can be represented in terms of the fundamental solution.

In this paper we apply the Adomian decomposition method to obtain a closed expression for the solution, also in terms of the Mittag-Leffler functions, of fractional systems with additive noise that could be a functional of fractional Brownian motion and with a function as the initial condition. Then the stability of these systems (that may be random) is analyzed applying the techniques of stochastic integration based on Young and Skorohod integrals. Namely, we study the stability of the solution of equations of the form

$$X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} AX(s)ds + Z_t, \quad t \geq 0. \tag{1.1}$$

Here $\beta \in (0,1)$, $A < 0$, ξ is the initial condition and Z is the Young integral

$$Z_t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d\theta_s,$$

where $\alpha \in (1,2)$ and θ is a Hölder continuous process that may represent the paths of a functional of fBm (even ξ may be random), as it is done in Theorem 4.3 below.

Unlike some papers, where the initial condition is a constant, we think that is important to have a process as the initial condition because of the memory of the system. Consequently, Z in (1.1) can be considered as a correction of the unknowledge of the initial condition that we could have.

The paper is organized as follows. The framework needed to prove our results is introduced in Section 2. In particular, we establish some properties of the Young integral that we need in the remaining of this paper. In Section 3 we obtain the form of the solution for the underlying equation and study its stability when the noise is null. Finally, in Section 4 we also consider the stability of stochastic fractional-order systems with a process as an initial condition and with a random noise that could be a fractional integral in the Young sense with respect to fractional Brownian motion.

2. Preliminaries

In this section we introduce the framework that we use to prove our results. Although some results are well-known, we give them here for the convenience of the reader.

2.1. The Mittag-Leffler function. An important tool of fractional calculus is the Mittag-Leffler function. The reader is refered to Podlubny [26] in order to see a more detailed exposition about this function. Here we state the properties that we use through this paper. This function is defined as

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ka+b)}, \quad a, b > 0, \text{ and } z \in \mathbb{R}.$$

A function f with domain $(0,\infty)$ is said to be completely monotonic if it possesses derivatives $f^{(n)}$ for all $n \in \mathbb{N} \cup \{0\}$, and if

$$(-1)^n f^{(n)}(t) \geq 0$$

for all $t > 0$ (see [21] and [31]). A useful property of the Mittag-Leffler function is that, for $z \geq 0$ and $a, b > 0$, $E_{a,b}(-z)$ is completely monotonic if and only if

$a \in (0,1]$ and $b \geq a$ (see Schneider [31]). In Podlubny [26] (Theorem 1.6), it is proved that if $a < 2$ there is a positive constant C such that

$$E_{a,b}(z) \leq \frac{C}{1+|z|}, \quad z \leq 0. \tag{2.1}$$

We will also use the next fact of differentiability for this function (see (1.83) in [26]):

$$\frac{d}{dt}\left(t^{b-1}E_{a,b}(\lambda t^a)\right) = t^{b-2}E_{a,b-1}(\lambda t^a), \quad a,b>0, \text{ and } \lambda \in \mathbb{R}. \tag{2.2}$$

2.2. Algebraic Young integration. We need to introduce some definitions on an algebraic integration scheme, specifically, the Young integration. Here we do it briefly, but if the reader needs a more detailed exposition on this integral, we refer to Young [32] and Gubinelli [9]. In the following, we use the approach used in Gubinelli [9] for the Young integral.

For an arbitrary real number $T > 0$ and $k = 1, 2, 3$, we denote by $C_k([0,t], \mathbb{R})$ the set of continuous functions $g : [0,T]^k \to \mathbb{R}$ such that $g_{t_1 \cdots t_k} = 0$ if $t_i = t_{i+1}$ for some $i \leq k-1$. Sometimes we denote $C_k([0,T],\mathbb{R})$ by $C_k(\mathbb{R})$ to simplify the notation.

An important tool for Young integration is the following operator: For $g \in C_1(\mathbb{R})$ and $h \in C_2(\mathbb{R})$, let $\delta : C_k(\mathbb{R}) \to C_{k+1}(\mathbb{R})$, $k = 1, 2$, be the operator defined by

$$(\delta g)_{st} = g_t - g_s, \quad \text{and} \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}, \quad \text{for each } s,u,t \in [0,T].$$

We need to introduce the norms for the functions in the spaces $C_1(\mathbb{R})$ and $C_2(\mathbb{R})$. For $0 \leq a_1 < a_2 \leq T$ and $f \in C_2([a_1,a_2];\mathbb{R})$, let

$$||f||_{\mu,[a_1,a_2]} := \sup_{r,t \in [a_1,a_2]} \frac{|f_{rt}|}{|t-r|^{\mu}}$$

and

$$C_2^{\mu}([a_1,a_2];\mathbb{R}) = \{f \in C_2([a_1,a_2];\mathbb{R}) : ||f||_{\mu,[a_1,a_2]} < \infty\}.$$

Note that an analogous definition can be extended for C_1^{μ}, thus, this space consists in the one-parameter Hölder continuous functions g with finite norm $||g||_{\mu} = ||\delta g||_{\mu}$. That is, $g \in C_1^{\mu}([a_1,a_2])$ if and only if g is a μ-Hölder continuous function on $[a_1,a_2]$. Analogously, for a function $h \in C_3([a_1,a_2];\mathbb{R})$ we have

$$||h||_{\gamma,\rho,[a_1,a_2]} = \sup_{s,u,t \in [a_1,a_2]} \frac{|h_{sut}|}{|u-s|^{\gamma}|t-u|^{\rho}}$$

and

$$||h||_{\mu,[a_1,a_2]} = \inf\left\{\sum_i ||h_i||_{\rho_i,\mu-\rho_i} : h = \sum_i h_i, 0 < \rho_i < \mu\right\},$$

where the infimum is taken over all the sequences $\{h_i \in C_3([a_1,a_2];\mathbb{R})\}$ such that $h = \sum_i h_i$ and for all the choices of the numbers $\rho_i \in (0,\mu)$. Thus, $|| \cdot ||_{\mu,[a_1,a_2]}$ is a norm in $C_3([a_1,a_2];\mathbb{R})$, and write

$$C_3^{\mu}([a_1,a_2];\mathbb{R}) := \{h \in C_3([a_1,a_2];\mathbb{R}) : ||h||_{\mu,[a_1,a_2]} < \infty\}.$$

With this definitions, we introduce the space

$$C_k^{1+}([a_1,a_2];\mathbb{R}) := \bigcup_{\mu>1} C_k^{\mu}([a_1,a_2];\mathbb{R}),$$

and we use the notation $\mathcal{Z}C_k^{1+}([a_1,a_2];\mathbb{R}) = C_k^{1+} \cap Ker\, \delta$, $k=2,3$.

The next result will be usefull because it provides several properties of δ and its inverse operator denoted by Λ (see [9]).

