

THE GAUSSIAN RADON TRANSFORM IN CLASSICAL WIENER SPACE*

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ABSTRACT. We study the Gaussian Radon transform in the classical Wiener space of Brownian motion. We determine explicit formulas for transforms of Brownian functionals specified by stochastic integrals. A Fock space decomposition is also established for Gaussian measure conditioned to closed affine subspaces in Hilbert spaces.

1. Introduction

The Gaussian Radon transform $G\phi$ of a function ϕ on a linear space associates to each closed affine subspace L the integral of ϕ with respect to a Gaussian measure on L . In this paper we study this transform for the classical Wiener space, which is the Banach space of continuous paths starting at a given basepoint. We also establish a Fock space decomposition in this context, expanding suitable functions ϕ in terms of symmetric tensors.

The classical Radon transform [14] associates to a function ϕ on Euclidean space \mathbb{R}^n its integrals over hyperplanes. An infinite dimensional analog of this transform was introduced in Mihai and Sengupta [13] (an early study of this type was carried out by Hertle [7]). Further development of this theory, including the establishment of counterparts of the classical Helgason support theorem [6], has been carried out in the works of Becnel [1, 2], Bogachev and Lukintsova [4, 12], and by us [8]. A comprehensive account of Gaussian measures on infinite dimensional spaces is presented in Bogachev [3]. The theory of abstract Wiener spaces, originating in the work of Gross [5], is presented in the monograph of Kuo [10] as well as in many other works.

The present paper is organized as follows. We begin in section 2 with a self-contained description of the Gaussian Radon transform in Banach spaces and introduce necessary notation and concepts. In infinite dimensions it is Gaussian measure μ that needs to be used as there is no useful notion of Lebesgue measure. An important issue is that $G\phi$ depends on the pointwise defined function ϕ and not simply its μ -almost-everywhere specified representative. For any bounded continuous function ϕ the transform $G\phi$ is meaningful without further specifications.

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In section 2 we also present a more detailed discussion of the case of affine subspaces of finite codimension. In section 3 we present a Fock space decomposition for functions with respect to Gaussian measure on closed affine subspaces. We turn next in section 4 to a description of the classical Wiener space, formulated in a manner useful for our objectives, along with the Gaussian Radon transform in this setting. In section 4 we apply the machinery developed to work out Gaussian Radon transforms of some specific functions on the classical Wiener space, such as those given by iterated Itô integrals.

2. The Gaussian Radon Transform

In this section we present a description of the Gaussian Radon transform, referring to [8] for further details and proofs. We begin with a review of the framework of an abstract Wiener space.

2.1. The Gaussian measure μ . We work with a real separable Banach space B , with norm denoted $|\cdot|$, and a Borel probability measure μ on B for which the distribution of every continuous linear functional $x^* : B \rightarrow \mathbb{R}$ is Gaussian with mean 0, the variance being positive if x^* is not zero. The characteristic function of x^* is therefore given by

$$\int_B e^{itx^*} d\mu = e^{-\frac{t^2}{2}\|x^*\|_{L^2(\mu)}^2}, \quad (2.1)$$

for all $t \in \mathbb{R}$, because the variance of x^* is $\mathbb{E}_\mu[x^{*2}]$, which is the L^2 -norm squared. By a Cameron-Martin space for (B, μ) we shall mean a Hilbert space H along with a linear isometry

$$I : H \rightarrow L^2(B, \mu)$$

whose image contains B^* , the set of all continuous linear functionals on B , as a dense subspace. For example, H could be the closure of B^* in $L^2(B, \mu)$, with I being the inclusion map. We use the notation

$$H_{B^*} = \{h \in H : I(h) \in B^*\}. \quad (2.2)$$

By assumption, the image $I(H_{B^*}) = B^*$ is dense in the range of the linear isometry I , and so H_{B^*} is dense in H . For $h \in H$ we can choose a sequence $x_n^* \in B^*$ converging to $I(h)$ in $L^2(\mu)$ -norm; the characteristic function of x_n^* is given by (2.1) with $x^* = x_n^*$, and then by using dominated convergence we have

$$\int_B e^{itI(h)} d\mu = e^{-\frac{t^2}{2}\|I(h)\|_{L^2(\mu)}^2}, \quad (2.3)$$

for all $t \in \mathbb{R}$. Thus $I(h)$ is Gaussian with mean 0 and variance $\|h\|^2$, for every $h \in H$. There is also a continuous linear injective map

$$i : H \rightarrow B : h \mapsto \int_B (Ih)(\omega)\omega d\mu(\omega), \quad (2.4)$$

where the integrand is a B -valued function and the integral is a Bochner integral. In particular

$$\langle x^*, i(h) \rangle = \int_B Ih(\omega)x^*(\omega) d\mu(\omega), \quad (2.5)$$

for all $x^* \in B^*$ and $h \in H$. Taking $x^* = I(h_1)$, where $h_1 \in H_{B^*}$, we obtain

$$\langle I(h_1), i(h) \rangle = \langle I(h_1), I(h) \rangle_{L^2(\mu)} = \langle h_1, h \rangle_H \tag{2.6}$$

for all $h \in H$, where on the left we have the evaluation of $I(h_1) \in B^*$ on $i(h)$ and in the second expression on the right we have stressed through the subscript that the inner product is in H . In particular, if $i(h) = 0$ then h is orthogonal to every $h_1 \in H_{B^*}$, and so, since H_{B^*} is dense in H , $h = 0$. Thus the mapping i is injective.

Another perspective on H_{B^*} is also useful: an element $h_1 \in H$ lies in the subspace H_{B^*} if and only if the mapping

$$H \rightarrow \mathbb{R} : h \mapsto \langle h, h_1 \rangle$$

is continuous with respect to the norm $|\cdot|$ on H obtained by the identification of H with the subspace $i(H)$ of B .

The system $i : H \rightarrow B$, along with the Gaussian measure μ on B , is called an *abstract Wiener space*. This notion was introduced by Gross [5].

2.2. The measure on affine subspaces. A closed affine subspace of H is of the form

$$L = p + L_0,$$

where L_0 is a closed subspace of H and p a point H . In [8, Theorem 2.1] we have constructed, for every closed affine subspace L of H , a Borel probability measure μ_L on B such that for any $h \in H_{B^*}$

$$\int_B e^{iI(h)} d\mu_L = e^{i\langle h_L, h \rangle - \frac{1}{2}\|P_{L_0}h\|^2} \tag{2.7}$$

where h_L is the point on L closest to $0 \in H$ and

$$P_{L_0} : H \rightarrow H \tag{2.8}$$

is the orthogonal projection onto the closed subspace

$$L_0 = -h_L + L. \tag{2.9}$$

The measure μ_L is concentrated on the closure \bar{L} of L in the Banach space B :

$$\mu_L(\bar{L}) = 1. \tag{2.10}$$

(This is also proved in [8, Theorem 2.1].) One other fact to note here is the nature of the closure \bar{L} . Suppose $L_0 = \ker f$ where $f \in H^*$ is the restriction to H of an element $f_B \in B^*$; then

$$\bar{L}_0 = \ker f_B. \tag{2.11}$$

If f is not continuous with respect to the norm on B then $\bar{L} = B$. This is [8, Proposition 5.2].

