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# MOMENTS ANALYSIS OF A MARKOV-MODULATED RISK MODEL WITH STOCHASTIC INTEREST RATES

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Abstract. In this paper we determine explicitly the closed form of the moments of a Markov-Modulated Risk Model with Stochastic Interest Rate. The moments are derived by means of Laplace-Stieltjes transforms. Equations and formulas are conveniently represented by using the 2-dimensional matrix formalism. This paper substantially extends the results of Kim and Kim (Insurance: Mathematics and Economics 40, 485-497, 2007) by allowing the possibility to work with a stochastic modulated interest rate and by considering a company having several business lines. A numerical example is provided to show possible applications of the model.

## 1. Introduction

The classical discounted aggregate claim process is defined as

$$L(t) = \sum_{n=1}^{N(t)} X_n e^{-\delta s_n}, \tag{1.1}$$

where $N(t)$ is the number of reported claims up to time $t$, $X_n$ the size of the nth claim, $s_n$ the epoch of the nth claim, $\delta$ is the constant force of interest.

Many contributions rely on common simplifying assumptions:
i) the claims occur according to a Poisson process;
ii) the epochs of claim occurrence and the size of claims are independent;
iii) the size of claims is described by a sequence of i.i.d. random variables.

The need of increasing the flexibility in the claims arrival and size led to the introduction of an environment modulating process. The milestone in Markov-Modulated risk models was marked by [1] who opened this new direction of research. Other scholars have, more recently, investigated different aspects of Markov-modulated risk processes and aggregate loss processes, see for example [2], [3], [19], [18], [15], [16], [13] and [10].

The paper by [13] presents explicit expressions for the first two moments of (1.1) and extended the results by [12] and [14] by considering a Markov modulated model.

In this paper we advance a more general model by considering a stochastic force of interest that is described by a Markov chain which is modulated by the

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environment process and, in this setting, the case of a firm having several lines of business.

The motivation of building a Markov-Modulated Risk Model with Stochastic Interest Rate (MMRPSIR) model is that in some applications a constant force of interest is a too restrictive hypothesis. The recent paper [9] contains matrix representation for formulas of moments of cash value of future payment streams arising from a discrete time multistate insurance contract where the evolution of the insured risk and the interest rate are random but independent of each other. The same independence assumption is formulated for ruin models with stochastic interest rates, see for example [4] with the exception that the interest rate process is assumed to have a dependent autoregressive structure. More recently in [26] different forces of interest were used depending on the sign of the surplus process, the latter is, obviously, dependent on the environment process. For this reason, in this paper we consider a general model where intensities of interest rates, number of claims and sizes of claims are dependent on the state of the same environment process describing economic circumstances. The hypothesized dependence may occur in real life problems of insurance and financial mathematics. Indeed, interesting and new applications suggesting the adoption of the MMRPSIR are described in next Remark 2.1 below.

The purpose of this paper is that of providing, in this more general setting, explicit expressions for the expectation and the variance of the aggregate discounted risk process when the dynamics of the interest rate and actuarial processes are not assumed to be independent. This may be of interest from a theoretical point of view because a new model is defined and studied, but also from a practical point of view because insurance companies are exposed to the interest rate risk as well.

The results generalize those by [13], [12] and [14]. They are obtained through Laplace-Stieltjes transform and are conveniently represented by using the technology of 2-dimensional matrices. We shall employ matrix notation extensively. The adopted 2-dimensional matrix formalism is important as the very complicated formulas can be expressed in a compact and simple way.

The rest of the article is organized as follows. In Section 2 the model is described and possible domains of application are discussed. In Section 3 the interest rate model is presented and only the results that are strictly relevant to our analysis are illustrated. Section 4 is devoted to the investigation of moments of the MMRMSIR. In Section 5 we present a numerical example with simulated data to validate the contribution of the previous sections. In Section 6, we present our conclusion. An appendix on 2-dimensional matrices and operations concludes the paper.

## 2. The Model

Let us consider an insurance company having $K$ business lines in which the policyholders are allocated.

Let $\{J(t), t \geq 0\}$ be a continuous time Markov chain with state space $E = \{1, 2, ..., m\}$ describing the environment process for the risk business. The process $J(t)$ may be thought as general economic conditions which are present at time $t$. We denote by $\mathbf{Q} = (q_{ij})_{i,j \in E}$ the infinitesimal generator of $J(t)$.

By $N^h(t)$ we denote the number of claims reported up to time $t$ in the business line $h \in \{1, 2, ..., K\}$. We assume that $N^h(t)$ is a doubly stochastic Poisson process with intensity $r(t) = r^h_{J(t)}$. Note that the intensity depends on the state of the economy and on the business line of reference.

By $X^h_n(J(s_n))$ we denote the nth claim size. It depends on the business line $h$ and on the state of the environment process $J(t)$ at the time $s_n$ of the nth claim occurrence. The claim sizes are independent given the state of the economy.

Let $\{\delta(t), t \geq 0\}$ be the process of the instantaneous rate of interest. We assume that $\delta(t)$ is a continuous time Markov chain with a finite state space $I$ and infinitesimal generator $\mathbf{\Pi}(i) = (\pi_{ab}(i))_{a,b\in I}$, $i \in E$ which depends on the state of the economy $i \in E$. Different generators act in correspondence of different states occupied by the environment process $J(t)$.

Each policyholder of the business line $h$ generates a discounted aggregate claim process $L^h(t) = \sum_{n=1}^{N^h(t)} X^h_n e^{-\delta(s_n)\cdot s_n}$.

If $n^{(h)}$ is the number of policyholders in the business line $h$, then the portfolio discounted aggregate claim process is defined by $\overline{L}(t) = \sum_{h=1}^{K} n^{(h)} L^h(t)$.

In this paper we are interested in getting explicit formulas for

$$\mathbb{E}[\overline{L}(t)] = \sum_{h=1}^{K} n^{(h)} \mathbb{E}[L^h(t)], \tag{2.1}$$

and

$$Var[\overline{L}(t)] = \sum_{h=1}^{K} (n^{(h)})^2 Var[L^h(t)] + 2 \sum_{h>k} n^{(h)} n^{(k)} Cov[L^h(t), L^k(t)]. \tag{2.2}$$

Notice that, if we have only one business line ($K = 1$) and the interest rate process is constant in time ($\delta(t) = \delta$, $\forall t \in [0, +\infty)$), then our model collapses in that of [13]. Additionally, if the environment process $J(t)$ is disregarded (that is $|E| = 1$), then we recover the models by [12] and [14].

This model is not only of interest for the described insurance problem. Two interesting financial applications are shortly described in next Remark 2.1.

*Remark* 2.1. Let us consider the example of a company seeking a source of funds. To this end, the company may issue a debt. The interest and the principal on the debt are paid back by using future cash flows generated by the company's business activity. Interests to pay on debt depend on the credit rating of the company that is a measure of the reliability the company has to honors its debt, see [7] and [8]. The higher the rating, the lower the interest rate the company should pay. Additionally, the higher the rating, the higher the cash flows generated by the company. In this application the environment or circumstance process $J(t)$ describes the rating evolution of the company, $\delta(t)$ is the stochastic interest rates which is a function of the rating process, $X^{(h)}_n$ denotes the nth inflow arising from the business lines $h$ of the company and $N^{(h)}(t)$ is the time of arrival of this inflow. In this model (2.1) and (2.2) would represent the expectation and the variance of the company's discounted aggregate cash inflow up to time $t$.

A similar example could be the case of a government selling public debt for the need of self-financing. The country has got a sovereign credit rating $J(t)$, the force of interest $\delta(t)$ depends on $J(t)$. The amount of public debt sold at the nth public auction $X_n$ is random and depend its self on the credit rating $J(t)$. In this example the model is simplified because the dates $s_n$ of the auctions are fixed by the government at the beginning of each year.

The two examples are only briefly described to show possible domains of applications other than the classical insurance problem. They are not further discussed in this paper for lack of space.

## 3. The Interest Rate Model

In this section we expose a simple and flexible interest rate model based on Markov chains. Markov chain modelling of interest rates is not a new proposal. Relevant contributions come from [25], [28], [24], [20] and [21]. We consider the force of interest $\{\delta(t),\, t \geq 0\}$ to be generated by a continuous time Markov chain with a finite state space $I$ and infinitesimal generator $\mathbf{\Pi}(i) = (\pi_{ab}(i))_{a,b\in I}$ which depends on the state of the environment process $J(t) = i \in E$. We assume that the infinitesimal generators are all stable and conservative.

This model is a natural generalization of that of [24] in which the dependence on the environment process was not considered. Here below, we present only results relevant to the aim of generalizing the risk process. Other results could be established by following the line of research in [24], but they are not of strict relevance to our purpose.