Proposition 2.1. *Let $0 \le a_1 < a_2 \le T$. Then we have:*

1. *There exists a unique linear map $\Lambda : \mathcal{Z}C_3^{1+}([a_1,a_2];\mathbb{R}) \to C_2^{1+}([a_1,a_2];\mathbb{R})$ such that $\delta\Lambda = Id_{\mathcal{Z}C_3^{1+}([a_1,a_2];\mathbb{R})}$. In addition, for any $\mu > 1$ and $h \in \mathcal{Z}C_3^{\mu}([a_1,a_2];\mathbb{R})$ we get*

$$||\Lambda h||_{\mu,[a_1,a_2]} \le \frac{1}{2^{\mu}-2}||h||_{\mu,[a_1,a_2]}.$$

2. *For $g \in C_2([a_1,a_2],\mathbb{R})$ such that $\delta g \in C_3^{1+}$, set $h = (Id - \Lambda\delta)g$. Then there is $f \in C_1(\mathbb{R})$ such that $h = \delta f$. Moreover*

$$h_{st} = (\delta f)_{st} = \lim_{|\Pi_{st}|\to 0} \sum_{i=0}^{n-1} g_{t_i t_{i+1}},$$

where the limit is over any partition $\Pi_{st} = \{t_0 = s, \ldots, t_n = t\}$ of $[s,t]$, whose mesh tends to zero.

The interpretation for Statement 1 is that, for any $h \in C_3^{1+}([a_1,a_2];\mathbb{R})$ such that $\delta h = 0$ there exists a unique $\psi = \Lambda(h) \in C_2^{1+}([a_1,a_2];\mathbb{R})$ such that $\delta\psi = h$.

In view of Proposition 2.1, we note that Λ can be related to the limit of some Riemann sums, which gives a link between an algebraic developments and some kind of generalized integration. Also we will use this result in order to give a Fubini theorem for a class of integrals with Hölder continuous integrators.

The following theorem will determine our notion of integral, whose proof can be found in [9] (see also [14]). Sometimes we write $\mathcal{J}_{st}(fdg)$ instead of $\int_s^t f_u dg_u$.

Theorem 2.2. *Let $f \in C_1^{\kappa}([0,T];\mathbb{R})$ and $g \in C_1^{\gamma}([0,T];\mathbb{R})$, with $\kappa + \gamma > 1$. Set*

$$\mathcal{J}_{st}(fdg) := f_s(\delta g)_{st} + \Lambda_{st}(\delta f \delta g), \quad 0 \le s \le t \le T,$$

with $(\delta f \delta g)_{sut} = (\delta f)_{su}(\delta g)_{ut}$, $u \in [0,T]$. Then

1. *Whenever f and g are smooth functions, $\mathcal{J}_{st}(fdg)$ coincides with the usual Riemann integral.*
2. *The generalized integral $\mathcal{J}(fdg)$ satisfies:*

$$|\mathcal{J}_{st}(fdg)| \le ||f||_{\infty}||g||_{\gamma}|t-s|^{\gamma} + c_{\gamma,\kappa}||f||_{k}||g||_{\gamma}|t-s|^{\gamma+\kappa},$$

where $c_{\gamma,k} = (2^{\gamma+\kappa}-2)^{-1}$.

3. *We have $\mathcal{J}(fdg) = [Id - \Lambda\delta](f\delta g)$ and*

$$\mathcal{J}_{st}(fdg) = \lim_{|\Pi_{st}|\to 0} \sum_{i=0}^{n-1} f_{t_i}\delta g_{t_i t_{i+1}},$$

where the limit is over any partition $\Pi_{st} = \{t_0 = s, \dots, t_n = t\}$ *of* $[s,t]$, *whose mesh tends to zero. In particular,* $\mathcal{J}_{st}(fdg)$ *coincides with the Young integral as defined in* [32].

Note that Statement 3 is a direct consequence of Proposition 2.1. This theorem is fundamental in our analysis because we have an estimation for the Young integral, and thus we are able to control its convergence, as we do it in some proofs below. It is important to point out that this integral has been extended by Zähle [33], Gubinelli [9], Lyons [17], etc.

The following result is a version of Fubini theorem for the Young integral.

Lemma 2.3. *(Fubini type theorem) Let* $\{\theta_s, s \le t\}$ *be a* γ*-Hölder continuous function and* $\beta > 0$. *Then for* $\varrho > 1$ *such that* $\varrho + \gamma - 2 > 0$, *we obtain*

$$\frac{1}{\Gamma(\varrho+\beta)}\int_0^t (t-s)^{\varrho+\beta-1} d\theta_s = \frac{1}{\Gamma(\varrho)\Gamma(\beta)}\int_0^t (t-r)^{\beta-1}\int_0^r (r-s)^{\varrho-1} d\theta_s dr. \quad (2.3)$$

Remark. Note that the integrals in (2.3) are well-defined because $\varrho - 1 + \gamma > 1$.

Proof. Let $\Pi_{0t} = \{s_0 = 0, s_1, \dots, s_n = t\}$ be a finite partition of the interval $[0,t]$ such that $|\Pi_{0t}| \to 0$, then Theorem 2.2 yields

$$\begin{aligned}
\frac{1}{\Gamma(\varrho+\beta)}\int_0^t (t-s)^{\varrho+\beta-1} d\theta_s &= \lim_{|\Pi_{0t}|\to 0}\left(\frac{1}{\Gamma(\varrho+\beta)}\sum_{i=0}^{n-1}(t-s_i)^{\varrho+\beta-1}(\theta_{s_{i+1}}-\theta_{s_i})\right)\\
&= \lim_{|\Pi_{0t}|\to 0}\left(\frac{1}{\Gamma(\varrho)\Gamma(\beta)}\sum_{i=0}^{n-1}\int_{s_i}^t (t-r)^{\beta-1}(r-s_i)^{\varrho-1}dr(\theta_{s_{i+1}}-\theta_{s_i})\right)\\
&= \lim_{|\Pi_{0t}|\to 0}\left(\frac{1}{\Gamma(\varrho)\Gamma(\beta)}\int_0^t (t-r)^{\beta-1}\right.\\
&\qquad\left.\times\left(\sum_{i=0}^{n-1} 1_{[s_i,t]}(r)(r-s_i)^{\varrho-1}(\theta_{s_{i+1}}-\theta_{s_i})\right)dr\right)\\
&= \lim_{|\Pi_{0t}|\to 0}\left(\frac{1}{\Gamma(\varrho)\Gamma(\beta)}\int_0^t (t-r)^{\beta-1}\sum_{s_i\le r}(r-s_i)^{\varrho-1}(\theta_{s_{i+1}}-\theta_{s_i})dr\right).
\end{aligned}$$

Using Proposition 2.1 and Theorem 2.2 we have that, for some $i_0 \in \{0, \dots, n-1\}$

$$\begin{aligned}
\sum_{s_i\le r}(r-s_i)^{\varrho-1}(\theta_{s_{i+1}}-\theta_{s_i}) = \mathcal{J}_{0r}((r-\cdot)^{\varrho-1}d\theta)&\\
+\sum_{i=0}^{n-1}\left(\Lambda\delta((r-\cdot)^{\varrho-1}\delta\theta)_{s_i\wedge r, s_{i+1}\wedge r}\right)&\\
+(r-s_{i_0})^{\varrho-1}(\theta_{s_{i_0+1}}-\theta_r)1_{(s_{i_0},s_{i_0+1}]}(r).&
\end{aligned}$$

Let us analyze the last two summands in the previous equality. Since $\varrho - 1 > 0$ and the conditions over θ we have

$$|(r-s_{i_0})^{\varrho-1}(\theta_{s_{i_0+1}}-\theta_r)1_{(s_{i_0},s_{i_0+1}]}(r)| \longrightarrow 0$$

as $|\Pi_{0t}| \to 0$. Note that the properties of the operators Λ and δ (see Proposition 2.1) allow us to write

$$\left|\sum_{i=0}^{n-1}\Big(\Lambda\delta((r-\cdot)^{\varrho-1}\delta\theta)_{s_i\wedge r, s_{i+1}\wedge r}\Big)\right| \leq C||\theta||_{\gamma,[0,t]}\sum_{k=0}^{n}|s_{i+1}-s_i|^{\varrho-1+\gamma},$$

for some $C>0$, since $\varrho-1+\gamma>1$ this quantity is bounded and converges to zero as $|\Pi_{0t}|\to 0$. Thus, the dominated convergence theorem implies that (2.3) holds. □

We will also need the following lemmas.