2.3. The mapping I_L . Let L be a closed affine subspace of H . The characteristic function of a Gaussian variable X is

$$\mathbb{E}[e^{itX}] = e^{it\mathbb{E}[X] - \frac{t^2}{2}\text{var}(X)},$$

where $\text{var}(X)$ is its variance. Then from (2.7), with th in place of h , we see that when $h \in H_{B^*}$ the random variable $I(h)$ has Gaussian distribution with respect to μ_L , having mean and variance

$$\mathbb{E}_{\mu_L}(I(h)) = \langle h_L, h \rangle \quad \text{and} \quad \text{Var}_{\mu_L}(I(h)) = \|P_{L_0}h\|^2. \quad (2.12)$$

Consider now the mapping

$$I_L^0 : H_{B^*} \rightarrow L^2(B, \mu_L) : h \mapsto I(h). \quad (2.13)$$

Although it appears to be just the mapping I , there is in fact a significant difference because here $I(h)$ is taken as an element of $L^2(\mu_L)$ (thus, for example, $I_L(h)$ might be 0 even if $I(h)$ is not 0). Clearly, I_L^0 is linear, and, moreover,

$$\|I_L^0(h)\|^2 = \mathbb{E}_{\mu_L}(I_L^0(h)^2) = \text{Var}_{\mu_L}(I(h)) + \left(\mathbb{E}_{\mu_L}(I(h))\right)^2 \leq (1 + \|h_L\|^2)\|h\|^2 \quad (2.14)$$

for all $h \in H_{B^*}$. Thus I_L^0 , being continuous with respect to the norm of H , extends to a continuous linear mapping

$$I_L : H \rightarrow L^2(B, \mu_L). \quad (2.15)$$

In an intuitive sense, $I_L(h)$ is the “restriction” of $I(h)$ to the closed affine subspace \overline{L} , the closure of L in B . Let us note again, however, that for a general element $h \in H$ the random variable

$$I(h)$$

is an element of $L^2(\mu)$, thus defined almost everywhere with respect to μ , and cannot generally be “restricted” to \overline{L} . For $h \in H$, the random variable $I_L(h)$ has characteristic function given by

$$\int_B e^{iI_L(h)} d\mu_L = e^{i\langle h_L, h \rangle - \frac{1}{2}\|P_{L_0}h\|^2}, \quad (2.16)$$

with h_L being as before, as can be seen by applying (2.7) to a sequence of elements $h_n \in H_{B^*}$ converging in H to h and evaluating the limit using the dominated convergence theorem.

As with (2.12), we see that $I_L(h)$ is Gaussian and

$$\mathbb{E}_{\mu_L}(I_L(h)) = \langle h_L, h \rangle \quad \text{and} \quad \text{Var}_{\mu_L}(I_L(h)) = \|P_{L_0}h\|^2. \quad (2.17)$$

(The difference between this and (2.12) is that now h is any element of H , not restricted to being in H_{B^*} .) In particular, *if h is orthogonal to L (more precisely, to the subspace $L - h_L$) then $I_L(h)$ has zero variance and is thus just the constant $\langle h, h_L \rangle$.*

From the variance formula in (2.17) with $h = th_1 + sh_2$, where $h_1, h_2 \in H$ and $t, s \in \mathbb{R}$, it follows that the covariance of $I_L(h_1)$ and $I_L(h_2)$ is given by

$$\text{Cov}(I_L(h_1), I_L(h_2)) = \langle P_{L_0}h_1, P_{L_0}h_2 \rangle. \quad (2.18)$$

As with any Gaussian we have

$$\int_B e^{zI_L(h)} d\mu = e^{z\langle h_L, h \rangle + \frac{z^2}{2} \|P_{L_0} h\|^2}, \quad (2.19)$$

for all $h \in H$ and all $z \in \mathbb{C}$.

2.4. The Gaussian Radon transform $G\phi$. Consider a measurable function ϕ on B . The Gaussian Radon transform $G\phi$ is a function that associates to a closed affine subspace $L \subset H$, the expectation of ϕ with respect to the measure μ_L :

$$G\phi|_L \stackrel{\text{def}}{=} G\phi(L) \stackrel{\text{def}}{=} \int_B \phi d\mu_L. \quad (2.20)$$

Thus the domain of $G\phi$ is the set of all L for which this integral is finite. If ϕ is a bounded Borel function then $G\phi$ is defined on the set of all closed affine subspaces.

We observe that G is defined for a pointwise specified function, and not on $L^2(\mu)$. For example, a function ψ might be 0 almost everywhere with respect to μ but $G\psi$ need not be 0.

As a special case consider $\phi = I(h)$ where $h \in H_{B^*}$; let us note that this means that $I(h)$ is a continuous function on B . From the formula (2.17) we conclude that the Gaussian Radon transform of $I(h)$ is

$$GI(h)|_L = \langle h_L, h \rangle \quad (2.21)$$

where h_L is the point on the closed affine subspace L that is of smallest norm.

2.5. Conditioning a function to an affine subspace. As we have seen above, for each vector $h \in H$ there is a function $I(h) \in L^2(\mu)$ and also a function $I_L(h) \in L^2(\mu_L)$. If $I_L(h) = 0$ then $I_L(th) = 0$ for all $t \in \mathbb{R}$ and so by (2.16), $P_{L_0} h = 0$ and $\langle h_L, h \rangle = 0$, and so h is a vector in L_0^\perp that is orthogonal to h_L . On the other hand, if $I(h) = 0$ then $h = 0$ (because $I : H \rightarrow L^2(\mu)$ is an isometry) and so $I_L(h) = 0$. Thus there is a continuous linear map

$$I_L \circ I^{-1} : L^2(\mu)_0 \rightarrow L^2(\mu_L) : I(h) \mapsto I_L(h) \quad (2.22)$$

where $L^2(\mu)_0$ is the image of I inside $L^2(\mu)$. This provides a way of obtaining functions in $L^2(\mu_L)$ from functions in $L^2(\mu)_0$.

2.6. The point h_L for finite codimension subspaces. Consider a closed affine subspace of H that is specified as a level surface of a finite number of linear functionals; specifically, consider the affine subspace $L_c \subset H$ given by

$$L_c = \{h \in H : \langle v_1, h \rangle = c_1, \dots, \langle v_m, h \rangle = c_m\}, \quad (2.23)$$

where $v_1, \dots, v_m \in H$ and $c = (c_1, \dots, c_m) \in \mathbb{R}^m$. (Such subspaces were used in the context of machine learning in our work [9], where formulas of the type discussed below in Proposition 2.1 were obtained.) Then we have the following result in linear algebra.

Proposition 2.1. *Let H be a real Hilbert space and $F : H \rightarrow \mathbb{R}^m$ a linear map given by*

$$F(h) = \left(\langle v_1, h \rangle, \dots, \langle v_m, h \rangle \right) \quad (2.24)$$

where v_1, \dots, v_m are linearly independent vectors in H and $c = (c_1, \dots, c_m) \in \mathbb{R}^m$. Then the mapping $FF^* : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is invertible and the point on the closed affine subspace $L_c = F^{-1}(c)$ closest to the origin 0 is

$$h_{L_c} = F^*(FF^*)^{-1}(c). \quad (2.25)$$

The matrix for FF^* is

$$FF^* = \begin{bmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_m, v_1 \rangle \\ \vdots & & \vdots \\ \langle v_1, v_m \rangle & \cdots & \langle v_m, v_m \rangle \end{bmatrix}, \quad (2.26)$$

and the point h_{L_c} can be expressed as

$$h_{L_c} = \sum_{k=1}^m a_k v_k \quad (2.27)$$

where

$$a_k = \sum_{j=1}^m [(FF^*)^{-1}]_{kj} c_j. \quad (2.28)$$

If the vectors v_j are orthonormal then

$$h_{L_c} = F^*(c) = \sum_{k=1}^m c_k v_k. \quad (2.29)$$

Proof. For any $b \in \mathbb{R}^m$ and $w \in H$ we have:

$$\langle b, F(w) \rangle_{\mathbb{R}^m} = \sum_{k=1}^m b_k \langle v_k, w \rangle = \left\langle \sum_{k=1}^m b_k v_k, w \right\rangle \quad (2.30)$$