Let us denote the transition probabilities of the process $(J(t), \delta(t))$ by

$$\phi_{(i,a);(j,b)}(t) = \mathbb{P}(J(t) = j, \delta(t) = b | J(0) = i, \delta(0) = a).$$

It will be useful to represent these probabilities by using 2-dimensional matrices, see the appendix below. To this end set $\mathbf{u} = \begin{bmatrix} |E| \\ |I| \end{bmatrix}$ and let $\mathbf{\Phi}_{[\mathbf{u}\ \mathbf{u}]}(t)$ be the 2-dimensional matrix with elements $\mathbf{\Phi}_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}(t) = \phi_{(i,a);(j,b)}(t)$, $i, j \in E, a, b \in I$.

**Lemma 3.1.** *For every $i, j \in E$ and $a, b \in I$ we have that*

$$\mathbf{\Phi}_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}(t) = \delta_{ij} e^{-q_i t} \big(e^{\mathbf{\Pi}(i)t}\big)_{a,b} + \sum_{k\neq i} \sum_{c\in I} \int_0^t \big(e^{\mathbf{\Pi}(i)\tau}\big)_{a,c} q_{i,k} e^{-q_i \tau} \mathbf{\Phi}_{\left[\begin{smallmatrix} k & j \\ c & b \end{smallmatrix}\right]}(t-\tau) d\tau \tag{3.1}$$

*Proof.* Let denote by $T_1$ the time of first transition of the environment process $J(t)$. Being the events $\{T_1 = k\}$ disjoint we have that

$$\begin{aligned} \mathbf{\Phi}_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}(t) &= \mathbb{P}(J(t) = j, \delta(t) = b, T_1 > t | J(0) = i, \delta(0) = a) \\ &+ \mathbb{P}(J(t) = j, \delta(t) = b, T_1 \leq t | J(0) = i, \delta(0) = a). \end{aligned}$$

Observe that

$$\begin{gathered} \mathbb{P}(J(t) = j, \delta(t) = b, T_1 > t | J(0) = i, \delta(0) = a) \\ = \mathbb{P}(\delta(t) = b | J(t) = j, T_1 > t, J(0) = i, \delta(0) = a) \\ \cdot \mathbb{P}(J(t) = j | T_1 > t, J(0) = i, \delta(0) = a) \mathbb{P}(T_1 > t | J(0) = i, \delta(0) = a) \\ = \big(e^{\mathbf{\Pi}(i)t}\big)_{a,b} \delta_{ij} e^{-q_i t}. \end{gathered}$$

On the other hand, the use of the strong Markov property gives

$$\mathbb{P}(J(t)=j,\delta(t)=b,T_1\le t|J(0)=i,\delta(0)=a)$$

$$=\int_0^t\sum_{k\neq i}\sum_{c\in I}\mathbb{P}(J(t)=j,\delta(t)=b,T_1\in(\tau,\tau+d\tau),J(T_1)=k,$$
$$\delta(T_1)=c\mid J(0)=i,\delta(0)=a)d\tau$$

$$=\sum_{k\neq i}\sum_{c\in I}\int_0^t\mathbb{P}(J(T_1)=k,\delta(T_1)=c,T_1\in(\tau,\tau+d\tau)|J(0)=i,\delta(0)=a)$$

$$\cdot\mathbb{P}(J(t)=j,\delta(t)=b|J(T_1)=k,\delta(T_1)=c)d\tau$$

$$=\sum_{k\neq i}\sum_{c\in I}\int_0^t\mathbb{P}(\delta(\tau)=c|J(\tau)=k,T_1=\tau,J(0)=i,\delta(0)=a)$$

$$\cdot\mathbb{P}(J(\tau)=k,T_1\in(\tau,\tau+d\tau)|J(0)=i,\delta(0)=a)$$

$$\cdot\mathbb{P}(J(t)=j,\delta(t)=b|J(\tau)=k,\delta(\tau)=c)d\tau$$

$$=\sum_{k\neq i}\sum_{c\in I}\int_0^t\left(e^{\mathbf{\Pi}(i)\tau}\right)_{a,c}q_{i,k}\,e^{-q_i\tau}\,\mathbf{\Phi}_{\left[\begin{smallmatrix}k&j\\c&b\end{smallmatrix}\right]}(t-\tau)d\tau.$$

□

**Theorem 3.2.** *The transition probability matrix* $\mathbf{\Phi}_{[\mathbf{u}\ \mathbf{u}]}(t)$ *satisfies the following matrix differential equation:*

$$\mathbf{\Phi}'_{[\mathbf{u}\ \mathbf{u}]}(t)=\mathbf{\Lambda}_{[\mathbf{u}\ \mathbf{u}]}\otimes\mathbf{\Phi}_{[\mathbf{u}\ \mathbf{u}]}(t),\tag{3.2}$$

*where for all* $i,j\in E$ *and* $a,b\in I$

$$\mathbf{\Lambda}_{\left[\begin{smallmatrix}i&j\\a&b\end{smallmatrix}\right]}=\begin{cases}-q_i-\pi_a(i) & \text{if } i=j,\ a=b\\ \pi_{a,b}(i) & \text{if } i=j,\ a\neq b\\ q_{i,j} & \text{if } i\neq j,\ a=b\\ 0 & \text{otherwise.}\end{cases}\tag{3.3}$$

*the symbol* $\otimes$ *denotes the product between two 2-dimensional matrices, see the appendix below.*

*Proof.* In equation (3.1) the change of variable $t-\tau=v$ yields:

$$\mathbf{\Phi}_{\left[\begin{smallmatrix}i&j\\a&b\end{smallmatrix}\right]}(t)=\delta_{ij}\left(e^{\mathbf{\Pi}(i)t}\right)_{a,b}e^{-q_it}$$
$$+\sum_{k\neq i}\sum_{c\in I}\int_0^t\left(e^{\mathbf{\Pi}(i)(t-v)}\right)_{a,c}q_{i,k}e^{-q_i(t-v)}\mathbf{\Phi}_{\left[\begin{smallmatrix}k&j\\c&b\end{smallmatrix}\right]}(v)dv.$$

Differentiating with respect to $t$, using Leibnitz's rule, we have

$$\frac{\partial \mathbf{\Phi}_{\left[\begin{smallmatrix} i & j \\ a & b\end{smallmatrix}\right]}(t)}{\partial t} = \delta_{ij}\big(\mathbf{\Pi}(i)e^{\mathbf{\Pi}(i)t}\big)_{a,b}e^{-q_i t} + \delta_{ij}\big(e^{\mathbf{\Pi}(i)t}\big)_{a,b}(-q_i)e^{-q_i t}$$

$$+\sum_{k\neq i}\sum_{c\in I}\Bigg(\big(e^{\mathbf{\Pi}(i)(t-t)}\big)_{a,c}q_{i,k}e^{-q_i(t-t)}\mathbf{\Phi}_{\left[\begin{smallmatrix} k & j \\ c & b\end{smallmatrix}\right]}(t) + \int_0^t \mathbf{\Phi}_{\left[\begin{smallmatrix} k & j \\ c & b\end{smallmatrix}\right]}(v)\cdot$$

$$\left[\Big(\mathbf{\Pi}(i)e^{\mathbf{\Pi}(i)(t-v)}\Big)_{a,c}q_{i,k}e^{-q_i(t-v)} + \big(e^{\mathbf{\Pi}(i)(t-v)}\big)_{a,c}q_{i,k}(-q_i)e^{-q_i(t-v)}\right]dv\Bigg)$$

$$= \delta_{ij}\big(\sum_{z\in E}\pi_{a,z}(i)\big(e^{\mathbf{\Pi}(i)t}\big)_{z,b}\big)e^{-q_i t}$$

$$+\delta_{ij}(-q_i)\big(e^{\mathbf{\Pi}(i)t}\big)_{a,b}e^{-q_i t} + \sum_{k\neq i}\sum_{c\in I}\Big(q_{i,k}\delta_{a,c}\mathbf{\Phi}_{\left[\begin{smallmatrix} k & j \\ c & b\end{smallmatrix}\right]}(t) + \int_0^t \mathbf{\Phi}_{\left[\begin{smallmatrix} k & j \\ c & b\end{smallmatrix}\right]}(v)$$

$$\left[\sum_{z\in E}\pi_{a,z}(i)\big(e^{\mathbf{\Pi}(i)(t-v)}\big)_{z,c}q_{i,k}e^{-q_i(t-v)} - q_i\big(e^{\mathbf{\Pi}(i)(t-v)}\big)_{a,c}q_{i,k}e^{-q_i(t-v)}\right]dv\Big)$$

$$= (-q_i)\Big[\delta_{ij}\big(e^{\mathbf{\Pi}(i)t}\big)_{a,b}e^{-q_i t} + \sum_{k\neq i}\sum_{c\in I}\int_0^t \big(e^{\mathbf{\Pi}(i)(t-v)}\big)_{a,c}q_{i,k}e^{-q_i(t-v)}\mathbf{\Phi}_{\left[\begin{smallmatrix} k & j \\ c & b\end{smallmatrix}\right]}(v)dv\Big]$$