Lemma 2.4. *Let $f \in C_1^{\tau}([0,t])$, $g \in C_1^{\lambda}([0,t])$ and $\theta \in C_1^{\gamma}([0,t])$, where $\tau, \lambda, \gamma \in (0,1)$ and $\tau+\gamma, \lambda+\gamma \in (1,2)$. Also let $\tilde{\theta} = \int_0^{\cdot} g(s)d\theta_s$ on $[0,t]$. Then*

$$\int_0^t f(s)d\tilde{\theta}_s = \int_0^t f(s)g(s)d\theta_s.$$

Proof. Let $\Pi_{0t} = \{s_0 = 0, s_1, \ldots, s_n = t\}$ be a finite partition of the interval $[0,t]$. Then by Theorem 2.2 (Statement 2) there is a constant $C>0$ such that

$$\begin{aligned}&\left|\sum_{k=0}^{n-1} f(s_k)(\tilde{\theta}_{s_{k+1}}-\tilde{\theta}_{s_k}) - \sum_{k=0}^{n-1} f(s_k)g(s_k)(\theta_{s_{k+1}}-\theta_{s_k})\right|\\ &\qquad = \left|\sum_{k=0}^{n-1} f(s_k)\int_{s_k}^{s_{k+1}}(g(s)-g(s_k))d\theta_s\right| \leq C||f||_{\infty}\sum_{k=0}^{n-1}|s_{k+1}-s_k|^{\lambda+\gamma} \longrightarrow 0\end{aligned}$$

as $|\Pi_{0t}|\to 0$. Therefore this is an immediate consequence of Theorem 2.2 (Statement 3). □

Lemma 2.5. *Let θ be a γ-Hölder continuous function on $[0,t]$, $A\in\mathbb{R}$, $\varrho\in(1,2)$ and $\beta>0$ such that $\varrho+\gamma-2>0$. Then we have*

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{A^k}{\Gamma(k\beta+\varrho)}\int_0^t (t-s)^{k\beta+\varrho-1}d\theta_s = \int_0^t (t-s)^{\varrho-1}E_{\beta,\varrho}(A(t-s)^{\beta})d\theta_s. \quad (2.4)$$

Remark 2.6. Note that the last integral in the right-hand side of (2.4) is well-defined. Indeed, let $k_0\in\mathbb{N}\cup\{0\}$ be such that $k\beta+\varrho-1\leq 1$ for $k\leq k_0$, and $k\beta+\varrho-1>1$ for $k>k_0$. Then for $s,\tilde{s}\in[0,t]$, we have

$$\begin{aligned}&|(t-s)^{\varrho-1}E_{\beta,\varrho}(A(t-s)^{\beta}) - (t-\tilde{s})^{\varrho-1}E_{\beta,\varrho}(A(t-\tilde{s})^{\beta})|\\ &\leq \sum_{k=0}^{k_0}\frac{|A|^k|s-\tilde{s}|^{k\beta+\varrho-1}}{\Gamma(k\beta+\varrho)} + \sum_{k=k_0+1}^{\infty}\frac{|A|^k|(t-s)^{k\beta+\varrho-1}-(t-\tilde{s})^{k\beta+\varrho-1}|}{\Gamma(k\beta+\varrho)}\\ &= I_1+I_2. \qquad (2.5)\end{aligned}$$

Note

$$I_1 \leq |s-\tilde{s}|^{\varrho-1}\sum_{k=0}^{k_0}\frac{(|A|t^{\beta})^k}{\Gamma(k\beta+\varrho)} \leq |s-\tilde{s}|^{\varrho-1}E_{\beta,\varrho}(|A|t^{\beta}). \quad (2.6)$$

Now, the mean value theorem gives

$$I_2 \leq |s-\tilde{s}| \sum_{k=k_0+1}^{\infty} \frac{|A|^k (k\beta+\varrho-1) t^{k\beta+\varrho-2}}{\Gamma(k\beta+\varrho)} \leq |s-\tilde{s}| t^{\varrho-2} E_{\beta,\varrho-1}(|A|t^\beta).$$

Hence (2.5) and (2.6) implies our claim due to $\gamma+\varrho-1>1$.

Proof of Lemma 2.5. Write, for $m \in \mathbb{N}$, the function

$$f_m(s) = \sum_{k=m}^{\infty} \frac{A^k}{\Gamma(k\beta+\varrho)} (t-s)^{k\beta+\varrho-1}, \quad s \in [0,t].$$

In order to prove our result, we only need to observe that, for m large enough

$$\begin{aligned} \sup_{s\leq t} |f_m(s)| &= \sup_{s\leq t} \left| \sum_{k=m}^{\infty} \frac{A^k}{\Gamma(k\beta+\varrho)} (t-s)^{k\beta+\varrho-1} \right| \\ &\leq t^{\varrho-1} \sum_{k=m}^{\infty} \frac{|A|^k t^{k\beta}}{\Gamma(k\beta+\varrho)} \longrightarrow 0 \end{aligned}$$

as $m \to \infty$, and analogously

$$\begin{aligned} \sup_{s\leq t} |f'_m(s)| &= \sup_{s\leq t} \left| \sum_{k=m}^{\infty} \frac{A^k (k\beta+\varrho-1)}{\Gamma(k\beta+\varrho)} (t-s)^{k\beta+\varrho-2} \right| \\ &\leq t^{\varrho-2} \sum_{k=m}^{\infty} \frac{|A|^k t^{k\beta}}{\Gamma(k\beta+\varrho-1)} \longrightarrow 0 \end{aligned}$$

as $m \to \infty$. This procedure, the definition of the Mittag-Leffler function and Theorem 2.2 implies the result. □

We need the following lemma in order to prove stability for some equations driven by fractional Brownian motion (see Theorem 4.3 below).

Let I an interval. Set $C^1(I)$ the class of continuously differentiable functions on I.