Hence the adjoint $F^* : \mathbb{R}^m \rightarrow H$ is given by

$$F^*(b) = \sum_{k=1}^m b_k v_k, \quad (2.31)$$

for all $b \in \mathbb{R}^m$. In particular, taking $b = e_j$, the j -th standard basis vector in \mathbb{R}^m we have

$$F^*(e_j) = v_j. \quad (2.32)$$

Since v_1, \dots, v_m are linearly independent we see that $\ker F^* = \{0\}$ which implies also that FF^* is invertible because any vector b that lies in $\ker(FF^*)$ satisfies

$$\|F^*b\|^2 = \langle b, FF^*b \rangle = 0.$$

Thus the definition of the vector h_{L_c} given in (2.25) is meaningful. The definition shows directly that h_{L_c} lies in $F^{-1}(c)$. Furthermore, if w is any point on $F^{-1}(c)$ then

$$\langle w, h_{L_c} \rangle = \langle F(w), (FF^*)^{-1}(c) \rangle_{\mathbb{R}^m} = \langle c, (FF^*)^{-1}(c) \rangle,$$

which, being independent of the choice of w is equal to the value obtained on taking w to be the point h_{L_c} ; consequently

$$\langle w - h_{L_c}, h_{L_c} \rangle = 0.$$

Thus h_{L_c} is a point on L_c that is orthogonal to all vectors in the subspace $L_c - h_{L_c}$. This implies that the vector h_{L_c} forms the perpendicular from the origin onto the affine subspace L_c . Hence h_{L_c} is the point on $F^{-1}(c)$ of smallest norm (which may also be seen by noting that $\|h_{L_c}\|^2 = \langle w, h_{L_c} \rangle \leq \|w\| \|h_{L_c}\|$ for all $w \in L_c$, from which it follows that $\|h_{L_c}\| \leq \|w\|$ for such w .)

To express things in coordinates we apply F to $F^*(b)$ as given in (2.31) to obtain

$$FF^*(b) = \sum_{k=1}^m b_k \left(\sum_{j=1}^m \langle v_k, v_j \rangle e_j \right) = \sum_{j=1}^m \left(\sum_{k=1}^m \langle v_k, v_j \rangle b_k \right) e_j. \quad (2.33)$$

Thus the (j, k) -th entry of the matrix for FF^* is $\langle v_k, v_j \rangle$. Using (2.25) we have

$$h_{L_c} = F^*(FF^*)^{-1}(c) = \sum_{j,k=1}^m [(FF^*)^{-1}]_{kj} c_j F^*(e_k) \quad (2.34)$$

and this establishes the expression (2.27) for h_{L_c} upon using $F^*(e_k) = v_k$ as observed in (2.32).

Finally, if the vectors v_j are orthonormal then $FF^* = I$ and then the expression for h_{L_c} reduces to $F^*(c)$. \square

The explicit formula for h_{L_c} given above makes it possible for us to write down a formula for the Gaussian Radon transform of $I(h)$ for $h \in H_{B^*}$:

$$\begin{aligned} GI(h)|_{L(c)} &= \langle h_{L_c}, h \rangle \\ &= \langle (FF^*)^{-1}(c), F(h) \rangle_{\mathbb{R}^m} \\ &= \sum_{j,k=1}^m [(FF^*)^{-1}]_{jk} c_k \langle v_j, h \rangle. \end{aligned} \quad (2.35)$$

Here we have used the formula (2.21) for $GI(h)|_L$ as well as the expression for h_{L_c} from (2.29). In the special case where the vectors v_j are orthonormal this simplifies further to

$$GI(h)|_{L(c)} = \sum_{k=1}^m c_k \langle v_k, h \rangle. \quad (2.36)$$

3. Fock Space Decomposition

There is a well known unitary isomorphism between L^2 of a Gaussian space and a completed tensor algebra. In this section we construct such an isomorphism for the conditioned measure μ_L . We work with a centered Gaussian measure μ on a separable real Banach space B , as discussed before, and H is a Cameron-Martin Hilbert space for (B, μ) .

3.1. Tensor powers. The algebraic tensor power

$$\otimes_{\text{alg}}^n H$$

which is a vector space spanned by vectors of the form $v_1 \otimes \dots \otimes v_n$ with $v_1, \dots, v_n \in H$; in slightly greater details, if $\{b_\alpha\}_{\alpha \in I}$ is a vector space basis of H then the abstract expressions $b_{\alpha_1} \otimes \dots \otimes b_{\alpha_n}$, with $\alpha_1, \dots, \alpha_n$ running over I , form a basis

of the vector space $\otimes_{\text{alg}}^n H$. On this vector space there is a unique inner product specified by the requirement that

$$\langle v_1 \otimes \dots \otimes v_n, w_1 \otimes \dots \otimes w_n \rangle_n = \prod_{j=1}^n \langle v_j, w_j \rangle. \quad (3.1)$$

The completion of $\otimes_{\text{alg}}^n H$ with respect to this inner product is the Hilbert space

$$\otimes^n H,$$

which is the n -th tensor power of H in the Hilbert-space sense. Let S_n denote the symmetric group S_n of all permutations on $\{1, \dots, n\}$. Then for each $\sigma \in S_n$ there is a unitary isomorphism U_σ on $H^{\otimes n}$ specified by:

$$U_\sigma : H^{\otimes n} \rightarrow H^{\otimes n} : v_1 \otimes \dots \otimes v_n \mapsto v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(n)}.$$

The set of all $x \in H^{\otimes n}$ that satisfy

$$U_\sigma x = x \quad \text{for all } \sigma \in S_n$$

is a closed subspace of $H^{\otimes n}$ called the n -th *symmetric tensor power* of H ; it is denoted

$$H^{\hat{\otimes} n}.$$

In the case $n = 0$ we take this space to be just \mathbb{R} with its usual inner-product.

3.2. Symmetric Fock space. Next we consider the *symmetric tensor algebra*, which is the vector space direct sum

$$S(H) = \sum_{n \geq 0} H^{\hat{\otimes} n}. \quad (3.2)$$

Thus each element of $S(H)$ is of the form

$$\sum_{n \geq 0} x_n,$$

where $x_n \in H^{\hat{\otimes} n}$ and all except finitely many x_n are 0.

On $S(H)$ there are different natural choices for an inner product; we work with the inner product specified by

$$\left\langle \sum_{n \geq 0} x_n, \sum_{m \geq 0} y_m \right\rangle = \sum_{n \geq 0} n! \langle x_n, y_n \rangle_n, \quad (3.3)$$

wherein $x_n, y_n \in H^{\hat{\otimes} n}$. The completion of $S(H)$ with respect to this inner product is a Hilbert space called the *symmetric Fock space* over H and we will denote it by

$$\mathcal{F}_s(H).$$

Finally, we can consider the complexification of this, which is a complex Hilbert space that we denote by

$$\mathcal{F}_s(H)_c.$$

3.3. Exponential vectors. Inside this Hilbert space are certain useful vectors called *exponential vectors* or *coherent state vectors*; these are of the form

$$\text{Exp}(v) = \sum_{n=0}^{\infty} \frac{1}{n!} v^{\otimes n}, \tag{3.4}$$

for any $v \in H$. We observe that

$$\langle \text{Exp}(v), \text{Exp}(w) \rangle = \sum_{n=0}^{\infty} n! \frac{1}{n!^2} \langle v^{\otimes n}, w^{\otimes n} \rangle_n = e^{\langle v, w \rangle} \tag{3.5}$$

for all $v, w \in H$. The other essential fact is that the linear span of the vectors $\text{Exp}(v)$ is dense in $\mathcal{F}_s(H)$.

