Communications on Stochastic Analysis
Vol. 8, No. 3 (2014) 307-315

Serials Publications
www.serialspublications.com

AN EXTENDED NOVIKOV-TYPE CRITERION FOR LOCAL MARTINGALES WITH JUMPS

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Abstract. For local martingales with nonnegative jumps, we prove a sufficient criterion for the corresponding exponential martingale to be a true martingale. The criterion is in terms of exponential moments of a convex combination of the optional and predictable quadratic variation. The result extends earlier known criteria.

1. Introduction

In [16], Novikov introduced a sufficient criterion for the exponential martingale of a continuous local martingale to be a uniformly integrable martingale. In this paper, we prove a similar result in the case where the local martingale is not continuous, but is assumed to have nonnegative jumps. The novelty of our criterion rests in that our result is stronger than previously known results, in that it combines optional and predictable components and in that our proof of the criterion demonstrates a straightforward two-step structure. We begin by fixing our notation and recalling some results from stochastic analysis.

Assume given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions, see [18] for the definition of this and other probabilistic concepts such as localising sequences, local martingales, the quadratic covariation et cetera. For any local martingale M, we say that M has initial value zero if $M_0 = 0$. For any local martingale M with initial value zero, we denote by $[M]$ the quadratic variation of M, that is, the unique increasing adapted process with initial value zero such that $M^2 - [M]$ is a local martingale.

If A is an adapted increasing process with initial value zero, we say that A is integrable if EA_∞ is finite, and we say that A is locally integrable if A^{T_n} is integrable for some localising sequence (T_n), that is, a sequence of stopping times increasing to infinity. If A is an adapted process with initial value zero and paths of finite variation, we say that A is locally integrable if the variation process is locally integrable. Whenever A is adapted, has initial value zero, is of finite variation and is locally integrable, there exists a predictable process $\Pi_p^* A$ with those same properties such that $A - \Pi_p^* A$ is a local martingale, see Definition VI.21.3 of [20]. We refer to $\Pi_p^* A$ as the dual predictable projection of A, or simply as the compensator of A.

Received 2014-3-3; Communicated by the editors.
2010 *Mathematics Subject Classification.* Primary 60G44; Secondary 60G40.
Key words and phrases. Martingale, exponential martingale, uniform integrability, Novikov.

If M is locally square integrable, it holds that $[M]$ is locally integrable, and we denote by $\langle M\rangle$ the compensator of $[M]$. We refer to $\langle M\rangle$ as the predictable quadratic variation of M. It then holds that $M^2 - \langle M\rangle$ is a local martingale.

For any local martingale with initial value zero, there exists by Theorem I.4.18 of [4] a unique decomposition $M = M^c + M^d$, where M^c is a continuous local martingale and M^d is a purely discontinuous local martingale, both with initial value zero. Here, we say that a local martingale with initial value zero is purely discontinuous if it has zero quadratic covariation with any continuous local martingale with initial value zero. We refer to M^c as the continuous martingale part of M, and refer to M^d as the purely discontinuous martingale part of M.

With M a local martingale with initial value zero and $\Delta M \geq 0$, the exponential martingale of M, also known as the Doléans-Dade exponential of M, is given by

$$\mathcal{E}(M)_t = \exp\left(M_t - \frac{1}{2}[M^c]_t\right)\prod_{0<s\leq t}(1+\Delta M_s)\exp(-\Delta M_s). \tag{1.1}$$

The process $\mathcal{E}(M)$ is the unique càdlàg solution in Z to the stochastic differential equation $Z_t = 1 + \int_0^t Z_{s-}\,\mathrm{d}M_s$, see Theorem II.37 of [18]. By Theorem 29.4 of [14], $\mathcal{E}(M)$ is always a local martingale with initial value one. We are interested in sufficient criteria to ensure that $\mathcal{E}(M)$ is a uniformly integrable martingale. This is a classical question in probability theory, with applications for example in finance, stochatic differential equations and statistical inference for continuously observed stochastic processes, see for example [19, 1, 6, 7, 11]. For the case when M is continuous, sufficient criteria ensuring that $\mathcal{E}(M)$ is a uniformly integrable martingale have been obtained in [16, 2, 8, 9, 15]. For the case when M has jumps, see [12, 3, 17, 22, 5, 21].

We now explain the particular result to be obtained in this paper. In [16], the following result was obtained: If M is a continuous local martingale with initial value zero and $\exp(\frac{1}{2}[M]_\infty)$ is integrable, then $\mathcal{E}(M)$ is a uniformly integrable martingale. This criterion is known as Novikov's criterion. In [10], it was shown that for a continuous local martingale M with initial value zero, the condition

$$\liminf_{\varepsilon\to 0}\varepsilon\log E\exp\left((1-\varepsilon)\frac{1}{2}[M]_\infty\right) < \infty \tag{1.2}$$