Lemma 2.7. *Let $\tau, \gamma \in (0,1)$ be such that $\tau+\gamma>1$. Suppose that $\theta \in C_1^\gamma([0,t])$ and $h \in C^1((0,t)) \cap C_1^\tau([0,t])$ is such that $\dot{h} \in L^1([0,t])$ and $h(0)=0$. Then*

$$\int_0^t h(t-s) d\theta_s = \int_0^t \dot{h}(s)(\theta_{t-s}-\theta_0) ds.$$

Proof. Set $\Pi_{0t} = \{s_0=0,\ldots,s_n=t\}$ a finite partition of the interval $[0,t]$ such that its mesh tends to zero. Using the hypotheses on θ and h, and the fundamental theorem of calculus, we obtain

$$\begin{aligned}\int_0^t h(t-s)d\theta_s &= \lim_{|\Pi_{0t}|\to 0}\sum_{k=0}^{n-1} h(t-s_k)(\theta_{s_{k+1}}-\theta_{s_k})\\ &= \lim_{|\Pi_{0t}|\to 0}\sum_{k=0}^{n-1}\int_0^{t-s_k}\dot{h}(s)ds(\theta_{s_{k+1}}-\theta_{s_k})\\ &= \lim_{|\Pi_{0t}|\to 0}\int_0^t \dot{h}(s)\sum_{k=0}^{n-1}1_{[0,t-s]}(s_k)(\theta_{s_{k+1}}-\theta_{s_k})ds\\ &= \int_0^t \dot{h}(s)(\theta_{t-s}-\theta_0)ds.\end{aligned}$$

Here in the last equality we have used the dominated convergence theorem. □

3. Linear Volterra Integral Equation

In this section we consider the linear Volterra integral equation

$$X(t)=\xi_t+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}AX(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}d\theta_s, \quad t\geq 0. \tag{3.1}$$

Here the initial condition $\xi=\{\xi_t, t\geq 0\}$ is bounded on compact sets and measurable, $\beta\in(0,1)$, $A\in\mathbb{R}$ is an arbitrary constant, $\alpha\in(1,2)$ and $\theta=\{\theta_s, s\geq 0\}$ is a γ-Hölder continuous function with $\gamma\in(0,1)$. Since ξ is bounded and measurable on compact sets, the existence and uniqueness for the solution of this equation is an immediate consequence of the Picard iterations and the properties of the Mittag-Leffler function (see, for example, the book of Podlubny [26]).

Now we deduce a closed expression for the solution to (3.1).

Lemma 3.1. *Let $\beta\in(0,1)$, $A\in\mathbb{R}$, $\alpha\in(1,2)$, $\{\theta_s, s\geq 0\}$ a γ-Hölder continuous function with $\gamma\in(0,1)$ and $\gamma+\alpha-1>1$. Then equation (3.1) has the solution*

$$\begin{aligned}X(t)=\xi_t &+ A\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^\beta)\xi_s ds\\ &+\int_0^t (t-s)^{\alpha-1}E_{\beta,\alpha}(A(t-s)^\beta)d\theta_s, \quad t\geq 0.\end{aligned}$$

Proof. Recall that (see (1.1))

$$Z_t=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}d\theta_s.$$

Iterating equation (3.1), using Lemma 2.3 and Fubini's theorem we are able to write

$$\begin{aligned} X(t) &= \xi_t + \frac{A}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\xi_s ds + Z_t + \frac{A}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} Z_s ds \\ &\quad + \frac{A^2}{\Gamma(\beta)^2}\int_0^t\int_0^s (t-s)^{\beta-1}(s-s_1)^{\beta-1}X(s_1)ds_1 ds \\ &= \xi_t + \frac{A}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\xi_s ds + Z_t + \frac{A}{\Gamma(\beta+\alpha)}\int_0^t (t-s)^{\beta+\alpha-1} d\theta_s \\ &\quad + \frac{A^2}{\Gamma(2\beta)}\int_0^t (t-s)^{2\beta-1}X(s)ds. \end{aligned} \tag{3.2}$$

Now we use induction on n. So, we assume that

$$X(t) = \xi_t + \sum_{k=1}^{n}\frac{A^k}{\Gamma(k\beta)}\int_0^t (t-s)^{k\beta-1}\xi_s ds + \sum_{k=0}^{n}\frac{A^k}{\Gamma(k\beta+\alpha)}\int_0^t (t-s)^{k\beta+\alpha-1}d\theta_s + \frac{A^{n+1}}{\Gamma((n+1)\beta)}\int_0^t (t-s)^{(n+1)\beta-1}X(s)ds$$

holds. Therefore, one more iteration provides

$$\begin{aligned} X(t) &= \xi_t + \sum_{k=1}^{n}\frac{A^k}{\Gamma(k\beta)}\int_0^t (t-s)^{k\beta-1}\xi_s ds + \sum_{k=0}^{n}\frac{A^k}{\Gamma(k\beta+\alpha)}\int_0^t (t-s)^{k\beta+\alpha-1}d\theta_s \\ &\quad + \frac{A^{n+1}}{\Gamma((n+1)\beta)}\int_0^t (t-s)^{(n+1)\beta-1}\xi_s ds \\ &\quad + \frac{A^{n+1}}{\Gamma(\alpha)\Gamma((n+1)\beta)}\int_0^t\int_0^s (t-s)^{(n+1)\beta-1}(s-s_1)^{\alpha-1}d\theta_{s_1}ds \\ &\quad + \frac{A^{n+1}}{\Gamma((n+1)\beta)}\frac{A}{\Gamma(\beta)}\int_0^t\int_0^s (t-s)^{(n+1)\beta-1}(s-s_1)^{\beta-1}X(s_1)ds_1 ds. \end{aligned}$$

Proceeding as in (3.2) (i.e., applying Lemma 2.3 and Fubini's theorem) we get

$$\begin{aligned} X(t) &= \xi_t + \sum_{k=1}^{n+1}\frac{A^k}{\Gamma(k\beta)}\int_0^t (t-s)^{k\beta-1}\xi_s ds + \sum_{k=0}^{n+1}\frac{A^k}{\Gamma(k\beta+\alpha)}\int_0^t (t-s)^{k\beta+\alpha-1}d\theta_s \\ &\quad + \frac{A^{n+2}}{\Gamma((n+2)\beta)}\int_0^t (t-s)^{(n+2)\beta-1}X(s)ds, \quad t \geq 0. \end{aligned}$$

Finally, the fact that X is a bounded function on $[0,t]$ gives that the last summand converges to zero as n goes to infinity. Thus, the result is an immediate consequence of Lemma 2.5. □

Remark. Note that in view of the iterative procedure in the proof of this lemma we have a slight generalization of the Gronwall inequality in this context. For example, suppose additionally that $A > 0$, and that

$$X(t) \leq \xi_t + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}AX(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}d\theta_s.$$

Consequently, in this case, we can also apply the scheme in the previous proof.

3.1. Stability of linear Volterra integral equations for several initial conditions. This part is dedicated to study equation (3.1) when $\theta \equiv 0$. Here we analyze the stability of such an equation considering various different kinds of initial conditions. This development is the basic tool to establish the stability of stochastic fractional integral equations of the form (3.1) (see Section 4 below).

In the remaining of this paper, we suppose that all processes are defined on a complete probability space $(\Omega, \mathcal{F}, P)$.

Definition 3.2. The solution X to equation (3.1) is said to be *asymptotically stable* if $X(t)$ goes to zero as $t \to \infty$.

Proposition 3.3. *Set*

$$X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} AX(s)ds, \quad t \geq 0. \tag{3.3}$$

Assume that the constant A is negative, $\beta \in (0,1)$ and the initial condition satisfies one of the following properties:

1. *It is continuous, and there is $\xi_\infty \in \mathbb{R}$ such that, given $\varepsilon > 0$, there exists $t_0 > 0$ such that $|\xi_s - \xi_\infty| \leq \varepsilon$, for any $s \geq t_0$.*
2. *We have that $\xi \in C^1(\mathbb{R}_+)$,*
$$\lim_{t \mapsto \infty} |\xi_t|/t^\beta = 0 \quad \textit{and} \quad |\xi'_t| \leq \frac{C}{t^{1-\upsilon}}, \quad \textit{for some } \upsilon \in (0,\beta). \tag{3.4}$$
3. *It is bounded on compact sets and can be written as*
$$\xi_t = \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} g(s)ds,$$
with $\eta \in (0, \beta+1)$, $g \in L^1([0,\infty)) \cap L^p([t_0,\infty))$ for some $p > \frac{1}{\eta} \vee 1$ and $t_0 > 0$.