3.4. Fock space decomposition. Now let us bring in the Banach space B with centered Gaussian measure μ , for which H is a Cameron-Martin space. There is a fundamental unitary isomorphism

$$U : \mathcal{F}_s(H)_c \rightarrow L^2(B, \mu) \tag{3.6}$$

which is uniquely specified by the requirement that it map $\text{Exp}(v)$ to a multiple of the exponential function $e^{I(v)}$; more specifically

$$U(\text{Exp}(h)) = e^{I(h) - \frac{1}{2}\|h\|^2} \quad \text{for all } h \in H. \tag{3.7}$$

The surjectivity of U follows from the fact that the random variables of the form $e^{I(h)}$ span a dense subspace of $L^2(B, \mu)$. The relation (3.5) implies that Exp is an isometry.

3.5. Fock space decomposition for μ_L . Now let L be a closed affine subspace of H . We denote by h_L the point on L closest to the origin

$$\|h_L\| = \inf\{\|v\| : v \in L\}. \tag{3.8}$$

The vector h_L is orthogonal to L ; more precisely,

$$h_L \in L_0^\perp. \tag{3.9}$$

We have then the following result on Fock space decomposition of $L^2(B, \mu_L)$:

Theorem 3.1. *Let H be a Cameron-Martin Hilbert space associated with the separable real Banach space B equipped with a centered Gaussian measure μ . Let L be a closed affine subspace of H , h_L the point on L closest to the origin, and L_0 the closed subspace of H given by*

$$L_0 = L - h_L. \tag{3.10}$$

Let μ_L be the probability measure on B given by (2.7) and $I_L : H \rightarrow L^2(B, \mu_L)$ the continuous linear map given in (2.15). Then there is a unique unitary isomorphism

$$U_L : \mathcal{F}_s(L_0)_c \rightarrow L^2(B, \mu_L) \tag{3.11}$$

for which

$$U_L(\text{Exp}(v)) = e^{I_L(v) - \frac{1}{2}\|v\|^2} \quad \text{for all } v \in L_0. \tag{3.12}$$

Taking $v = 0$ we see that U_L maps the ‘tensor’ $1 \in \mathbb{C} \subset \mathcal{F}_s(L_0)_c$ to the constant function $1 \in L^2(B, \mu_L)$. Moreover, replacing v by tv and ‘comparing the coefficient’ of t on both sides we have

$$U_L(v) = I_L(v) \quad \text{for all } v \in L_0, \quad (3.13)$$

showing that U_L reduces to the mapping I_L on $L_0 \subset \mathcal{F}_s(L_0)_c$.

Proof. Let us verify that U_L preserves inner-products for the coherent vectors $\text{Exp}(v)$. For any v and w in the closed subspace $L_0 \subset H$ we have

$$\begin{aligned} & \int_B e^{I_L(v) - \frac{1}{2}\|v\|^2} e^{I_L(w) - \frac{1}{2}\|w\|^2} d\mu_L \\ &= e^{-\frac{1}{2}(\|v\|^2 + \|w\|^2)} \int_B e^{I_L(v+w)} d\mu_L \\ &= e^{-\frac{1}{2}(\|v\|^2 + \|w\|^2) + \langle v+w, h_L \rangle + \frac{1}{2}\|P_{L_0}(v+w)\|^2} \\ & \quad \text{(using the Gaussian formula (2.19))} \\ &= e^{-\frac{1}{2}(\|v\|^2 + \|w\|^2) + \frac{1}{2}\|v+w\|^2} \\ & \quad \text{(since } v, w \in L_0, \text{ and } h_L \in L_0^\perp; \text{ also } P_{L_0}v = v \text{ and } P_{L_0}w = w) \\ &= e^{\langle v, w \rangle}, \end{aligned} \quad (3.14)$$

and this is equal to the inner product

$$\langle \text{Exp}(v), \text{Exp}(w) \rangle = e^{\langle v, w \rangle}.$$

This shows that U_L , specified on the vectors of the form $\text{Exp}(v)$ by (3.12), is well-defined as an isometric linear mapping on the complex linear span of $\{\text{Exp}(v) : v \in H\}$ in $\mathcal{F}_s(H)_c$. Thus, since this subspace is dense in $\mathcal{F}_s(H)_c$, the mapping U_L extends uniquely to a continuous linear mapping $\mathcal{F}_s(H)_c \rightarrow L^2(B, \mu_L)$. The functions $e^{I_L(v)}$ span a dense subspace in $L^2(B, \mu_L)$ and so U_L maps $\mathcal{F}_s(H)_c$ onto $L^2(B, \mu_L)$. \square

4. The Classical Wiener Space and the Conditioned Measure

In this section we review facts about the classical Wiener space and then extend some of the considerations to the measure μ_L . For more details and further development of the theory we refer to the works of Gross [5] and Kuo [10].

4.1. The classical Wiener space. The Banach space here is

$$\mathcal{C}_0 = C_0[0, 1],$$

the space of continuous functions $\omega : [0, 1] \rightarrow \mathbb{R}$, with initial value $\omega(0) = 0$, with the norm being the supremum norm, and the measure μ being standard Wiener measure. In this section we shall review standard facts with a view to use in the context of the Gaussian Radon transform. For each $t \in [0, 1]$ we have the evaluation map:

$$B_t : \mathcal{C}_0 \rightarrow \mathbb{R} : \omega \mapsto B_t(\omega) = B(t; \omega) = \omega(t). \quad (4.1)$$

The process

$$t \mapsto B_t$$

is standard Brownian motion under the measure μ . This means that each B_t is a Gaussian random variable of mean 0 and variance t , and for any $t_1, \dots, t_n \in [0, 1]$ the variable $(B_{t_1}, \dots, B_{t_n})$ is an \mathbb{R}^n -valued Gaussian, and

$$\mathbb{E}_\mu(B_s B_t) = \min\{s, t\} \quad \text{for all } s, t \in [0, 1]. \quad (4.2)$$

4.2. Reproducing kernel. A convenient choice for the Cameron-Martin space \mathcal{H}_0 is given as follows. For each $s \in [0, 1]$ let K_s be the function on $[0, 1]$ given by

$$K_s : [0, 1] \rightarrow \mathbb{R} : t \mapsto K_s(t) \stackrel{\text{def}}{=} \min\{s, t\} = \int_0^t 1_{[0, s]}(u) du = \langle 1_{[0, s]}, 1_{[0, t]} \rangle_{L^2([0, 1])} \quad (4.3)$$

and let

$$\mathcal{H}_{00} = \text{linear span of } \{K_s : s \in [0, 1]\}.$$

It is useful to observe that the derivative \dot{K}_s equals 1 on $[0, s)$ and 0 on $(s, 1]$:

$$\dot{K}_s = 1_{[0, s]} \quad \text{everywhere on } [0, 1] \text{ except at } s = t. \quad (4.4)$$

On the vector space \mathcal{H}_{00} we have the inner product

$$\langle h, k \rangle = \int_0^1 \dot{h}(t) \dot{k}(t) dt. \quad (4.5)$$

Then

$$\langle K_s, K_t \rangle = \min\{s, t\} \quad (4.6)$$

for all $s, t \in [0, 1]$. Thus

$$\langle K_s, K_t \rangle = K_t(s),$$

and so, by taking linear combinations,

$$\langle K_s, x \rangle = x(s) \quad \text{for all } x \in \mathcal{H}_{00} \text{ and } s \in [0, 1]. \quad (4.7)$$