$$+\sum_{z\in E}\pi_{a,z}(i)\Big(\big(e^{\mathbf{\Pi}(i)t}\big)_{z,b}\delta_{ij}e^{-q_i t}$$

$$+\sum_{k\neq i}\sum_{c\in I}\int_0^t \big(e^{\mathbf{\Pi}(i)(t-v)}\big)_{z,c}q_{i,k}e^{-q_i(t-v)}\mathbf{\Phi}_{\left[\begin{smallmatrix} k & j \\ c & b\end{smallmatrix}\right]}(v)dv\Big) + \sum_{k\neq i}\sum_{c\in I}q_{i,k}\delta_{a,c}\mathbf{\Phi}_{\left[\begin{smallmatrix} k & j \\ c & b\end{smallmatrix}\right]}(t)$$

$$= -q_i\mathbf{\Phi}_{\left[\begin{smallmatrix} i & j \\ a & b\end{smallmatrix}\right]}(t) + \sum_{k\neq i}\sum_{c\in I}q_{ik}\delta_{a,c}\mathbf{\Phi}_{\left[\begin{smallmatrix} k & j \\ c & b\end{smallmatrix}\right]}(t) + \sum_{z\in E}\pi_{a,z}(i)\mathbf{\Phi}_{\left[\begin{smallmatrix} i & j \\ z & b\end{smallmatrix}\right]}(t)$$

$$= \Big(-q_i - \pi_a(i)\Big)\mathbf{\Phi}_{\left[\begin{smallmatrix} i & j \\ a & b\end{smallmatrix}\right]}(t) + \sum_{z\neq a}\pi_{a,z}(i)\mathbf{\Phi}_{\left[\begin{smallmatrix} i & j \\ z & b\end{smallmatrix}\right]}(t) + \sum_{k\neq i}q_{ik}\mathbf{\Phi}_{\left[\begin{smallmatrix} k & j \\ a & b\end{smallmatrix}\right]}(t).$$

The application of the 2-dimensional matrix formalism gives the matrix equation (3.2). □

As a matter of example, if we assume that $\mid E \mid= 2$ and $\mid I \mid= 3$, then the matrix $\mathbf{\Lambda}$ has the following form:

$$\mathbf{\Lambda} =$$

$$\begin{bmatrix} \begin{bmatrix} -q_1-\pi_1(1) & \pi_{1,2}(1) & \pi_{1,3}(1) \\ \pi_{2,1}(1) & -q_1-\pi_2(1) & \pi_{2,3}(1) \\ \pi_{3,1}(1) & \pi_{3,2}(1) & -q_1-\pi_3(1) \end{bmatrix} & \begin{bmatrix} q_{1,2} & 0 & 0 \\ 0 & q_{1,2} & 0 \\ 0 & 0 & q_{1,2} \end{bmatrix} \\ \begin{bmatrix} q_{2,1} & 0 & 0 \\ 0 & q_{2,1} & 0 \\ 0 & 0 & q_{2,1} \end{bmatrix} & \begin{bmatrix} -q_2-\pi_1(2) & \pi_{1,2}(2) & \pi_{1,3}(2) \\ \pi_{2,1}(2) & -q_2-\pi_2(2) & \pi_{2,3}(2) \\ \pi_{3,1}(2) & \pi_{3,2}(2) & -q_2-\pi_3(2) \end{bmatrix} \end{bmatrix}$$

The element corresponding to the indices $i=1$, $a=2$, $j=1$, $b=3$ is

$$\mathbf{\Lambda}_{\left[\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}\right]} = \pi_{2,3}(1)$$

*Remark* 3.3. Matrix $\mathbf{\Lambda}$ has the following properties:
i) the diagonal elements of $\mathbf{\Lambda}$ are nonpositive;
ii) the off-diagonal elements are nonnegative;
iii) the row sums of $\mathbf{\Lambda}$ are all zero.
Consequently we recognize $\mathbf{\Lambda}$ as the infinitesimal generator of a finite Markov chain with state space $E \times I$.

## 4. Analysis of the Risk Process

In this section we extend the results by [13] considering our more general MM-RPSIR with several business lines.

**Definition 4.1.** We define the *z-translated discounted aggregate claim process* relative to the business line $h$ by

$$L^h(t,z) := \sum_{n=1}^{N^{(h)}(t)} X_n^h e^{-\delta(s_n)(s_n+z)}. \tag{4.1}$$

The process (4.1) is obtained by multiplying each one of the discounted claim $X_n^h e^{-\delta(s_n)(s_n)}$ by the corresponding random factor $e^{-\delta(s_n)z}$. Notice that, if $z=0$, then $L^h$ coincides with $L^h(t)$.

Introduce the 2-dimensional matrix of functions $\mathbf{F}_{[\mathbf{u}\ \mathbf{u}]}(x_h, x_k; t, z)$ with elements

$$f_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}(x_h, x_k; t, z) = \mathbb{E}_{(i,a)}[e^{-(x_h L^h(t,z) + x_k L^k(t,z))} 1_{\{J(t)=j, \delta(t)=b\}}]. \tag{4.2}$$

From now on, all matrices will be intended to be 2-dimensional matrices of row order $\mathbf{u}$ and column order $\mathbf{u}$. To safe space we avoid the repetition of the dimensional indices.

**Proposition 4.2.** *For each couple $(h,k)$ of business lines and $\forall x_h, x_k, t, z \geq 0$,*

$$\mathbf{F}(x_h, x_k; t+z, 0) = \mathbf{F}(x_h, x_k; t, 0) \otimes \mathbf{F}(x_h, x_k; z, t) \tag{4.3}$$

*where $\otimes$ denotes the product between two 2-dimensional matrices.*

*Proof.*

$$\begin{aligned}
&f_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}(x_h, x_k; t+z, 0) = \mathbb{E}_{(i,a)}[e^{-(x_h L^h(t+z,0) + x_k L^k(t+z,0))} 1_{\{J(t+z)=j, \delta(t+z)=b\}}] \\
&= \mathbb{E}_{(i,a)}[\mathbb{E}_{(i,a)}[e^{-(x_h L^h(t+z,0) + x_k L^k(t+z,0))} 1_{\{J(t+z)=j, \delta(t+z)=b\}} | J(t), \delta(t)]] \\
&= \mathbb{E}_{(i,a)}[\mathbb{E}_{(i,a)}[e^{-(x_h (L^h(t+z,0) - L^h(t,0) + L^h(t,0)))} \\
&\quad \cdot e^{-(x_k (L^k(t+z,0) - L^k(t,0) + L^k(t,0)))} 1_{\{J(t+z)=j, \delta(t+z)=b\}} | J(t), \delta(t)]] \\
&= \mathbb{E}_{(i,a)}[e^{-x_h L^h(t,0)} e^{-x_k L^k(t,0)} \mathbb{E}_{(i,a)}[e^{-(x_h (L^h(t+z,0) - L^h(t,0)))} \\
&\quad e^{-(x_k (L^k(t+z,0) - L^k(t,0)))} 1_{\{J(t+z)=j, \delta(t+z)=b\}} | J(t), \delta(t)]].
\end{aligned}$$

Denote by $\mathcal{D}(Y)$ the probability distribution of random variable $Y$. Then

$$\mathcal{D}(L^h(t+z,0)-L^h(t,0)|J(t)=i,\delta(t)=a)=\mathcal{D}(L^h(z,t)|J(0)=i,\delta(0)=a). \quad (4.4)$$

To prove the truth of relation (4.4) it is sufficient to represent

$$\begin{aligned} L^h(t+z,0)-L^h(t,0) &= \sum_{n=N^h(t)+1}^{N^h(t+z)} X_n^h e^{-\delta(s_n)s_n} \\ &= \sum_{m=1}^{N^h(t+z)-N^h(t)} X^h_{m+N^h(t)} e^{-\delta(s_{m+N^h(t)})s_{m+N^h(t)}}, \end{aligned}$$

and to note that $\forall i\in E$, $\forall a\in I$

$$\begin{gathered} i)\,\mathcal{D}(X^h_{m+N^h(t)}\mid J(t)=i)=\mathcal{D}(X^h_m\mid J(0)=i); \\ ii)\,\mathcal{D}(s_{m+N^h(t)}\mid J(t)=i)=\mathcal{D}(s_m+t\mid J(0)=i); \\ iii)\,\mathcal{D}(N^h(t+z)-N^h(t)\mid J(t)=i)=\mathcal{D}(N^h(z)\mid J(0)=i); \\ iv)\,\mathcal{D}(\delta(s_{m+N^h(t)})\mid J(t)=i,\delta(t)=a)=\mathcal{D}(\delta(s_m)\mid J(0)=i,\delta(0)=a). \end{gathered}$$