Then the solution of (3.3) *is asymptotically stable.*

Remark. Let A and ξ be random. Note that if $A(\omega)$ and $\xi(\omega)$ satisfy the conditions of Proposition 3.3 pathwise, then the solution of (3.3) is asymptotically stable ω by ω (i.e., $X(\omega,\cdot)$ is asymptotically stable for any $\omega \in \Omega$).

Proof. We divide the proof into three parts.

Step 1. Here we assume that ξ is as in Statement 1. Lemma 3.1 leads to write

$$\begin{aligned} |X(t)| &= \left| \xi_t + A \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) \xi_s ds \right| \\ &= \left| \xi_t + A \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta)(\xi_s - \xi_t) ds \right. \\ &\quad \left. + A\xi_t \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) ds \right|. \end{aligned}$$

Now, using (4.8) in [12], we obtain

$$1 + A \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) ds = E_{\beta,1}(At^\beta), \tag{3.5}$$

thus

$$|X(t)| \leq \left|\xi_t E_{\beta,1}(At^\beta)\right| + \left|A\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^\beta)(\xi_s-\xi_t)ds\right|.$$

Let $\varepsilon > 0$ be arbitrary and fix $t_0 > 0$ such that $|\xi_s - \xi_t| < \varepsilon$, for $s, t > t_0$. Then for $t > t_0$,

$$|X(t)| \leq I_1(t) + I_2(t) + I_3(t),$$

with

$$I_1(t) = \left|\xi_t E_{\beta,1}(At^\beta)\right|, \quad I_2(t) = \left|A\int_0^{t_0} (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^\beta)(\xi_s-\xi_t)ds\right|$$

and

$$I_3(t) = \left|A\int_{t_0}^t (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^\beta)(\xi_s-\xi_t)ds\right|.$$

Using that ξ is bounded by hypothesis and (2.1), we have

$$I_1(t) = \left|\xi_t E_{\beta,1}(At^\beta)\right| \leq C\frac{|\xi_t|}{1+|A|t^\beta}, \text{ for some } C > 0, \tag{3.6}$$

and this quantity converges to zero as t tends to $+\infty$.

Let $M > 0$ be such that $|\xi_s| \leq M/2$, for any $s > 0$, and therefore

$$I_2(t) \leq M|A|\int_0^{t_0} (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^\beta)ds \leq CM\int_0^{t_0} \frac{(t-s)^{\beta-1}}{1+|A|(t-s)^\beta}ds,$$

thus I_2 converges to zero as t tends to $+\infty$, due to dominanted convergence theorem.

Finally, using the property of the limit of ξ, we obtain, by [26] (equality (1.99))

$$\begin{aligned} I_3(t) &\leq \varepsilon|A|\int_{t_0}^t (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^\beta)ds \\ &\leq \varepsilon|A|\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^\beta)ds \\ &= \varepsilon|A|t^\beta E_{\beta,\beta+1}(At^\beta) < \varepsilon C. \end{aligned}$$

This finishes the proof of Step 1.

Step 2. Now ξ is as in Statement 2. As in Step 1 we get

$$\begin{aligned} |X(t)| &\leq \left|\xi_t E_{\beta,1}(At^\beta)\right| + \left|A\int_0^t (t-s)^{\beta-1}E_{\beta,\beta}(A(t-s)^\beta)(\xi_s-\xi_t)ds\right| \\ &\leq \left|\xi_t E_{\beta,1}(At^\beta)\right| + C|A|\int_0^t (t-s)^{\beta}E_{\beta,\beta}(A(t-s)^\beta)s^{\upsilon-1}ds = I_1(t)+I_2(t). \end{aligned}$$

The first term goes to zero as $t \to \infty$ due to (3.6) and (3.4). For the second one we use the definition of Mittag-Leffler function:

$$\begin{aligned}
I_2(t) &= C|A| \int_0^t (t-s)^{\beta} \sum_{k=0}^{\infty} \frac{A^k (t-s)^{k\beta}}{\Gamma(k\beta+\beta)} \ s^{\upsilon-1} ds \\
&= C|A| \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\beta+\beta)} \int_0^t (t-s)^{(k+1)\beta} s^{\upsilon-1} ds \\
&= C|A| \sum_{k=0}^{\infty} \frac{A^k}{\Gamma((k+1)\beta)} \frac{\Gamma((k+1)\beta+1)\Gamma(\upsilon)}{\Gamma((k+1)\beta+1+\upsilon)} \ t^{(k+1)\beta+\upsilon} \\
&= C|A| \sum_{k=0}^{\infty} \frac{A^k (k+1)\beta\Gamma(\upsilon)}{\Gamma((k+1)\beta+1+\upsilon)} \ t^{(k+1)\beta+\upsilon} \\
&= C|A|\Gamma(\upsilon) \sum_{k=0}^{\infty} \frac{A^k [(k+1)\beta+\upsilon-\upsilon]}{((k+1)\beta+\upsilon)\Gamma((k+1)\beta+\upsilon)} \ t^{(k+1)\beta+\upsilon} \\
&= C|A|\Gamma(\upsilon) \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\beta+\upsilon}}{\Gamma((k+1)\beta+\upsilon)} - C|A|\upsilon\Gamma(\upsilon) \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\beta+\upsilon}}{\Gamma((k+1)\beta+\upsilon+1)} \\
&= -C\Gamma(\upsilon) \sum_{k=0}^{\infty} \frac{A^{k+1} t^{(k+1)\beta+\upsilon}}{\Gamma((k+1)\beta+\upsilon)} + C\Gamma(\upsilon+1) \sum_{k=0}^{\infty} \frac{A^{k+1} t^{(k+1)\beta+\upsilon}}{\Gamma((k+1)\beta+\upsilon+1)} \\
&= -C\Gamma(\upsilon) t^{\upsilon} E_{\beta,\upsilon}(At^{\beta}) + \frac{C\Gamma(\upsilon)t^{\upsilon}}{\Gamma(\upsilon)} + C\Gamma(\upsilon+1) t^{\upsilon} E_{\beta,\upsilon+1}(At^{\beta}) \\
&\quad - \frac{C\Gamma(\upsilon+1)t^{\upsilon}}{\Gamma(\upsilon+1)} \\
&= t^{\upsilon} C\Gamma(\upsilon) \left[\upsilon E_{\beta,\upsilon+1}(At^{\beta}) - E_{\beta,\upsilon}(At^{\beta}) \right], \quad t \geq 0,
\end{aligned}$$

and this quantity converges to zero if $\upsilon < \beta$.