Thus evaluation at s is a linear functional on \mathcal{H}_{00} that is given by inner-product against the function K_s . The relation (4.7) is a *reproducing kernel* relation, with the function

$$(s, t) \mapsto \min\{s, t\}$$

being the reproducing kernel function. Let us also note that point-evaluation is continuous on \mathcal{H}_{00} :

$$|x(s)| = |\langle K_s, x \rangle| \leq \|K_s\| \|x\| = s^{1/2} \|x\| \leq \|x\| \quad (4.8)$$

for all $x \in \mathcal{H}_{00}$; in fact,

$$|x|_{\text{sup}} \leq \|x\|. \quad (4.9)$$

4.3. Cameron-Martin Hilbert space. We take as the Cameron-Martin Hilbert space the completion of \mathcal{H}_{00} :

$$\mathcal{H}_0 = \overline{\mathcal{H}_{00}}. \quad (4.10)$$

We shall now examine the nature of the elements of \mathcal{H}_0 . Suppose (x_n) is a Cauchy sequence in \mathcal{H}_0 . Thus,

$$\|\dot{x}_n - \dot{x}_m\|_{L^2[0,1]} = \|x_n - x_m\| \rightarrow 0$$

as $n, m \rightarrow \infty$, and so there is a function $y \in L^2[0, 1]$ for which

$$x_n \rightarrow y \quad \text{in } L^2[0, 1].$$

From the bound (4.9) it follows that (x_n) is uniformly Cauchy and so converges uniformly to a continuous function $x : [0, 1] \rightarrow \mathbb{R}$; furthermore, taking $n \rightarrow \infty$ in

$$x_n(s) = \langle K_s, \dot{x}_n \rangle = \int_0^s x_n(s) ds,$$

we have

$$x(s) = \langle K_s, y \rangle = \int_0^s y(u) du \quad \text{for all } s \in [0, 1]. \quad (4.11)$$

Then x is absolutely continuous and, by Lebesgue's theorem, x is differentiable almost everywhere with $x' = y$ almost everywhere. The Hilbert space \mathcal{H}_0 consists of all absolutely continuous functions $x : [0, 1] \rightarrow \mathbb{R}$, with initial value 0 and with $\int_0^1 \dot{x}(t)^2 dt < \infty$, where the derivative \dot{x} is defined almost everywhere.

Finally let us note that the reproducing kernel relation (4.7) extends by continuity to

$$\langle K_s, x \rangle = x(s) \quad \text{for all } x \in \mathcal{H}_0 \text{ and } s \in [0, 1]. \quad (4.12)$$

4.4. The Paley-Wiener map I . The mapping I is given by the Wiener integral

$$I : \mathcal{H}_0 \rightarrow L^2(\mathcal{C}_0, \mu) : h \mapsto I(h) = \int_0^1 \dot{h}(t) dB_t. \quad (4.13)$$

This is best understood in our context by examining the values of I on the special elements $K_s \in \mathcal{H}_{00}$:

$$I(K_s) = B_s \in \mathcal{C}_0^*, \quad (4.14)$$

from which we have

$$\begin{aligned} \langle I(K_s), I(K_t) \rangle_{L^2(\mu)} &= \mathbb{E}_\mu[B_s B_t] \\ &= \min\{s, t\} \quad (\text{by (4.2)}) \\ &= \langle K_s, K_t \rangle_H. \end{aligned} \quad (4.15)$$

4.5. The imbedding i . We also need to understand the imbedding

$$i : \mathcal{H}_0 \rightarrow \mathcal{C}_0 : h \mapsto i(h) = \int_{\mathcal{C}_0} \langle h, \omega \rangle \omega d\mu(\omega).$$

For any $t \in [0, 1]$ we have

$$\begin{aligned}
 h(t) &= \int_0^1 1_{[0,t]}(s) \dot{h}(s) ds \\
 &= \langle K_t, h \rangle_E \\
 &= \langle I(K_t), i(h) \rangle \\
 &\quad \text{(evaluating } I(K_t) \in \mathcal{C}_0^* \text{ on } i(h) \in \mathcal{C}_0 \text{, using (2.6))} \\
 &= \langle B_t, i(h) \rangle \quad \text{(because } I(K_t) = B_t \text{ as seen in (4.14))} \\
 &= i(h)(t) \quad \text{(because } B_t : \mathcal{C}_0 \rightarrow \mathbb{R} \text{ is evaluation at } t\text{).}
 \end{aligned}
 \tag{4.16}$$

Thus

$$i(h) = h,$$

viewed as an element of \mathcal{C}_0 .

4.6. The subspace $\mathcal{H}_{\mathcal{C}_0^*}$ of continuous functionals. Recall that the special function K_s has the property that

$$I(K_s)(\omega) = B_s(\omega) = \omega(s) = \int_0^1 \omega d\delta_s \quad \text{for all } \omega \in \mathcal{C}_0,$$

where δ_s is the delta measure at $s \in [0, 1]$. Taking linear combinations we have, for any $h = \sum_j c_j K_{s_j} \in \mathcal{H}_{00}$,

$$I(h)(\omega) = \int_0^1 \omega d\lambda_h \quad \text{for all } \omega \in \mathcal{C}_0, \tag{4.17}$$

for a signed Borel measure $\lambda_h = \sum_j c_j \delta_{s_j}$ on $[0, 1]$. A general element of the dual space \mathcal{C}_0^* is given by integration against a Borel signed measure on $[0, 1]$ and corresponds to a function $h \in \mathcal{H}_0$ for which the derivative \dot{h} is of bounded variation.

4.7. Codimension one affine subspaces. The simplest type of closed affine subspace of \mathcal{H}_0 is of the form

$$\{h \in \mathcal{H}_0 : \langle f, h \rangle = c\},$$

for some $f \in \mathcal{H}_0$ and $c \in \mathbb{R}$. As a special case of interest consider the function $f = K_1 : t \mapsto t$. The reproducing kernel property (4.12) then implies

$$\langle f, h \rangle = \langle K_1, h \rangle = h(1),$$

and so the affine subspace above is

$$\{h \in \mathcal{H}_0 : h(1) = c\}.$$

By (2.11) the closure in the Banach space \mathcal{C}_0 of $\ker \langle f, \cdot \rangle_H$ is $\ker I(f)$:

$$\{\omega \in \mathcal{C}_0 : \omega(1) = 0\} = \overline{\{\omega \in \mathcal{H}_0 : \omega(1) = 0\}} \tag{4.18}$$

On the left here we have the set of continuous loops based at 0.

4.8. The mapping I_L . Consider a closed affine subspace L of \mathcal{C}_0 . Let h_L be the point on L closest to the origin and

$$P_{L_0} : \mathcal{H}_0 \rightarrow \mathcal{H}_0$$

is the orthogonal projection onto the closed subspace $L_0 = L - h_L$. Then, as we have discussed before in the context of (2.7), there is a Gaussian measure μ_L on \mathcal{C}_0 for which

$$\int_{\mathcal{C}_0} e^{iI(h)} d\mu_L = e^{i\langle h_L, h \rangle - \frac{1}{2}\|P_{L_0}h\|^2} \quad \text{for all } h \in \mathcal{H}_{\mathcal{C}_0^*}, \tag{4.19}$$

where

$$I : \mathcal{H}_0 \rightarrow L^2(\mathcal{C}_0, \mu) : h \mapsto \int_0^1 \dot{h}(t) dB_t \tag{4.20}$$

is the Paley-Wiener map. Next, for the measure μ_L we observe that the mapping

$$\mathcal{H}_{\mathcal{C}_0^*} \rightarrow L^2(B, \mu_L) : h \mapsto I(h)$$

is continuous linear and extends uniquely to a continuous linear mapping

$$I_L : \mathcal{H}_0 \rightarrow L^2(B, \mu_L). \tag{4.21}$$

We think of $I_L(h)$ as the random variable $I(h)$ *conditioned to lie on the affine subspace L* . Its distribution is determined by the characteristic function

$$\int_{\mathcal{C}_0} e^{iI_L(h)} d\mu_L = e^{i\langle h_L, h \rangle - \frac{1}{2}\|P_{L_0}h\|^2} \quad \text{for all } h \in \mathcal{H}_0. \tag{4.22}$$