By using relation (4.4) we have that

$$\begin{aligned} & f_{\left[\begin{smallmatrix} i & j \\ a & b\end{smallmatrix}\right]}(x_h,x_k;t+z,0)=\mathbb{E}_{(i,a)}\big[e^{-(x_hL^h(t,0)+x_kL^k(t,0))} \\ & \mathbb{E}_{(J(t),\delta(t))}[e^{-(x_h(L^h(z,t)+x_kL^k(z,t)))}1_{\{J(z)=j,\delta(z)=b\}}]\big] \\ & =\mathbb{E}_{(i,a)}[e^{-(x_hL^h(t,0)+x_kL^k(t,0))}\cdot f_{\left[\begin{smallmatrix} J(t) & j \\ \delta(t) & b\end{smallmatrix}\right]}(x_h,x_k;z,t)] \\ & =\sum_{c\in E}\sum_{d\in I}\mathbb{E}_{(i,a)}[e^{-(x_hL^h(t,0)+x_kL^k(t,0))}1_{\{J(t)=c,\delta(t)=d\}}\cdot f_{\left[\begin{smallmatrix} c & j \\ d & b\end{smallmatrix}\right]}(x_h,x_k;z,t)] \\ & =\sum_{c\in E}\sum_{d\in I} f_{\left[\begin{smallmatrix} i & c \\ a & d\end{smallmatrix}\right]}(x_h,x_k;t,0)\cdot f_{\left[\begin{smallmatrix} c & j \\ d & b\end{smallmatrix}\right]}(x_h,x_k;z,t). \end{aligned}$$

□

*Remark* 4.3. The z-translated discounted aggregate claim process $L^h(t,z)$ is introduced in order to get a convenient representation of the increment $L^h(t+z,0)-L^h(t,0)$ that is possible to write in a recursive form through 2-dimensional matrices and represented by a simple matrix product.

*Remark* 4.4. If we consider only a business line and the interest rate process with a constant force of interest, i.e. $\delta(t)=\delta$, then we recover Proposition 1 in [13].

Let us denote by $\hat{g}^h_{i,a}(x_h,t)=\mathbb{E}_{(i,a)}[e^{-x_hX_1^he^{-at}}]$, then the following proposition holds true.

**Proposition 4.5.** *For each couple $(h,k)$ of business lines and $\forall x_h\geq 0, x_k\geq 0, t\geq 0$,*

$$\mathbf{F}(x_h,x_k;z,t)=\mathbf{I}+\sum_{w\in\{h,k\}}\mathbf{R}^w\otimes\big(\hat{\mathbf{G}}^w(x_w,t)-\mathbf{I}\big)\cdot z+\hat{\mathbf{Q}}\cdot z+\hat{\mathbf{\Pi}}\cdot z+\mathbf{o}(z) \quad (4.5)$$

*where for all* $i,j \in E, a,b \in I, w \in \{h,k\}$

$$\mathbf{I} = \left(i_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}\right), \text{ and } i_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]} = \begin{cases} 1 & \text{if } i=j,\ a=b \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{R}^w = \left(r^w_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}\right) \text{ and } r^w_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]} = \begin{cases} r^w_i & \text{if } i=j,\ a=b \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\mathbf{G}}^w(x_w,t) = \left(\hat{g}^w_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}(x_w,t)\right) \text{ and } \hat{g}^w_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}(x_w,t) = \begin{cases} \hat{g}^w_{i,a}(x_w,t) & \text{if } i=j,\ a=b \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\mathbf{Q}} = \left(q_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}\right) \text{ and } q_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]} = \begin{cases} q_{i,j} & \text{if } a=b,\ \forall i,j \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{\mathbf{\Pi}} = \left(\pi_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}\right) \text{ and } \pi_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]} = \begin{cases} \pi_{a,b}(i) & \text{if } i=j,\ a=b \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Let us denote by $N^\delta(z)$ and by $N^J(z)$ the number of transitions of the interest rate process $\delta(t)$ and of the environment process $J(t)$ until time $z$, respectively. Let us introduce the multivariate counting process

$$\mathbf{N}(z) = (N^h(z), N^k(z), N^\delta(z), N^J(z)), \tag{4.6}$$

and let us denote by $\|\mathbf{N}(z)\|$ the Euclidean norm of the vector $\mathbf{N}(z)$. Moreover, denote by $\mathbf{e}_1 = (1,0,0,0)$,$\mathbf{e}_2 = (0,1,0,0)$,$\mathbf{e}_3 = (0,0,1,0)$ and $\mathbf{e}_4 = (0,0,0,1)$ the canonical basis.

Consider the following representation:

$$\begin{aligned}
&f_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}(x_h,x_k;z,t) = \mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}] \\
&= \mathbb{P}_{(i,a)}(\mathbf{N}(z)=\mathbf{e}_1)\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=\mathbf{e}_1] \\
&+ \mathbb{P}_{(i,a)}(\mathbf{N}(z)=\mathbf{e}_2)\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=\mathbf{e}_2] \\
&+ \mathbb{P}_{(i,a)}(\mathbf{N}(z)=\mathbf{e}_3)\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=\mathbf{e}_3] \\
&+ \mathbb{P}_{(i,a)}(\mathbf{N}(z)=\mathbf{e}_4)\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=\mathbf{e}_4] \\
&+ \mathbb{P}_{(i,a)}(\|\mathbf{N}(z)\|>1)\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\|\mathbf{N}(z)\|>1] \\
&+ \mathbb{P}_{(i,a)}(\mathbf{N}(z)=0)\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=0],
\end{aligned} \tag{4.7}$$

It is not difficult to establish that $\mathbb{P}_{(i,a)}(\mathbf{N}(z)=\mathbf{e}_1) = r^h_i z + o(z)$ moreover, when computing $\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=\mathbf{e}_1]$ it should be noticed that, because $N^h(z)=1$, then $L^h(z,t) = X^h_1 e^{-\delta(s_1)(s_1+t)}$ and, consequently

$$\begin{aligned}
&\lim_{z\to 0}\mathbb{E}_{(i,a)}[e^{-x_hX_1^{(h)}e^{-\delta(s_1)(s_1+t)}}]1_{\{i=j\}}1_{\{a=b\}} \\
&= \mathbb{E}_{(i,a)}[e^{-x_hX_1^{(h)}e^{-at}}]1_{\{i=j\}}1_{\{a=b\}} \equiv \hat{g}^{(h)}_{i,a}(x_h,t)1_{\{i=j\}}1_{\{a=b\}}.
\end{aligned}$$

Summarizing we obtained that

$$\begin{aligned}
&\mathbb{P}_{(i,a)}(\mathbf{N}(z)=\mathbf{e}_1)\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=\mathbf{e}_1] \\
&= \left(r^h_i z + o(z)\right)\hat{g}^{(h)}_{i,a}(x_h,t)1_{\{i=j\}}1_{\{a=b\}}, \text{ as } z\to 0.
\end{aligned} \tag{4.8}$$

By symmetry we have that

$$\begin{aligned}&\mathbb{P}_{(i,a)}(\mathbf{N}(z)=\mathbf{e}_2)\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=\mathbf{e}_2]\\&=\big(r_i^k z+o(z)\big)\hat{g}_{i,a}^{(k)}(x_k,t)1_{\{i=j\}}1_{\{a=b\}},\ \text{ as } z\to 0.\end{aligned}\tag{4.9}$$

Similarly, it is possible to establish that as $z\to 0$

$$\mathbb{P}_{(i,a)}(\mathbf{N}(z)=\mathbf{e}_3,\delta(z)=w)=\pi_{aw}(i)z+o(z),$$

$$\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=\mathbf{e}_3,\delta(z)=w]=1_{\{i=j\}}1_{\{w=b\}},$$

and consequently

$$\begin{aligned}&\mathbb{P}_{(i,a)}(\mathbf{N}(z)=\mathbf{e}_3)\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=\mathbf{e}_3]\\&=\pi_{ab}(i)z1_{\{i=j\}}+o(z),\ \text{ as } z\to 0.\end{aligned}\tag{4.10}$$

By noting that $\lim_{z\to 0}\mathbb{P}_{(i,a)}(\mathbf{N}(z)=\mathbf{e}_4,J(z)=k,)=q_{ik}z+o(z)$ and

$$\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=\mathbf{e}_4,J(z)=k]=1_{\{b=a\}}1_{\{j=k\}}$$

it is possible to obtain that

$$\begin{aligned}&\mathbb{P}_{(i,a)}(\mathbf{N}(z)=\mathbf{e}_4)\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=\mathbf{e}_4]\\&=q_{ij}z1_{\{a=b\}}+o(z),\ \text{ as } z\to 0.\end{aligned}\tag{4.11}$$

Moreover

$$\begin{aligned}&\mathbb{P}_{(i,a)}(\|\mathbf{N}(z)\|>1)\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\|\mathbf{N}(z)\|>1]\\&=o(z),\ \text{ as } z\to 0,\end{aligned}\tag{4.12}$$

being $\lim_{z\to 0}\frac{\mathbb{P}_{(i,a)}(\|\mathbf{N}(z)\|>1)}{z}=0$.