Step 3. Finally we consider Statement 3. Proceeding as in Lemma 3.1, the solution is

$$\begin{aligned}
X(t) &= \int_0^t (t-s)^{\eta-1} E_{\beta,\eta}(A(t-s)^{\beta}) g(s) ds \\
&= \int_0^t s^{\eta-1} E_{\beta,\eta}(As^{\beta}) g(t-s) ds \\
&= I_1(t) + I_2(t), \quad t \geq 0,
\end{aligned}$$

where

$$I_1(t) = \int_0^{t_0} s^{\eta-1} E_{\beta,\eta}(As^{\beta}) g(t-s) ds \quad \text{and} \quad I_2(t) = \int_{t_0}^t s^{\eta-1} E_{\beta,\eta}(As^{\beta}) g(t-s) ds.$$

In the second case we obtain

$$I_2(t) = \int_0^{t-t_0} (t-s)^{\eta-1} E_{\beta,\eta}(A(t-s)^{\beta}) g(s) ds.$$

Since $(t-s) \in [t_0, t]$, given $\varepsilon > 0$ there exists t_0 such that

$$\left|(t-s)^{\eta-1} E_{\beta,\eta}(A(t-s)^{\beta})\right| \leq \varepsilon,$$

by (2.1). So,

$$|I_2(t)| = \left| \int_0^{t-t_0} (t-s)^{\eta-1} E_{\beta,\eta}(A(t-s)^{\beta}) g(s) ds \right| \leq \varepsilon \int_0^{\infty} |g(s)| ds,$$

and $g \in L^1([0,\infty))$.

Hölder's inequality implies that

$$|I_1(t)| \leq C \left[\int_0^{t_0} |g(t-s)|^p ds \right]^{\frac{1}{p}} \left[\int_0^{t_0} s^{q(\eta-1)} ds \right]^{\frac{1}{q}}.$$

If $\eta \geq 1$ the proof is trivial by the dominated convergence theorem. Therefore we assume that $\eta < 1$. If $p = \frac{1}{\eta} + \rho$ for some $\rho > 0$, then $q < \frac{1}{1-\eta}$, since $\frac{1}{p} + \frac{1}{q} = 1$. So, using the dominated convergence theorem again, since $g \in L^p([t_0, \infty[)$ we have

$$|I_1(t)| \leq C \left[\int_0^{t_0} |g(t-s)|^p ds \right]^{\frac{1}{p}} = C \left[\int_{t-t_0}^{t} |g(s)|^p ds \right]^{\frac{1}{p}} \longrightarrow 0,$$

as t tends to infinity. □

Remark. In Step 3, if g is continuous and $\lim_{t\mapsto\infty} g(t) = 0$, as g is bounded, by the dominated convergence theorem I_1 tends to zero. Thus, in this case, hypothesis $g \in L^p([t_0, \infty))$ can be omitted.

In the following of this section $\xi = \{\xi_t, t \geq 0\}$ is a measurable and bounded stochastic process.

Now we introduce the second concept of stability that we study.

Definition 3.4. The solution X to equation (3.1) is said to be *stable in the mean* if $\mathbf{E}|X(t)| \to 0$ as $t \to \infty$.

As an immediate consequence of Proposition 3.3 we have.

Corollary 3.5. *Assume that A is a negative constant and $\beta \in (0,1)$. The initial condition ξ is a stochastic process which is measurable and bounded over compact sets, and satisfies one of the following properties:*

1. *It is $L^1(\Omega)$-continuous, and given $\varepsilon > 0$, there exists $t_0 > 0$ and a random variable ξ_∞ such that $\mathbf{E}|\xi_s - \xi_\infty| \leq \varepsilon$, for any $s \geq t_0$.*
2. *We have that $\xi(\omega) \in C^1(\mathbb{R}_+)$ for all $\omega \in \Omega$, $\lim_{t\mapsto\infty} \mathbf{E}|\xi_t|/t^{\beta} = 0$, and*
$$|\xi_t'| \leq \frac{C}{t^{1-\upsilon}}, \quad \textit{for some } \upsilon \in (0,\beta) \textit{ and } C \in L^1(\Omega).$$
3. *It can be written as*
$$\xi_t = \frac{1}{\Gamma(\eta)} \int_0^t (t-s)^{\eta-1} g(s) ds,$$
with $\eta \in (0, \beta+1)$, $g \in L^1(\Omega \times [0,\infty)) \cap L^p(\Omega \times [t_0, \infty))$, $p > \frac{1}{\eta} \vee 1$ and $t_0 > 0$.

Then the solution of (3.3) is stable in the mean.

4. Stability of Random Linear Volterra Integral Equations

We consider the equation

$$X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} AX(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\ f(s)d\theta_s, \quad t \geq 0. \tag{4.1}$$

In the remaining of this section we suppose thst ξ is a measurable function, $\beta \in (0,1)$, A is a negative constant, $\alpha \in (1,2)$, and $\{\theta_s, s \geq 0\}$ is a γ-Hölder continuous function with $\gamma \in (0,1)$ and $\theta_0 = 0$. We also suppose that $\beta+1 > \alpha$ and $\alpha+\gamma > 2$.

We now analyze several stability criteria for the solution X of (4.1). Remember that these criteria are given in Definitions 3.2 and 3.4.

Proposition 4.1. *Let $f \in C^1(\mathbb{R}_+)$ be a Lipschitz function, $t_0 > 0$ and $p > \frac{1}{\alpha-1}$ such that $f \in L^1([0,\infty)) \cap L^p([t_0,\infty))$ and $\dot{f} \in L^1([0,\infty))$. Also assume that the hypotheses of Proposition 3.3 (resp. Corollary 3.5) hold and that θ is a bounded function (resp. bounded process in the mean). Then the solution of equation (4.1) is asymptotically stable (resp. stable in the mean).*

Remark. From Lemma 2.4 we are dealing with

$$X(t) = \xi_t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} AX(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} d\tilde{\theta}_s, \quad t \geq 0,$$

where $\tilde{\theta}_\cdot = \int_0^\cdot f(s)d\theta_s$.

Proof. First of all, by Lemmas 3.1 and 2.4, the solution of (4.1) is

$$\begin{aligned} X(t) &= \xi_t + A \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta)\xi_s ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\beta,\alpha}(A(t-s)^\beta) f(s) d\theta_s, \quad t \geq 0, \end{aligned}$$

due to f being Lipschitz. Now, from Proposition 3.3 (resp. Corollary 3.5), we only need to study the convergence of the term

$$I(t) = \int_0^t (t-s)^{\alpha-1} E_{\beta,\alpha}(A(t-s)^\beta) f(s) d\theta_s.$$

We observe that Proposition 2.8 in [14] is still true for this integral because of its proof. So, Theorem 2.2 (Statement 3) implies

$$I(t) = I_1(t) + I_2(t),$$

with

$$I_1(t) = \int_0^t \theta_s (t-s)^{\alpha-2} E_{\beta,\alpha-1}(A(t-s)^\beta) f(s) ds$$

and

$$I_2(t) = -\int_0^t \theta_s (t-s)^{\alpha-1} E_{\beta,\alpha}(A(t-s)^\beta) \dot{f}(s) ds,$$

where we have used that $\theta_0 = 0$.