5. Explicit Formulas for Transforms

In this section we work out the Gaussian Radon transforms of some Brownian functionals. We use the framework and notation set up in the preceding sections. In particular, we will work with the classical Wiener measure μ on \mathcal{C}_0 , the Banach space of continuous paths $[0, 1] \rightarrow \mathbb{R}$ initiating at 0, the Hilbert space $\mathcal{H}_0 = \overline{\mathcal{H}}_{00}$ discussed in subsection 4.3, and the continuous linear map

$$I_L : \mathcal{H}_0 \rightarrow L^2(\mathcal{C}_0, \mu_L)$$

for closed affine subspaces $L \subset \mathcal{C}_0$.

We will determine $G\phi$ for ϕ a linear combination of multiple Itô integrals

$$U(g^{\otimes n}) = \int_0^1 \dots \int_0^1 g(s_1) \dots g(s_n) dB_{s_1} \dots dB_{s_n},$$

where $g \in L^2[0, 1]$ and U is the Wiener chaos isomorphism

$$U : \mathcal{F}_s(H) \rightarrow L^2(\mathcal{H}_0) \tag{5.1}$$

discussed in (3.6).

5.1. An affine subspace of pinched paths. Let us consider a concrete and simple example. Let us take for L the closed affine subspace

$$L_T(c) = \{h \in \mathcal{H}_0 : h(T) = c\} = \{h \in \mathcal{H}_0 : \langle K_T, h \rangle = c\}, \tag{5.2}$$

where T is a fixed time in $(0, 1]$ and $c \in \mathbb{R}$, and, as usual,

$$K_T : [0, 1] \rightarrow \mathbb{R} : s \mapsto \min\{T, s\}.$$

Geometrically, a point h in the affine subspace $L_T(c)$ is a path initiating at 0 and passing through the point c at time T . If $T = 1$ this would be a *bridge* running from 0 to c .

We can display $L_T(c)$ also as

$$L_T(c) = \frac{c}{T}K_T + K_T^\perp, \tag{5.3}$$

where T in the first term on the right side arises as $\|K_T\|^2$ and the point on $L_T(c)$ closest to the origin is

$$h_{L_T(c)} = \frac{c}{T}K_T. \tag{5.4}$$

The orthogonal projection $P_{L_T(c)} : \mathcal{H}_0 \rightarrow \mathcal{H}_0$ onto the closed subspace K_T^\perp is given by

$$P_{L_T(c)}(h) = h - \frac{1}{T}\langle h, K_T \rangle K_T. \tag{5.5}$$

5.2. The bridge process. Let us recall the linear mapping

$$I_{L_T(c)} : \mathcal{H}_0 \rightarrow L^2(\mathcal{C}_0, \mu_L)$$

given in (2.15). The distribution of the random variable

$$I_{L_T(c)}(h) \in L^2(\mathcal{C}_0, \mu_L)$$

is determined through its characteristic function, which is given by (4.22) as

$$\int_{\mathcal{C}_0} e^{iuI_{L_T(c)}(h)} d\mu_{L_T(c)} = e^{iu\frac{c}{T}\langle K_T, h \rangle - \frac{u^2}{2}\|h - \frac{1}{T}\langle h, K_T \rangle K_T\|^2} \tag{5.6}$$

for all $h \in \mathcal{H}_0$ and $u \in \mathbb{R}$. The variance term in the exponent works out to

$$\text{var}\left(I_{L_T(c)}(h)\right) = \|h\|^2 - \frac{1}{T}\langle h, K_T \rangle^2. \tag{5.7}$$

Recalling from (4.13) that

$$I(h) = \int_0^1 \dot{h}(t) dB_t, \tag{5.8}$$

let us use the notation

$$I_{L_T(c)}(h) \stackrel{\text{def}}{=} \int_0^1 \dot{h}(t) dB_t \quad \text{for } h \in \mathcal{H}_0. \tag{5.9}$$

There is a slight conflict between this usage and (4.13) because $I_{L_T(c)}(h)$ is an element of $L^2(\mathcal{C}_0, \mu_{L_T(c)})$ whereas $I(h)$ is an element of the Wiener L^2 -space $L^2(\mathcal{C}_0, \mu)$. But, as noted in (2.13), the mapping $I_{L_T(c)}$ is defined to agree with I on the elements $h \in \mathcal{H}_{00}$. Furthermore, $I(h)$ is a *continuous* function on \mathcal{C}_0 when $h \in \mathcal{H}_{00}$ (which is contained in H_{B^*} in this context) and so is uniquely specified not simply almost-everywhere but at all points of \mathcal{C}_0 .

Let $f = \dot{h}$, where $h \in \mathcal{H}_0$; the characteristic function given in (5.6) and the variance formula (5.7) show that $\int_0^1 f(t) dB_t$ is Gaussian with mean and variance

$$\begin{aligned} \mathbb{E}_{\mu_{L_T(c)}}\left(\int_0^1 f(t) dB_t\right) &= \frac{c}{T} \int_0^T f(t) dt \\ \text{var}_{\mu_{L_T(c)}}\left(\int_0^1 f(t) dB_t\right) &= \int_0^1 f(t)^2 dt - \frac{1}{T} \left(\int_0^T f(t) dt\right)^2. \end{aligned} \tag{5.10}$$

If we take for h the function K_t , then $f = 1_{[0,t]}$ and so

$$I(K_t) = \int_0^1 \dot{K}_t(s) dB_s = \int_0^t dB_s = B_t.$$

Then $I_{L_T(c)}(K_t)$ is Gaussian with mean and variance given by

$$\begin{aligned} \mathbb{E}_{\mu_{L_T(c)}}(B_t) &= \frac{c}{T} \langle K_T, K_t \rangle = \frac{c}{T} \min\{T, t\} \\ \text{var}_{\mu_{L_T(c)}}(B_t) &= t - \frac{1}{T} \min\{t, T\}^2 = T \left(\frac{t}{T} - \min\{1, t/T\}\right)^2. \end{aligned} \tag{5.11}$$

The expectation term is the Gaussian Radon transform; thus the value of the Gaussian Radon transform GB_t on the affine subspace $L_T(c)$ is

$$GB_t|_{[B_T=c]} = \frac{c}{T} \min\{T, t\}, \tag{5.12}$$

where we have sacrificed some notational accuracy for notational intuition in writing $[B_T = c]$ to denote the affine subspace $L_T(c)$.

Using the covariance formula (2.18) we have

$$\begin{aligned} \text{Cov}_{\mu_{L_T(c)}}(B_t, B_s) &= \langle P_{L_0}K_t, P_{L_0}K_s \rangle \\ &= \left\langle K_t - \frac{1}{T} \langle K_t, K_T \rangle K_T, K_s - \frac{1}{T} \langle K_s, K_T \rangle K_T \right\rangle \\ &= \langle K_t, K_s \rangle - \frac{1}{T} \langle K_t, K_T \rangle \langle K_s, K_T \rangle \\ &= \min\{s, t\} - \frac{1}{T} \min\{t, T\} \min\{s, T\} \\ &= T \left(\min\left\{ \frac{s}{T}, \frac{t}{T} \right\} - \min\left\{ 1, \frac{t}{T} \right\} \min\left\{ 1, \frac{s}{T} \right\} \right). \end{aligned} \tag{5.13}$$

This covariance structure identifies the process

$$t \mapsto I_{L_T(c)}(K_t)$$

as a *Brownian bridge process* for $t \in [0, T]$ and a Brownian motion for $t > T$ starting at c when $t = T$.