Using the previously established results, we have that as $z\to 0$

$$\begin{aligned}\mathbb{P}_{(i,a)}(\mathbf{N}(z)=0)&=1-\big(r_i^{(h)}z+o(z)+r_i^{(k)}z+o(z)\\&+\sum_{w\neq i}\pi_{aw}(i)z+o(z)+\sum_{k\neq i}q_{ik}z+o(z)+o(z)\big)\\&=1-\big(r_i^{(h)}+r_i^{(k)}-\pi_{aa}(i)-q_{ii}\big)z+o(z).\end{aligned}$$

Finally,

$$\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\|\mathbf{N}(z)\|=0]=1_{\{b=a\}}1_{\{j=i\}}.$$

In this way we proved that

$$\begin{aligned}&\mathbb{P}_{(i,a)}(\mathbf{N}(z)=0)\cdot\mathbb{E}_{(i,a)}[e^{-(x_hL^h(z,t)+x_kL^k(z,t))}1_{\{J(z)=j,\delta(z)=b\}}|\mathbf{N}(z)=0]\\&=\Big(1-\big(r_i^{(h)}+r_i^{(k)}-\pi_{aa}(i)-q_{ii}\big)z+o(z)\Big)1_{\{a=b\}}1_{\{i=j\}},\ \text{ as } z\to 0.\end{aligned}\tag{4.13}$$

By substitution of (4.8), (4.9), (4.10), (4.11), (4.12) and (4.13) in (4.7) and by little algebra we conclude that, as $z \to 0$

$$\begin{aligned} & f_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}(x_h, x_k; z, t) \\ & = [1 + r_i^{(h)} z(\hat{g}_{i,a}^{(h)}(x_h, t) - 1) + r_i^{(k)} z(\hat{g}_{i,a}^{(k)}(x_k, t) - 1)] 1_{\{i=j\}} 1_{\{a=b\}} \\ & + q_{ij} z 1_{\{a=b\}} + \pi_{ab}(i) z 1_{\{i=j\}} + o(z) 1_{\{i=j\}} 1_{\{a=b\}}. \end{aligned}$$

The application of the 2-dimensional matrix formalism gives (4.5). □

**Proposition 4.6.** *For each couple* $(h,k)$ *of business lines, for* $x_h, x_k \geq 0$, $\mathbf{F}(x_h, x_k; t, 0)$ *is the solution of the following initial value problem:*

$$\frac{\partial}{\partial t}\mathbf{F}(x_h, x_k; t, 0) = \mathbf{F}(x_h, x_k; t, 0) \otimes \left[\hat{\mathbf{Q}} + \hat{\mathbf{\Pi}} + \sum_{w \in \{h,k\}} \mathbf{R}^w \otimes \big(\hat{\mathbf{G}}^w(x_w, t) - \mathbf{I}\big)\right] \quad (4.14)$$

$$\mathbf{F}(x_h, x_k; 0, 0) = \mathbf{I} \quad (4.15)$$

*Proof.* The consideration of formulas (4.3) and (4.5) when $z = \Delta t$ and the subsequent substitution of (4.5) in (4.3) leads to

$$\begin{aligned} \mathbf{F}(x_h, x_k; t + \Delta t, 0) &= \mathbf{F}(x_h, x_k; t, 0) \\ & \otimes \Big\{\mathbf{I} + \sum_{w \in \{h,k\}} \big[\mathbf{R}^w \otimes \big(\hat{\mathbf{G}}^w(x_w, t) - \mathbf{I}\big) + \hat{\mathbf{Q}} + \hat{\mathbf{\Pi}}\big] \cdot \Delta t + \mathbf{o}(\Delta t)\Big\} \end{aligned} \quad (4.16)$$

Add now $-\mathbf{F}(x_h, x_k; t, 0)$ and divide by $\Delta t$ on the left and right sides of (4.16) and consider $\Delta t \to 0$; this produces equation (4.14). The initial condition (4.15) is obtained by setting $z = 0$ and $t = 0$ in equation (4.5). □

Let $h, k$ be two business lines; if we define $\mathbf{E}^{(r,s)}_{[\mathbf{u}\ \mathbf{u}]}(t)$ to be the 2-dimensional matrix with elements

$$\mathbf{E}^{(r,s)}_{\left[\begin{smallmatrix} |i| & |j| \\ |a| & |b| \end{smallmatrix}\right]}(t) = \mathbb{E}_{(i,a)}[\big(L^h(t,0)\big)^r \big(L^k(t,0)\big)^s 1_{\{J(t)=j, \delta(t)=b\}}],$$

where $r, s \in \{0,1,2\}$, $0 < r + s < 3$, by

$$\mu^{(r,s)}_{(i,a)}(t) := \mathbb{E}_{(i,a)}[\big(L^h(t,0)\big)^r \big(L^k(t,0)\big)^s] = \sum_{j \in E} \sum_{b \in I} \mathbf{E}^{(r,s)}_{\left[\begin{smallmatrix} |i| & |j| \\ |a| & |b| \end{smallmatrix}\right]}(t),$$

and by

$$\mathbf{M}^{(r,s)}_{\left[\begin{smallmatrix} |i| & |j| \\ |a| & |b| \end{smallmatrix}\right]}(t) = \frac{\partial^r}{\partial x_h^r} \frac{\partial^s}{\partial x_k^s} f_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]}(x_h, x_k; t, 0)\Bigg|_{x_h = x_k = 0}, \quad (4.17)$$

it is simple to realize that $\mathbf{M}^{(r,s)}_{\left[\begin{smallmatrix} |i| & |j| \\ |a| & |b| \end{smallmatrix}\right]}(t) = (-1)^{r+s} \mathbf{E}^{(r,s)}_{\left[\begin{smallmatrix} |i| & |j| \\ |a| & |b| \end{smallmatrix}\right]}(t)$, where, by convention, we consider $\frac{\partial^0}{\partial x^0} f(x) = f(x)$.

Next theorem provides explicit formulas for moments of claims for each business lines and the cross product moment among the business lines.

**Theorem 4.7.** *For each couple of business lines* $(h, k)$, *for all* $t \geq 0$ *it results that*

$$\mathbf{M}^{(1,0)}(t) = \int_0^t e^{\boldsymbol{\Lambda}\cdot y} \otimes \mathbf{R}^h \otimes \mathbf{Ge}^h(y) \otimes e^{\left[\hat{\mathbf{Q}}+\hat{\boldsymbol{\Pi}}\right]\cdot(t-y)} dy \tag{4.18}$$

$$\mathbf{M}^{(0,1)}(t) = \int_0^t e^{\boldsymbol{\Lambda}\cdot y} \otimes \mathbf{R}^k \otimes \mathbf{Ge}^k(y) \otimes e^{\left[\hat{\mathbf{Q}}+\hat{\boldsymbol{\Pi}}\right]\cdot(t-y)} dy, \tag{4.19}$$

$$\begin{aligned}\mathbf{M}^{(1,1)}(t) = &\int_0^t e^{\boldsymbol{\Lambda}\cdot y} \otimes \mathbf{R}^h \otimes \mathbf{Ge}^h(y) \otimes \mathbf{Int}^k(y,t) dy,\\ &+ \int_0^t e^{\boldsymbol{\Lambda}\cdot y} \otimes \mathbf{R}^k \otimes \mathbf{Ge}^k(y) \otimes \mathbf{Int}^h(y,t) dy,\end{aligned} \tag{4.20}$$

*where* $\mathbf{Int}^w(y,t)$ *is the matrix with generic element* $\mathbf{Int}^w_{\left[\begin{smallmatrix} i & j \\ a & b\end{smallmatrix}\right]}(y,t)$ *equal to*

$$\frac{-r_j^w g_{j,b}^{(w)} e^{-y\cdot\boldsymbol{\Lambda}_{\left[\begin{smallmatrix} i & j \\ a & b\end{smallmatrix}\right]}+t\cdot\boldsymbol{\Lambda}_{\left[\begin{smallmatrix} j & j \\ b & b\end{smallmatrix}\right]}}}{\boldsymbol{\Lambda}_{\left[\begin{smallmatrix} i & j \\ a & b\end{smallmatrix}\right]} - \boldsymbol{\Lambda}_{\left[\begin{smallmatrix} j & j \\ b & b\end{smallmatrix}\right]} - b} \Big( e^{(\boldsymbol{\Lambda}_{\left[\begin{smallmatrix} i & j \\ a & b\end{smallmatrix}\right]}-\boldsymbol{\Lambda}_{\left[\begin{smallmatrix} j & j \\ b & b\end{smallmatrix}\right]}-b)t} - e^{(\boldsymbol{\Lambda}_{\left[\begin{smallmatrix} i & j \\ a & b\end{smallmatrix}\right]}-\boldsymbol{\Lambda}_{\left[\begin{smallmatrix} j & j \\ b & b\end{smallmatrix}\right]}-b)y} \Big).$$