To fix ideas we suppose that θ is bounded because the case $\sup_{t\geq 0}\mathbf{E}(|\theta_t|)<\infty$ is proven similarly. We first deal with I_1. Using that $\sup_{t\geq 0}|\theta_t|<\infty$, we have

$$\begin{aligned}
|I_1(t)| &\leq C\int_0^t (t-s)^{\alpha-2}\left|E_{\beta,\alpha-1}(A(t-s)^\beta)\right||f(s)|ds\\
&= C\int_0^{t_0} s^{\alpha-2}\left|E_{\beta,\alpha-1}(As^\beta)\right||f(t-s)|ds\\
&\quad +C\int_{t_0}^t s^{\alpha-2}\left|E_{\beta,\alpha-1}(As^\beta)\right||f(t-s)|ds\\
&= \tilde{I}_{1,1}(t)+\tilde{I}_{1,2}(t).
\end{aligned}$$

Given $\varepsilon>0$ we fix t_0 such that for any $s>t_0$

$$s^{\alpha-2}\left|E_{\beta,\alpha-1}(As^\beta)\right|\leq\varepsilon.$$

Hence we have that

$$\tilde{I}_{1,2}(t)\leq\varepsilon C\int_{t_0}^t|f(t-s)|ds\leq\varepsilon C\int_0^\infty|f(s)|ds.$$

Now, $q<\frac{1}{2-\alpha}$ because $p>\frac{1}{\alpha-1}$. So, using that $|E_{\beta,\alpha-1}(As^\beta)|\leq C$, we can write

$$\begin{aligned}
\tilde{I}_{1,1}(t) &= C\int_0^{t_0}s^{\alpha-2}|f(t-s)|ds\\
&\leq C\left(\int_0^{t_0}s^{q(\alpha-2)}ds\right)^{\frac{1}{q}}\left(\int_0^{t_0}|f(t-s)|^pds\right)^{\frac{1}{p}}\leq C\left(\int_{t-t_0}^t|f(s)|^pds\right)^{\frac{1}{p}},
\end{aligned}$$

which converges to zero as t tends to infinity because of our hypotheses on f.

Next we study I_2. For $t>t_0$,

$$|I_2(t)|\leq C\int_0^t(t-s)^{\alpha-1}E_{\beta,\alpha}(A(t-s)^\beta)|\dot{f}(s)|ds=C\left[\tilde{I}_{2,1}(t)+\tilde{I}_{2,2}(t)\right],$$

with

$$\tilde{I}_{2,1}(t)=\int_0^{t_0}s^{\alpha-1}E_{\beta,\alpha}(As^\beta)|\dot{f}(t-s)|ds$$

and

$$\tilde{I}_{2,2}(t)=\int_{t_0}^t s^{\alpha-1}E_{\beta,\alpha}(As^\beta)|\dot{f}(t-s)|ds.$$

Since $\alpha-1>0$,

$$\tilde{I}_{2,1}(t)\leq C\int_0^{t_0}|\dot{f}(t-s)|ds=C\int_{t-t_0}^t|\dot{f}(s)|ds\longrightarrow 0,$$

as t goes to infinity since $\dot{f}\in L^1([0,\infty))$.

Finally, the fact that $\beta>\alpha-1$, leads to choose $t_0>0$ such that

$$\tilde{I}_{2,2}(t)\leq\varepsilon C\int_{t_0}^t|\dot{f}(t-s)|ds\leq\varepsilon C\int_0^\infty|\dot{f}(s)|ds<\varepsilon C. \tag{4.2}$$

Thus, the proof is finished. □

Remark. Observe that, in view of the proof of last proposition, if the assumptions of Proposition 3.3 are true and θ is a bounded process with probability 1, then equation (4.1) is pathwise asymptotically stable. Also note that, in the case where $\beta \leq \alpha - 1$, we get

$$\frac{d}{ds}\left(s^{\alpha-1}E_{\beta,\alpha}(As^{\beta})\right) = s^{\alpha-2}E_{\beta,\alpha-1}(As^{\beta}) > 0, \quad s \geq 0,$$

thus (4.2) could be false, and then equation (4.1) could be unstable.

4.1. Stability of random integral equations driven by a fractional Brownian motion. Here we study stability for some stochastic integral equations driven by fractional Brownian motion.

Remember that the fractional Brownian motion B^H is a centered Gaussian process with covariance law

$$R_H(s,t) = \mathbf{E}(B_s^H B_t^H) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right), \quad H \in (0,1),$$

$s, t \geq 0$. In particular, B^H has γ-Hölder continuous paths for any exponent $\gamma < H$ on compact sets.

Let $C_{loc}^{\gamma'}(\mathbb{R}_+)$ be the set of locally γ'-Hölder continuous functions (i.e., the family of all functions that are γ'-Hölder continuous on any compact interval of $\mathbb{R}_+$). The following is a consequence of Proposition 4.1.

Corollary 4.2. *Suppose that hypotheses of Proposition 4.1 hold, with ξ as in Corollary 3.5. Let*

$$\theta_t = \int_0^t z(s)dB_s^{\gamma},$$

with $\gamma > \frac{1}{2}$, $z \in L^{1/\gamma}([0,\infty)) \cap C_{loc}^{\gamma'}([0,\infty))$, $\gamma' + \gamma > 1$. Then the solution X of equation (4.1) is stable in the mean.

Remark. Note that, for fixed $t > 0$, θ_t is well-defined. Indeed, we can choose $\tilde{\gamma} \in (1/2, \gamma)$ such that $\tilde{\gamma} + \gamma' > 1$, and remember that B^{γ} is $\tilde{\gamma}$-Hölder continuous on $[0,t]$. So, θ is defined pathwise.

Proof. We only need to note that Proposition 3 and Remark 1 in Alòs and Nualart [1], and Proposition 2 in Russo and Vallois [29] allow us to observe that the last Young integral is also a Skorohod one. As a consequence, [1] (inequality (11)) implies

$$\sup_{t\geq 0} \mathbf{E}|\theta_t| \leq C\left(\int_0^{\infty} |z(r)|^{\frac{1}{\gamma}} dr\right)^{\gamma}, \quad C > 0.$$

Thus, the result follows from Proposition 4.1. □

Consider now the equation

$$X(t) = \xi_t + \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}AX(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ f(s)dB_s^{\gamma}, \quad t \geq 0. \tag{4.3}$$

Here $\gamma \in (0,1)$.

Theorem 4.3. *For all $\omega \in \Omega$, let $f(\omega,\cdot) \in C^1([0,\infty))$ be such that*

$$\left(r \mapsto r^{\gamma}\left[\mathbf{E}|f(r)|^2\right]^{\frac{1}{2}}\right) \in L^1([0,\infty)) \cap L^p([t_0,\infty)) \text{ for some } p > \frac{1}{\alpha-1}$$

and $t_0 > 0$, and $\left(r \mapsto r^{\gamma}[\mathbf{E}|\dot{f}(r)|^2]^{\frac{1}{2}}\right) \in L^1([0,\infty))$. Then the hypotheses of Proposition 4.1 for the case that ξ is as in Corollary 3.5 imply that the solution of equation (4.3) is stable in the mean.