5.3. Multiple bridges. Instead of conditioning the path to pass through a specified point at a particular time T we can condition the process to pass through a sequence of specific points at specified times $T_1 < \dots < T_m$. For this the relevant affine subspace of \mathcal{H}_0 is

$$L_T(c) = \{h \in \mathcal{H}_0 : h(T_1) = c_1, \dots, h(T_m) = c_m\} \tag{5.14}$$

where now $T = (T_1, \dots, T_m) \in [0, 1]^m$ with $0 < T_1 < \dots < T_m$ and c is the point $(c_1, \dots, c_m) \in \mathbb{R}^m$. The special point $h_{L_T(c)}$ on $L_T(c)$ closest to the origin is given by the formula in Proposition 2.1. The affine subspace $L_T(c)$ is the level set

$$L_T(c) = F^{-1}(c),$$

where

$$F : \mathcal{H}_0 \rightarrow \mathbb{R}^m : h \mapsto (\langle K_{T_1}, h \rangle, \dots, \langle K_{T_m}, h \rangle). \tag{5.15}$$

By Proposition 2.1 the matrix FF^* has (j, k) -the entry given by

$$(FF^*)_{jk} = \langle K_{T_k}, K_{T_j} \rangle = \min\{T_j, T_k\}. \tag{5.16}$$

Using this and the expression for $GI(h)$ obtained in (2.35) we have

$$GI(h)|_{L_T(c)} = \sum_{j=1}^m [(FF^*)^{-1}]_{jk} c_k \langle K_{T_j}, h \rangle. \tag{5.17}$$

A more transparent formula is obtained if we use a different representation for the affine subspace $L_T(c)$. To this end let

$$w_1 = \frac{K_{T_1}}{\sqrt{T_1}}, \quad w_2 = \frac{K_{T_2} - K_{T_1}}{\sqrt{T_2 - T_1}}, \quad \dots, \quad w_m = \frac{K_{T_m} - K_{T_{m-1}}}{\sqrt{T_m - T_{m-1}}}.$$

Then

$$L_T(c) = \{h \in \mathcal{H}_0 : \langle w_1, h \rangle = b_1, \dots, \langle w_m, h \rangle = b_m\}$$

where

$$b_1 = \frac{c_1}{\sqrt{T_1}}, \quad b_2 = \frac{c_2 - c_1}{\sqrt{T_2 - T_1}}, \dots, b_m = \frac{c_m - c_{m-1}}{\sqrt{T_m - T_{m-1}}}.$$

The advantage of this representation for $L_T(c)$ is that the vectors w_1, \dots, w_m are orthonormal. Then using the formula for $GI(h)$ given in (2.36) we have

$$\begin{aligned} GI(h)|_{L_T(c)} &= \sum_{k=1}^m b_k \langle w_k, h \rangle \\ &= \frac{c_1}{T_1} \int_0^{T_1} \dot{h}(t) dt + \sum_{k=2}^m \frac{c_k - c_{k-1}}{T_k - T_{k-1}} \int_{T_{k-1}}^{T_k} \dot{h}(t) dt. \end{aligned} \tag{5.18}$$

Expressing this in different notation, and writing f for \dot{h} , we have

$$\mathbb{E}_{\mu_{L_T(c)}} \left(\int_0^1 f(t) dB_t \right) = \frac{c_1}{T_1} \int_0^{T_1} f(t) dt + \sum_{k=2}^m \frac{c_k - c_{k-1}}{T_k - T_{k-1}} \int_{T_{k-1}}^{T_k} f(t) dt. \tag{5.19}$$

This generalizes the first equation in (5.11). As a check we observe that when $f = 1_{[0, T_1]}$ the formula (5.19) yields

$$\mathbb{E}_{\mu_{L_T(c)}} (B_{T_1}) = c_1,$$

and, more generally, taking $f = 1_{[0, T_j]}$, we obtain

$$\mathbb{E}_{\mu_{L_T(c)}} (B_{T_k}) = c_1 + \sum_{k=2}^j \frac{c_k - c_{k-1}}{T_k - T_{k-1}} (T_k - T_{k-1}) = c_j,$$

which is consistent with the process $t \mapsto B_t$ passing through the point c_k at time T_k , for each $k \in \{1, \dots, m\}$, almost surely with respect to the measure $\mu_{L_T(c)}$.

5.4. Bridges in higher dimension. For this let us simply note that for an \mathbb{R}^n -valued Brownian motion process the Cameron-Martin Hilbert space is \mathcal{H}_0^n , where \mathcal{H}_0 is the Hilbert space for the one-dimensional process. A bridge process, or more precisely a process that is required to pass through a point $(p_1, \dots, p_n) \in \mathbb{R}^n$ at a specified time T , corresponds then to a closed affine subspace of \mathcal{H}_0^n specified by level sets of the n orthogonal vectors in \mathcal{H}_0 given by

$$v_1 = (K_T, 0, \dots, 0), \dots, v_n = (0, \dots, 0, K_T) \in \mathcal{H}_0^n.$$

We can obtain Gaussian Radon transforms of functions of the form $I(h)$ by applying Proposition 2.1 in a manner similar to what we did in the preceding subsection. We shall not explore this further.

5.5. Multiple stochastic integrals. We refer to the monograph of Kuo [11, sec. 9.6] for a development of Itô's theory of multiple stochastic integrals. For any symmetric function $F \in L^2[0, 1]^n$, the multiple Itô stochastic integral is given by $n!$ times the iterated integral; thus:

$$\begin{aligned} J_n(F) &\stackrel{\text{def}}{=} \int_{[0,1]^n} F(t_1, \dots, t_n) dB_{t_n} \dots dB_{t_1} \\ &\stackrel{\text{def}}{=} n! \int_0^1 \dots \int_0^{t_{n-2}} \left[\int_0^{t_{n-1}} F(t_1, \dots, t_n) dB_{t_n} \right] \dots dB_{t_1}. \end{aligned} \tag{5.20}$$

For any $f \in L^2[0, 1]$, the integral of $f^{\otimes n}$ can be expressed as a Hermite polynomial:

$$J_n(f^{\otimes n}) = H_n \left(\int_0^1 f(s) dB_s; \|f\|_{L^2[0,1]}^2 \right) \tag{5.21}$$

where the Hermite polynomial

$$H_n(x; u^2)$$

is specified through the generating formula

$$e^{tx - \frac{u^2}{2}t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x; u^2). \tag{5.22}$$

(For a proof of (5.20) and more details we refer to [11].) The left side remains unchanged under the following transformation:

$$(t, x, u^2) \mapsto (\lambda^{-1}t, \lambda x, \lambda u^2) \tag{5.23}$$

for any $\lambda \in \mathbb{R}$. As a result we have

$$H_n(\lambda x; \lambda^2 u^2) = \lambda^n H_n(x; u^2). \tag{5.24}$$

Using the generating formula (5.22) we have

$$e^{I(h) - \frac{1}{2}\|h\|_0^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(I(h); \|h\|_0^2). \tag{5.25}$$

Writing $f = \dot{h}$, which is in $L^2[0, 1]$ and using

$$I(h) = \int_0^1 \dot{h}(t) dB_t = \int_0^1 f(t) dB_t$$

and

$$\|h\|_0^2 = \|\dot{h}\|_{L^2[0,1]}^2 = \|f\|_{L^2[0,1]}^2,$$

we have from (5.25) and (5.21):

$$e^{I(h) - \frac{1}{2}\|h\|_0^2} = \sum_{n=0}^{\infty} \frac{1}{n!} J_n(\dot{h}^{\otimes n}). \quad (5.26)$$

We have noted that for $h \in \mathcal{H}_0$ the Wiener integral $\int_0^1 \dot{h}(s) dB_s$ is a continuous function on \mathcal{C}_0 , and hence so is $J_n(\dot{h}^{\otimes n})$. This connects up with the Fock space isomorphism

$$U : \mathcal{F}_s(\mathcal{H}_0)_c \rightarrow L^2(\mathcal{C}_0; \mu) : \text{Exp}(h) \mapsto e^{I(h) - \frac{1}{2}\|h\|_0^2} \quad (5.27)$$

for $h \in \mathcal{H}_0$.