*Proof.* Let us consider the first order partial derivatives of (4.14) and (4.15) with respect to $x_h$ and $x_k$,

$$\begin{aligned}&\frac{\partial}{\partial x_h}\frac{\partial}{\partial x_k}\frac{\partial}{\partial t}\mathbf{F}(x_h, x_k; t, 0)\\ &= \frac{\partial}{\partial x_h}\frac{\partial}{\partial x_k}\mathbf{F}(x_h, x_k; t, 0) \otimes \big[\hat{\mathbf{Q}} + \hat{\boldsymbol{\Pi}} + \sum_{w\in\{h,k\}} \mathbf{R}^w \otimes \big(\hat{\mathbf{G}}^w(x_w,t) - \mathbf{I}\big)\big]\\ &\quad + \frac{\partial}{\partial x_k}\mathbf{F}(x_h, x_k; t, 0) \otimes \Big[\mathbf{R}^h \otimes \frac{\partial}{\partial x_h}\hat{\mathbf{G}}^h(x_h,t)\Big]\\ &\quad + \frac{\partial}{\partial x_h}\mathbf{F}(x_h, x_k; t, 0) \otimes \Big[\mathbf{R}^k \otimes \frac{\partial}{\partial x_k}\hat{\mathbf{G}}^k(x_k,t)\Big],\end{aligned} \tag{4.21}$$

$$\frac{\partial}{\partial x_h}\frac{\partial}{\partial x_k}\mathbf{F}(x_h, x_k; 0, 0) = \mathbf{O}. \tag{4.22}$$

Let us now compute, $\forall w \in \{h, k\}$, the following derivative:

$$\frac{\partial}{\partial x_w}\hat{\mathbf{G}}^w_{\left[\begin{smallmatrix} |i| & |j| \\ |a| & |b|\end{smallmatrix}\right]}(x_w, t) = \mathbb{E}_{(i,a)}[-X_1^{(w)} e^{-at} e^{-x_w X_1^{(w)} e^{-at}}] 1_{\{i=j\}} 1_{\{a=b\}}.$$

If we denote by $g_{i,a}^{(w)} = \mathbb{E}_{(i,a)}[X_1^{(w)}]$, then we have

$$\begin{aligned}&\frac{\partial}{\partial x_w}\hat{\mathbf{G}}^w_{\left[\begin{smallmatrix} |i| & |j| \\ |a| & |b|\end{smallmatrix}\right]}(x_w, t)\Bigg|_{x_w=0}\\ &= -g_{i,a}^{(w)} e^{-at} 1_{\{i=j\}} 1_{\{a=b\}} := Ge^w_{\left[\begin{smallmatrix} |i| & |j| \\ |a| & |b|\end{smallmatrix}\right]}(t) = \begin{cases} -g_{i,a}^{(w)} e^{-at} & \text{if } i = j,\ a = b \\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{4.23}$$

Interchanging the derivatives of (4.21) and combining (4.17) and (4.23) we get

$$\begin{aligned}\frac{d}{dt}\mathbf{M}^{(1,1)}(t) &= \mathbf{M}^{(1,1)}(t) \otimes \big[\hat{\mathbf{Q}} + \hat{\mathbf{\Pi}} + \sum_{w\in\{h,k\}} \mathbf{R}^{(w)} \otimes \big(\hat{\mathbf{G}}^{(w)}(0,t) - \mathbf{I}\big) \\ &+ \mathbf{M}^{(0,1)}(t) \otimes \mathbf{R}^{(h)} \otimes \mathbf{Ge}^{(h)}(t) + \mathbf{M}^{(1,0)}(t) \otimes \mathbf{R}^{(k)} \otimes \mathbf{Ge}^{(k)}(t).\end{aligned} \tag{4.24}$$

It is simple to see that $\hat{\mathbf{G}}^{(w)}(0,t) = \mathbf{I}$, consequently equation (4.24) simplifies in

$$\begin{aligned}\frac{d}{dt}\mathbf{M}^{(1,1)}(t) &= \mathbf{M}^{(1,1)}(t) \otimes \big[\hat{\mathbf{Q}} + \hat{\mathbf{\Pi}}\big] \\ &+ \mathbf{M}^{(0,1)}(t) \otimes \mathbf{R}^{(h)} \otimes \mathbf{Ge}^{(h)}(t) + \mathbf{M}^{(1,0)}(t) \otimes \mathbf{R}^{(k)} \otimes \mathbf{Ge}^{(k)}(t).\end{aligned} \tag{4.25}$$

To find a closed form expression for $\mathbf{M}^{(1,1)}(t)$ it is necessary first to find explicit representations of $\mathbf{M}^{(1,0)}(t)$ and $\mathbf{M}^{(0,1)}(t)$. To this end, consider $x_k = 0$ inside equation (4.14) and compute the partial derivative with respect to $x_h$. This produces

$$\begin{aligned}\frac{\partial}{\partial x_h}\frac{\partial}{\partial t}\mathbf{F}(x_h,0;t,0) &= \frac{\partial}{\partial x_h}\mathbf{F}(x_h,0;t,0) \otimes \Bigg[\hat{\mathbf{Q}} + \hat{\mathbf{\Pi}} \\ &+ \sum_{w\in\{h,k\}} \mathbf{R}^{(w)} \otimes \big(\hat{\mathbf{G}}^{(w)}(x_w,t) - \mathbf{I}\Bigg] + \mathbf{F}(x_h,0;t,0) \otimes \mathbf{R}^{(h)} \otimes \frac{\partial}{\partial x_h}\hat{\mathbf{G}}^{(h)}(x_h,t).\end{aligned}$$

Now consider again that $\hat{\mathbf{G}}^{(k)}(0,t) = \mathbf{I}$ and let $x_h \to 0$ to have

$$\frac{d}{dt}\mathbf{M}^{(1,0)}(t) = \mathbf{M}^{(1,0)}(t) \otimes \Big[\hat{\mathbf{Q}} + \hat{\mathbf{\Pi}}\Big] + \mathbf{F}(0,0;t,0) \otimes \mathbf{R}^{(h)} \otimes \mathbf{Ge}^{(h)}(t). \tag{4.26}$$

Additionally, from Theorem 3.2 we have that

$$\mathbf{F}_{\left[\begin{smallmatrix} i & j \\ a & b\end{smallmatrix}\right]}(0,0;t,0) = \mathbb{E}_{(i,a)}[1_{\{J(t)=j,\delta(t)=b\}}] = e^{t\cdot\mathbf{\Lambda}}{}_{\left[\begin{smallmatrix} |i| & |j| \\ |a| & |b|\end{smallmatrix}\right]}$$

and by substitution in (4.26) it produces

$$\frac{d}{dt}\mathbf{M}^{(1,0)}(t) = \mathbf{M}^{(1,0)}(t) \otimes \big[\hat{\mathbf{Q}} + \hat{\mathbf{\Pi}}\big] + e^{t\cdot\mathbf{\Lambda}} \otimes \mathbf{R}^{(h)} \otimes \mathbf{Ge}^{(h)}(t), \tag{4.27}$$

from which it is possible to arrive at

$$\frac{d}{dt}\mathbf{M}^{(1,0)}(t) \otimes e^{-t\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]} = e^{t\cdot\mathbf{\Lambda}} \otimes \mathbf{R}^{(h)} \otimes \mathbf{Ge}^{(h)}(t) \otimes e^{-t\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]},$$

and consequently into

$$\mathbf{M}^{(1,0)}(t) = \int_0^t e^{y\cdot\mathbf{\Lambda}} \otimes \mathbf{R}^{(h)} \otimes \mathbf{Ge}^{(h)}(y) \otimes e^{(t-y)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]}\,dy. \tag{4.28}$$

By symmetric arguments it is possible to prove (4.19). Next, by using formulas (4.28) and (4.19) it is possible to rewrite equation (4.25) as

$$\frac{d}{dt}\mathbf{M}^{(1,1)}(t) = \mathbf{M}^{(1,1)}(t) \otimes \big[\hat{\mathbf{Q}} + \hat{\mathbf{\Pi}}\big] + \mathbf{C}(t), \tag{4.29}$$

where

$$\mathbf{C}(t) = \mathbf{M}^{(0,1)}(t) \otimes \mathbf{R}^{(h)} \otimes \mathbf{Ge}^{(h)}(t) + \mathbf{M}^{(1,0)}(t) \otimes \mathbf{R}^{(k)} \otimes \mathbf{Ge}^{(k)}(t). \tag{4.30}$$

The solution to the differential equation (4.29) is

$$\mathbf{M}^{(1,1)}(t) = \int_0^t \mathbf{C}(z) \otimes e^{(t-z)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]} dz. \tag{4.31}$$