Proof. As in the previous proposition, we only need to analyze

$$I(t) = \int_0^t (t-s)^{\alpha-1} E_{\beta,\alpha}(A(t-s)^{\beta}) f(s) dB_s^{\gamma}.$$

Using the property of differentiability for the Mittag-Leffler function (2.2), Lemma 2.7 and Remark 2.6 we get

$$\begin{aligned} I(t) &= \int_0^t \partial_r \left[r^{\alpha-1} E_{\beta,\alpha}(Ar^{\beta}) f(t-r)\right] B_{t-r}^{\gamma} dr \\ &= \int_0^t r^{\alpha-2} E_{\beta,\alpha-1}(Ar^{\beta}) f(t-r) B_{t-r}^{\gamma} dr - \int_0^t r^{\alpha-1} E_{\beta,\alpha}(Ar^{\beta}) \dot{f}(t-r) B_{t-r}^{\gamma} dr \\ &:= I_1(t) + I_2(t). \end{aligned}$$

Given $\varepsilon > 0$, there exists $t_0 > 0$ such that $\left|t_0^{\alpha-2} E_{\beta,\alpha-1}(At_0^{\beta})\right| \le \varepsilon$. Take

$$I_1(t) = I_{1,1}(t) + I_{1,2}(t), \quad t \ge t_0,$$

with

$$I_{1,1}(t) = \int_{t_0}^t r^{\alpha-2} E_{\beta,\alpha-1}(Ar^{\beta}) f(t-r) B_{t-r}^{\gamma} dr$$

and

$$I_{1,2}(t) = \int_0^{t_0} r^{\alpha-2} E_{\beta,\alpha-1}(Ar^{\beta}) f(t-r) B_{t-r}^{\gamma} dr.$$

Then

$$\begin{aligned} \mathbf{E}\left|I_{1,1}(t)\right| &\le \varepsilon \int_{t_0}^t \mathbf{E}\left|f(t-r) B_{t-r}^{\gamma}\right| dr \le \varepsilon \int_0^{\infty} \mathbf{E}\left|f(r) B_r^{\gamma}\right| dr \\ &\le \varepsilon \int_0^{\infty} \left[\mathbf{E}\left|f(r)\right|^2\right]^{\frac{1}{2}} \left[\mathbf{E}\left|B_r^{\gamma}\right|^2\right]^{\frac{1}{2}} dr = \varepsilon \int_0^{\infty} r^{\gamma} \left[\mathbf{E}\left|f(r)\right|^2\right]^{\frac{1}{2}} dr. \end{aligned}$$

Using that $p > \frac{1}{\alpha-1}$ (which implies $q < \frac{1}{2-\alpha}$) and Hölder's inequality we have that

$$\begin{aligned} \mathbf{E}\left|I_{1,2}(t)\right| &\le C\left(\int_0^{t_0} r^{q(\alpha-2)} dr\right)^{\frac{1}{q}} \left(\int_0^{t_0} \left(\left[\mathbf{E}\left|f(t-r)\right|^2\right]^{\frac{1}{2}} \left[\mathbf{E}\left|B_{t-r}^{\gamma}\right|^2\right]^{\frac{1}{2}}\right)^p dr\right)^{\frac{1}{p}} \\ &\le C\left(\int_{t-t_0}^t \left[\mathbf{E}\left|f(r)\right|^2\right]^{\frac{p}{2}} r^{p\gamma} dr\right)^{\frac{1}{p}} \longrightarrow 0. \end{aligned}$$

On the other hand, we observe that

$$|I_2(t)| \le I_{2,1}(t) + I_{2,2}(t),$$

with $t > t_0$ and

$$I_{2,1}(t) = \int_{t_0}^{t} r^{\alpha-1} E_{\beta,\alpha}(Ar^{\beta}) \left| \dot{f}(t-r) B^{\gamma}_{t-r} \right| dr$$

and

$$I_{2,2}(t) = \int_{0}^{t_0} r^{\alpha-1} E_{\beta,\alpha}(Ar^{\beta}) \left| \dot{f}(t-r) B^{\gamma}_{t-r} \right| dr.$$

From the fact that $\alpha - 1 - \beta < 0$ and proceeding as before we obtain, for some $t_0 > 0$,

$$I_{2,1}(t) \leq \varepsilon \int_0^{\infty} \mathbf{E} \left| \dot{f}(r) B^{\gamma}_{r} \right| dr$$

and

$$I_{2,2}(t) \leq C \int_0^{t_0} \mathbf{E} \left| \dot{f}(t-r) B^{\gamma}_{t-r} \right| dr = C \int_{t-t_0}^{t} \mathbf{E} \left| \dot{f}(r) B^{\gamma}_{r} \right| dr.$$

In the first case the choice of ε is arbitrary, and in the other case the integral goes to zero. □

Example 4.4. A class of (deterministic) functions f that satisfies assumptions of Theorem 4.3 is the familly of all the *Schwartz functions*, since they have the property of decreasing rapidly to zero as t goes to ∞ (see Rudin [28] for a wide exposition).

Remark. We could deal with other type of stochastic integral to study the stability of (4.3) using different conditions over f in Theorem 4.3. For instance, let $f \in L^{1/\gamma}([0,\infty))$ be a Lipschitz function. Now, consider the Wiener integral (see [5] or [23])

$$I(t) = \int_0^t (t-s)^{\alpha-1} E_{\beta,\alpha}(A(t-s)^{\beta}) f(s) dB^{\gamma}_{s}, \quad t \geq 0.$$

Using [1] (inequality (11)) again, and proceeding analogously as in the results of this section

$$\begin{aligned}
\mathbf{E}|I(t)| &\leq C_{\gamma,1} \left[\int_0^t (t-s)^{\frac{\alpha-1}{\gamma}} \left(E_{\beta,\alpha}(A(t-s)^{\beta}) \right)^{\frac{1}{\gamma}} |f(s)|^{\frac{1}{\gamma}} ds \right]^{\gamma} \\
&\leq C_{\gamma,1} \left[\int_0^{t_0} s^{\frac{\alpha-1}{\gamma}} \left(E_{\beta,\alpha}(As^{\beta}) \right)^{\frac{1}{\gamma}} |f(t-s)|^{\frac{1}{\gamma}} ds \right]^{\gamma} \\
&\quad + C_{\gamma,1} \left[\int_{t_0}^{t} s^{\frac{\alpha-1}{\gamma}} \left(E_{\beta,\alpha}(As^{\beta}) \right)^{\frac{1}{\gamma}} |f(t-s)|^{\frac{1}{\gamma}} ds \right]^{\gamma} \\
&= C_{\gamma,1} \left[I_1^{\gamma}(t) + I_2^{\gamma}(t) \right],
\end{aligned}$$

where $C_{\gamma,1} > 0$ is a constant. Then we have

$$I_1(t) \leq C t_0^{\frac{\alpha-1}{\gamma}} \int_0^{t_0} |f(t-s)|^{\frac{1}{\gamma}} ds = C t_0^{\frac{\alpha-1}{\gamma}} \int_{t-t_0}^{t} |f(s)|^{\frac{1}{\gamma}} ds \longrightarrow 0, \quad \text{as } t \to \infty.$$

Since $\alpha - 1 - \beta < 0$, then given $\varepsilon > 0$ there is t_0 such that

$$I_2(t) \leq \varepsilon C \int_0^\infty |f(s)|^{\frac{1}{\gamma}} ds \leq \varepsilon C.$$

Thus, in this case, there is also stability in the mean.

Example 4.5. Proceeding as in the proof of Proposition 3.3 (Statement 1), we can see that the solution of (4.3) with

$$\xi_t = \begin{cases} -1 & \text{if} \quad t < 1, \\ 1 & \text{if} \quad t \geq 1, \end{cases}$$

is stable in the mean. Indeed, we only need to observe that, for $t > 1$ we get

$$\begin{aligned} &1 - A \int_0^1 (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) ds + A \int_1^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) ds \\ &= 1 - 2A \int_0^1 (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) ds + A \int_0^t (t-s)^{\beta-1} E_{\beta,\beta}(A(t-s)^\beta) ds. \end{aligned}$$

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