5.6. Gaussian Radon transform of multiple Itô integrals. The Gaussian Radon transform of $J_n(\dot{h}^{\otimes n})$ at the closed affine subspace $L_T(c)$ is

$$\begin{aligned} GJ_n(\dot{h}^{\otimes n}) \Big|_{L_T(c)} &= \mathbb{E}_{\mu_{L_T(c)}} \left[H_n \left(\int_0^1 f(s) dB_s; \|f\|_{L^2[0,1]}^2 \right) \right] \\ &= \mathbb{E} \left[H_n \left(I_{L_T(c)}(h); \|\dot{h}\|_{L^2[0,1]}^2 \right) \right]. \end{aligned} \quad (5.28)$$

Let X be any Gaussian variable. Then from the generating formula (5.22) for the Hermite polynomials we have

$$\mathbb{E} \left(e^{tX - \frac{u^2}{2}t^2} \right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbb{E}[H_n(X; u^2)] \quad (5.29)$$

for all $t, u \in \mathbb{R}$. On the other hand from the Gaussian formula

$$\mathbb{E}[e^{tX}] = e^{\mathbb{E}[X]t + \frac{1}{2}\text{var}(X)t^2} \quad (5.30)$$

we have

$$\mathbb{E}[e^{tX - \frac{u^2}{2}t^2}] = e^{\mathbb{E}[X]t - \frac{1}{2}(u^2 - \text{var}(X))t^2}. \quad (5.31)$$

Writing

$$v^2 = u^2 - \text{var}(X)$$

we then have

$$\mathbb{E}[e^{tX - \frac{u^2}{2}t^2}] = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(\mathbb{E}(X); v^2). \quad (5.32)$$

Here we assume that $u^2 \geq \text{var}(X)$. Comparing with (5.29), we have the following shift-of-variance relation

$$\mathbb{E}(H_n(X; u^2)) = H_n(\mathbb{E}(X); v^2). \quad (5.33)$$

Now let us apply this to the Gaussian Radon transform of the iterated integral of $J_n(\dot{h}^{\otimes n})$, for $h \in \mathcal{H}_{00}$, given in (5.28):

$$\begin{aligned}
 GJ_n(\dot{h}^{\otimes n})\Big|_{L_T(c)} &= \mathbb{E} \left[H_n \left(I_{L_T(c)}(h); \|\dot{h}\|_{L^2[0,1]}^2 \right) \right] \\
 &= H_n \left(\mathbb{E}(I_{L_T(c)}(h)); \|\dot{h}\|_{L^2[0,1]}^2 - \text{var}(I_{L_T(c)}(h)) \right) \\
 &= H_n \left(\frac{c}{T} \int_0^T \dot{h}(t) dt; \frac{1}{T} \left(\int_0^T \dot{h}(t) dt \right)^2 \right) \\
 &\quad \text{(using (5.10))} \\
 &= H_n \left(\frac{c}{T} h(T); \frac{1}{T} h(T)^2 \right) \\
 &= h(T)^n H_n \left(\frac{c}{T}; \frac{1}{T} \right)
 \end{aligned} \tag{5.34}$$

where in the last line we used the scaling relation (5.24).

Writing

$$f = \dot{h},$$

we can rewrite this as

$$GJ_n(f^{\otimes n})\Big|_{L_T(c)} = \left(\int_0^T f(t) dt \right)^n H_n \left(\frac{c}{T}; \frac{1}{T} \right). \tag{5.35}$$

This is the Gaussian Radon transform of the multiple Itô integral

$$n! \int_0^1 \dots \int_0^{t_{n-2}} \left[\int_0^{t_{n-1}} f(t_1) \dots f(t_n) dB_{t_n} \right] \dots dB_{t_1}$$

evaluated at the affine subspace given by

$$L_T(c) = \{h \in \mathcal{H}_0 : h(T) = c\},$$

corresponding geometrically to paths that start at the point 0 and pass through c at time T . In the special case of $T = 1$ and $f = 1$, the multiple integral $J_n(f^{\otimes n})$ is, by (5.21), equal to

$$J_n(1^{\otimes n}) = H_n(B_1; 1).$$

Now $B_1 = c$ almost surely with respect to the Brownian bridge measure $\mu_{L_1(c)}$ and so $J_n(1^{\otimes n}) = H_n(c)$, the traditional Hermite polynomial evaluate at c . This coincides with the value given by the right hand side of (5.35).

We observe that the integral on the right hand side of (5.35) is

$$\int_{[0,T]^n} f(t_1) \dots f(t_n) dt_1 \dots dt_n.$$

By the algebraic process of polarization we can express a symmetric tensor product

$$f_1 \hat{\otimes} \dots \hat{\otimes} f_n = \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma^{-1}(1)} \otimes \dots \otimes f_{\sigma^{-1}(n)} \tag{5.36}$$

as a linear combination of terms of the form $g^{\otimes n}$, where the elements g are linear combinations of f_1, \dots, f_n . Then from (5.35) we obtain the more general form

$$GJ_n(f_1 \hat{\otimes} \dots \hat{\otimes} f_n)|_{L_T(c)} = \left(\int_{[0,T]^n} f_1(t_1) \dots f_n(t_n) dt_1 \dots dt_n \right) H_n \left(\frac{c}{T}; \frac{1}{T} \right). \quad (5.37)$$

Here each f_j is of the form \dot{h}_j , where $h_j \in \mathcal{H}_{c_0^*}$, and in particular could be in the subspace $\mathcal{H}_{00} \subset \mathcal{H}_0$. Writing F for $f_1 \hat{\otimes} \dots \hat{\otimes} f_n$, viewed as an element of $L^2([0,1]^n)$, or taking F to be a linear combination of such functions, we then have

$$GJ_n(F)|_{L_T(c)} = \left(\int_{[0,T]^n} F(t_1, \dots, t_n) dt_1 \dots dt_n \right) H_n \left(\frac{c}{T}; \frac{1}{T} \right). \quad (5.38)$$

Here on the left we have the Gaussian Radon transform of the multiple Itô stochastic integral $\int_{[0,1]^n} F(t_1, \dots, t_n) dB_{t_n} \dots dB_{t_1}$ as defined in (5.20), evaluated on the affine subspace $L_T(c)$ corresponding to Brownian paths satisfying $B(T) = c$.

6. Concluding Remarks

In this paper we have studied Gaussian measure conditioned to be supported on closed affine subspaces of a Hilbert space. We proved a Fock space decomposition for such measures. We examined the Gaussian Radon transform, with special attention to finite-codimension affine subspaces. We investigated the transform and the conditioned measure for the case of the classical Wiener space, observing how Brownian bridges can be understood in terms of affine subspaces of finite codimension. We computed Gaussian Radon transforms of multiple Itô stochastic integrals explicitly in terms of Hermite polynomials.

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