Combining expressions (4.28), (4.19), (4.30) and (4.31) we obtain

$$\begin{aligned}\mathbf{M}^{(1,1)}(t) = \int_0^t \int_0^z \Big( & e^{y\cdot\mathbf{\Lambda}} \otimes \mathbf{R}^{(k)} \otimes \mathbf{Ge}^{(k)}(y) \otimes e^{(z-y)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]} dy \otimes \mathbf{R}^{(h)} \\ & \otimes \mathbf{Ge}^{(h)}(z) + e^{y\cdot\mathbf{\Lambda}} \otimes \mathbf{R}^{(h)} \otimes \mathbf{Ge}^{(h)}(y) \otimes e^{(z-y)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]} dy \otimes \mathbf{R}^{(k)} \\ & \otimes \mathbf{Ge}^{(k)}(z) \Big) \otimes e^{(t-z)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]} dz.\end{aligned} \tag{4.32}$$

Changing the domain of integration, the integral below can be written as

$$\begin{aligned}& \int_0^t \int_0^z \Bigg( \Big[ e^{y\cdot\mathbf{\Lambda}} \otimes \mathbf{R}^{(k)} \otimes \mathbf{Ge}^{(k)}(y) \otimes e^{(z-y)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]} \Big] dy \\ & \qquad \otimes \mathbf{R}^{(h)} \otimes \mathbf{Ge}^{(h)}(z) \otimes e^{(t-z)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]} \Bigg) dz \\ & \quad = \int_0^t dy \Bigg( e^{y\cdot\mathbf{\Lambda}} \otimes \mathbf{R}^{(k)} \otimes \mathbf{Ge}^{(k)}(y) \\ & \qquad \otimes \int_y^t e^{(z-y)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]} \otimes \mathbf{R}^{(h)} \otimes \mathbf{Ge}^{(h)}(z) \otimes e^{(t-z)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]} dz \Bigg).\end{aligned} \tag{4.33}$$

Consider now the generic $\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]$ element of the second integral in formula (4.33):

$$\begin{aligned}& \left( \int_y^t e^{(z-y)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]} \otimes \mathbf{R}^{(h)} \otimes \mathbf{Ge}^{(h)}(z) \otimes e^{(t-z)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]} dz \right)_{\left[\begin{smallmatrix} i & j \\ a & b \end{smallmatrix}\right]} \\ & \sum_{k_1,k_2,k_3\in E} \sum_{c_1,c_2,c_3\in I} \int_y^t e^{(z-y)\cdot\left[\hat{\mathbf{Q}}_{\left[\begin{smallmatrix} i & k_1 \\ a & c_1 \end{smallmatrix}\right]}+\hat{\mathbf{\Pi}}_{\left[\begin{smallmatrix} i & k_1 \\ a & c_1 \end{smallmatrix}\right]}\right]} \cdot \mathbf{R}^{(h)}_{\left[\begin{smallmatrix} k_1 & k_2 \\ c_1 & c_2 \end{smallmatrix}\right]} \\ & \cdot \mathbf{Ge}^{(h)}_{\left[\begin{smallmatrix} k_2 & k_3 \\ c_2 & c_3 \end{smallmatrix}\right]}(z) e^{(t-z)\cdot\left[\hat{\mathbf{Q}}_{\left[\begin{smallmatrix} k_3 & j \\ c_3 & b \end{smallmatrix}\right]}+\hat{\mathbf{\Pi}}_{\left[\begin{smallmatrix} k_3 & j \\ c_3 & b \end{smallmatrix}\right]}\right]} dz \\ & = \sum_{k_1,k_2,k_3\in E} \sum_{c_1,c_2,c_3\in I} \int_y^t e^{\left(q_{i,k_1} 1_{\{a=c_1\}} + \pi_{a,c_1}(i) 1_{\{i=k_1\}} 1_{\{a=c_1\}}\right)(z-y)} r^{(h)}_{k_1} 1_{\{k_1=k_2\}} \\ & \qquad \cdot 1_{\{c_1=c_2\}} \big( -g^{(h)}_{k_2,c_2} e^{-c_2 z} 1_{\{k_2=k_3\}} 1_{\{c_2=c_3\}} \big) \\ & \qquad \cdot \big( e^{(q_{k_3,j} 1_{\{c_3=b\}} + \pi_{c_3,b}(k_3) 1_{\{k_3=j\}} 1_{\{c_3=b\}})(t-z)} \big) dz\end{aligned}$$

$$= \int_y^t e^{\left(q_{i,j}1_{\{a=b\}}+\pi_{a,b}(i)1_{\{i=j\}}1_{\{a=b\}}\right)(z-y)} r_j^{(h)}\left(-g_{j,b}^{(h)}e^{-bz}\right)e^{\left(q_{j,j}+\pi_{b,b}(j)\right)}dz$$

$$= r_j^{(h)}\left(-g_{j,b}^{(h)}\right)\int_y^t e^{(z-y)\cdot\mathbf{\Lambda}\left[\begin{smallmatrix} i & j\\ a & b\end{smallmatrix}\right]}e^{-bz}e^{(t-z)\cdot\mathbf{\Lambda}\left[\begin{smallmatrix} j & j\\ b & b\end{smallmatrix}\right]}dz$$

$$= \frac{-r_j^h g_{j,b}^{(h)} e^{-y\cdot\mathbf{\Lambda}\left[\begin{smallmatrix} i & j\\ a & b\end{smallmatrix}\right]+t\cdot\mathbf{\Lambda}\left[\begin{smallmatrix} j & j\\ b & b\end{smallmatrix}\right]}}{\mathbf{\Lambda}\left[\begin{smallmatrix} i & j\\ a & b\end{smallmatrix}\right]-\mathbf{\Lambda}\left[\begin{smallmatrix} j & j\\ b & b\end{smallmatrix}\right]-b}\left(e^{(\mathbf{\Lambda}\left[\begin{smallmatrix} i & j\\ a & b\end{smallmatrix}\right]-\mathbf{\Lambda}\left[\begin{smallmatrix} j & j\\ b & b\end{smallmatrix}\right]-b)t}-e^{(\mathbf{\Lambda}\left[\begin{smallmatrix} i & j\\ a & b\end{smallmatrix}\right]-\mathbf{\Lambda}\left[\begin{smallmatrix} j & j\\ b & b\end{smallmatrix}\right]-b)y}\right).$$

A repetition of the computation on the second addend of (4.32) and a subsequent substitution of the result in (4.32) completes the proof. □

**Corollary 4.8.** *For each couple of business lines $(h,k)$, for all $t \geq 0$ it results that*

$$\mathbf{M}^{(2,0)}(t) = 2\int_0^t e^{\mathbf{\Lambda}\cdot y}\otimes\mathbf{R}^h\otimes\mathbf{Ge}^h(y)\otimes\mathbf{Int}^h(y,t)dy + \int_0^t e^{\mathbf{\Lambda}\cdot y}\otimes\mathbf{R}^h\otimes\left(\mathbf{Ge}^h(y)\right)^2\otimes e^{(t-y)\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]}dy \tag{4.34}$$

$$\mathbf{M}^{(0,2)}(t) = 2\int_0^t e^{\mathbf{\Lambda}\cdot y}\otimes\mathbf{R}^k\otimes\mathbf{Ge}^k(y)\otimes\mathbf{Int}^k(y,t)dy + \int_0^t e^{\mathbf{\Lambda}\cdot y}\otimes\mathbf{R}^k\otimes\left(\mathbf{Ge}^k(y)\right)^2\otimes e^{(t-y)\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]}dy \tag{4.35}$$

*Proof.* We only sketch the proof for reason of similarity to that of Theorem 2. Take the second order derivatives of (4.14) with respect to $x_h$, then let $x_h \to 0$ and consider that $\hat{\mathbf{G}}^{(w)}(0,t) = \mathbf{I}$ to have

$$\frac{d}{dt}\mathbf{M}^{(2,0)}(t) = \mathbf{M}^{(2,0)}(t)\otimes\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]+\mathbf{D}(t) \tag{4.36}$$

where

$$\mathbf{D}(t) = 2\mathbf{M}^{(1,0)}(t)\otimes\mathbf{R}^{(h)}\otimes\mathbf{Ge}^{(h)}(t)+e^{\mathbf{\Lambda t}}\otimes\mathbf{R}^{(h)}\otimes\left(\mathbf{Ge}^{(h)}(t)\right)^2. \tag{4.37}$$

The solution to the differential equation (4.36) is

$$\mathbf{M}^{(2,0)}(t) = \int_0^t \mathbf{D}(z)\otimes e^{(t-z)\cdot\left[\hat{\mathbf{Q}}+\hat{\mathbf{\Pi}}\right]}dz. \tag{4.38}$$

The substitution in (4.37) in (4.38) and the computation of the double integral, as already done in Theorem 2, gives the result. □

## 5. A Numerical Example

In this section we illustrate some of the results of our model in a simple case. Let us assume that the environment process $J(t)$ is a Markov chain with state space $E = \{1,2\}$ and generator

$$\mathbf{Q} = \begin{bmatrix} -0.02 & 0.02\\ 0.03 & -0.03\end{bmatrix}.$$

We assume also that there are only two business lines. The claim occurrence rates in the business line one are $r_1^1 = 8$ and $r_2^1 = 6$ whereas the claim occurrence rates in the second business line are $r_1^2 = 4$ and $r_2^2 = 2$. Therefore we aasume that the claims occur more frequently in the first business line than in the second and fixed the business line, the frequency is higher for state 1 of the environment process than for state two.

We assume that the sizes of the claims are exponentially distributed with parameter 0.1 and 0.5 in the first and second business line, respectively. In this example, we suppose also that the distributions of claim sizes do not change depending on the environment process.

The state space of the force of interest is $I = \{0.02, 0.06, 0.1\}$ and its evolution is according to the generator

$$\mathbf{\Pi(1)} = \begin{bmatrix} -1 & 1 & 0 \\ 0.5 & -1 & 0.5 \\ 0 & 1 & -1 \end{bmatrix},$$

if the environment process is in state 1. On the contrary, if the environment process is in state 2 then the generator is

$$\mathbf{\Pi(2)} = \begin{bmatrix} -1.3 & 1.3 & 0 \\ 0.8 & -1.6 & 0.8 \\ 0 & 1.3 & -1.3 \end{bmatrix}.$$

With these inputs the matrix $\mathbf{\Lambda}$ in equation (3.3) assumes the following form:

$$\begin{bmatrix} \begin{bmatrix} -1.02 & 1 & 0 \\ 0.5 & -1.02 & 0.5 \\ 0 & 1 & -1.02 \end{bmatrix} & \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.02 \end{bmatrix} \\ \begin{bmatrix} 0.03 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.03 \end{bmatrix} & \begin{bmatrix} -1.33 & 1.3 & 0 \\ 0.8 & -1.63 & 0.8 \\ 0 & 1.3 & -1.33 \end{bmatrix} \end{bmatrix}$$

Figure 1 illustrates the behavior of the conditional means of the discounted aggregate claim process as a function of time. The continuous line on the left panel indicates $\mu_{(1,1)}^{(1,0)}(t)$ for the business line 1 and the dotted line indicates $\mu_{(2,1)}^{(1,0)}(t)$ still for the business line 1. On the right panel the same results are represented for the business line 2.

The figures show that there is an ordering relation between the conditional means of $L(t)$:

$$\mu_{(1,1)}^{(1,0)}(t) > \mu_{(2,1)}^{(1,0)}(t) > \mu_{(1,1)}^{(0,1)}(t) > \mu_{(2,1)}^{(0,1)}(t).$$

The conditional means increase faster if the environment process is in state 1 and if we analyze the business line 1. The same ordering relation applies for the standard deviations of $L(t)$, this last behavior is summarized by Figure 5.

Figure 2 shows the corresponding results of Figure 1 as regard to the standard deviations of the discounted aggregate claim process as a function of time.

Finally in Figure 3 we illustrate the behavior of the conditional means of $L(t)$ for a constant force of interest $\delta = 0.0595$. The value 0.0595 is equal to the

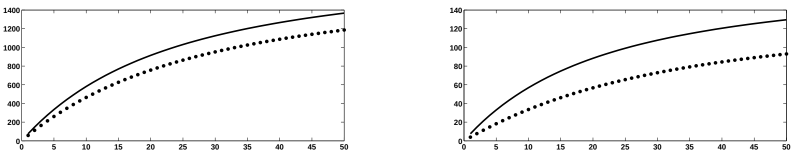


FIGURE 1. Conditional means of $L(t)$ given $i = 1, a = 1$ (continuous lines) and $i = 2, a = 1$ (dotted lines), respectively. In the left panel the conditional means refer to the business line 1 while in the right panel to the business line 2

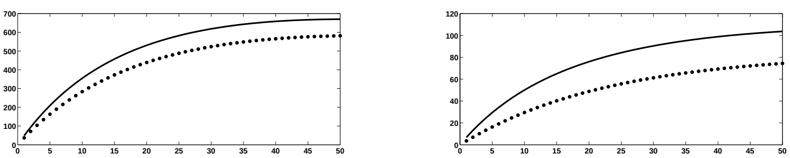


FIGURE 2. Conditional standard deviations of $L(t)$ given $i = 1, a = 1$ (continuous lines) and $i = 2, a = 1$ (dotted lines), respectively. In the left panel the conditional standard deviations refer to the business line 1 while in the right panel to the business line 2

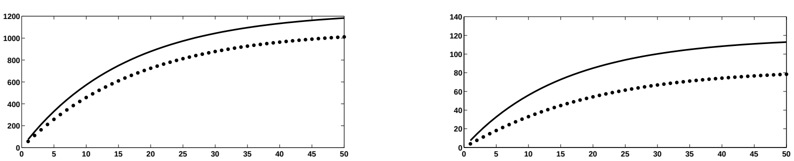


FIGURE 3. Conditional means of $L(t)$ given $i = 1, a = 1$ (continuous lines) and $i = 2, a = 1$ (dotted lines) in the case of deterministic force of interest, respectively. In the left panel the conditional means refer to the business line 1 while in the right panel to the business line 2

arithmetic mean of the values of the element of the state space $I$. A comparison of Figure 3 with Figure 1 reveals that the consideration of a constant force of interest generates an important distortion on the conditional means of $L(t)$. Indeed, as an example for $t = 50$ in the case of stochastic interest rate $\mu^{(1,0)}_{(1,1)}(50) \approx 1400$ (cf. Figure 1 continuous line left panel) whereas in the case of constant force of interest $\mu^{(1,0)}_{(1,1)}(50) \approx 1200$ (cf. Figure 3 continuous line left panel). The error is $\frac{1400-1200}{1400} \approx 14\%$. The same order of magnitude of errors is obtained also for the other cases.

## 6. Conclusion

We have determined explicit expressions for the first and second order moments of a Markov-Modulated Risk Model with Stochastic Interest Rates (MMRMSIR) when the insurance company works on several dependent business lines. We have generalized the results obtained by Kim and Kim (Insurance: Mathematics and Economics 40, 485-497, 2007). By using the technology of 2-dimensional matrices, we have shown how it is possible to manage our more complex framework in a natural way. The theoretical results have been applied to simulated data to show the applicability of the model.

Our paper leaves many open issues. For instance:

i) the execution of a real life application;

ii) the investigation of the ruin probabilities in the (MMRMSIR);

iii) the extension to a semi-Markov-modulated risk model.

These will be objects of investigation of our future works.

## 7. Appendix: 2-dimensional Matrices

Multidimensional matrices were defined by [27] and [22]. More recent results and applications are available in [17] and [23], [5] and [6].

In this appendix, following the approach by [22], we expose definitions and notations we used in the previous focusing only on 2-dimensional matrices.

**Definition 7.1.** Let $C$ be a field, a *matrix of dimension zero* is one element of the field.

**Definition 7.2.** A *1-dimensional matrix* $A_{[r_1 \ c_1]}$ is a matrix whose elements are 0-dimensional. $r_1$ and $c_1$ denote respectively the row and column orders of the matrix.

Inductively it is possible to define a 2-dimensional matrix as:

**Definition 7.3.** A *2-dimensional matrix* $A_{[\mathbf{r} \ \mathbf{c}]}$ is a matrix whose elements are 1-dimensional matrices. The vectors $\mathbf{r} = \left[\begin{smallmatrix} r_1 \\ r_2 \end{smallmatrix}\right]$ and $\mathbf{c} = \left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right]$ denote the numbers of rows and columns at the two dimension levels, respectively.

The vectors $\mathbf{r}$ and $\mathbf{c}$ are said row order and column order of the 2-dimensional matrix $\mathbf{A}$.

Given two 2-dimensional matrices $A_{[\mathbf{r} \ \mathbf{a}]}$ and $B_{[\mathbf{b} \ \mathbf{c}]}$ where $\mathbf{a} = \mathbf{b}$, their product $C_{[\mathbf{r} \ \mathbf{c}]} = A_{[\mathbf{r} \ \mathbf{a}]} \otimes A_{[\mathbf{b} \ \mathbf{c}]}$ is the 2-dimensional matrix of order $\mathbf{r}$, $\mathbf{c}$ whose simple element is

$$C_{\left[\begin{smallmatrix} i_1 & j_1 \\ i_2 & j_2 \end{smallmatrix}\right]} = \sum_{d_1=1}^{a_1} \sum_{d_2=1}^{a_2} A_{\left[\begin{smallmatrix} i_1 & d_1 \\ i_2 & d_2 \end{smallmatrix}\right]} \cdot B_{\left[\begin{smallmatrix} d_1 & j_1 \\ d_2 & j_2 \end{smallmatrix}\right]}.$$

Here $i_1 \in \{1, \ldots, r_1\}$, $i_2 \in \{1, \ldots, r_2\}$, $j_1 \in \{1, \ldots, c_1\}$, $j_2 \in \{1, \ldots, c_2\}$.

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