suffices to ensure that $\mathcal{E}(M)$ is a uniformly integrable martingale. This is an extension of the result in [16]. And in [21], optimal constants $\alpha(a)$ and $\beta(a)$ for $a > -1$ were identified such that when $\Delta M 1_{(\Delta M\neq 0)} \geq a$, integrability of $\exp(\alpha(a)[M]_\infty)$ and $\exp(\beta(a)[M]_\infty)$ suffices to ensure that $\mathcal{E}(M)$ is a uniformly integrable martingale, and it was noted that for the case $a = 0$, $\alpha(a) = \beta(a) = \frac{1}{2}$. Thus, the case where $\Delta M \geq 0$ presents a higher level of regularity than the general case. In this note, we prove that when $\Delta M \geq 0$, the condition

$$\liminf_{\varepsilon\to 0}\varepsilon\log E\exp\left((1-\varepsilon)\frac{1}{2}(\alpha[M]_\infty + (1-\alpha)\langle M\rangle_\infty)\right) < \infty \tag{1.3}$$

suffices to ensure that $\mathcal{E}(M)$ is a uniformly integrable martingale, thus extending the results of [16] and [10]. Note that while sufficiency of simple Novikov-type criteria such as those given in [21] follow from the results of [12], the condition

(1.3) does not. Also, to the best of the knowledge of the author, the condition (1.3) is the first one obtained applying both the quadratic variation and the predictable quadratic variation at the same time, thus leading to the consideration of novel proof methodology.

2. Main Results and Proofs

In this section, we will prove the following theorem.

Theorem 2.1. *Let M be a locally square integrable local martingale with initial value zero and $\Delta M \geq 0$. Fix $0 \leq \alpha \leq 1$ and assume that*

$$\liminf_{\varepsilon \to 0} \varepsilon \log E \exp\left((1-\varepsilon)\frac{1}{2}(\alpha[M]_\infty + (1-\alpha)\langle M\rangle_\infty)\right) < \infty. \tag{2.1}$$

Then $\mathcal{E}(M)$ is a uniformly integrable martingale. If $\alpha = 1$, it is not necessary that M be locally square integrable. Furthermore, for all $0 \leq \alpha \leq 1$, the constant $1/2$ in (2.1) is optimal.

Optimality of the constant $1/2$ follows by standard counterexamples, see [16] or [21]. We begin by considering the proof of the case $\alpha = 1$, where local square integrability is not required. Our proof in this case rests on the following two elementary martingale lemmas and the following real analysis lemma. Lemma 2.2 and Lemma 2.3 follow by applications of the optional sampling theorem, while Lemma 2.4 follows from elementary calculations.

Lemma 2.2. *Let M be a local martingale with initial value zero and $\Delta M \geq 0$. Then $E\mathcal{E}(M)_\infty \leq 1$, and $\mathcal{E}(M)$ is a uniformly integrable martingale if and only if $E\mathcal{E}(M)_\infty = 1$.*

Lemma 2.3. *Let M be a local martingale with initial value zero. Let $\mathcal{C}$ denote the set of all bounded stopping times. If there exists $a > 1$ such that $(M_T)_{T \in \mathcal{C}}$ is bounded in $\mathcal{L}^a$, then M is a uniformly integrable martingale.*

Lemma 2.4. *Let $x \geq 0$. It then holds that*

$$0 \leq \log \frac{1+\lambda x}{(1+x)^\lambda} \leq \frac{\lambda(1-\lambda)}{2}x^2 \quad \text{and} \tag{2.2}$$

$$0 \leq \log \frac{(1+x)^a}{1+ax} \leq \frac{a(a-1)}{2}x^2 \tag{2.3}$$

for $0 \leq \lambda \leq 1$ and $a \geq 1$.

Proof of Theorem 2.1 for the case $\alpha = 1$. In this case, we wish to show that when $\liminf_{\varepsilon \to 0} \varepsilon \log E \exp(((1-\varepsilon)/2)[M]_\infty)$ is finite, $\mathcal{E}(M)$ is a uniformly integrable martingale. We first prove that $\mathcal{E}(M)$ is a uniformly integrable martingale under the stronger condition that $\exp((1+\varepsilon)\frac{1}{2}[M]_\infty)$ is integrable for some $\varepsilon > 0$. Fix such an $\varepsilon > 0$, and let $a, r > 1$. Applying (2.3) of Lemma 2.4, we then have

$$\begin{aligned}\mathcal{E}(M)_t^a &= \mathcal{E}(arM)_t^{1/r} \exp\left(\frac{a(ar-1)}{2}[M^c]_t + \sum_{0<s\leq t} \log \frac{(1+\Delta M_s)^a}{(1+ar\Delta M_s)^{1/r}}\right) \\ &\leq \mathcal{E}(arM)_t^{1/r} \exp\left(\frac{a(ar-1)}{2}[M]_t\right).\end{aligned} \tag{2.4}$$

Now let T be a bounded stopping time. Note that as arM has nonnegative jumps, $\mathcal{E}(arM)$ is a nonnegative supermartingale and so $0 \leq E\mathcal{E}(arM)_T \leq 1$. Let $y = ar$ and let s be the dual exponent to r, such that $s = r/(r-1)$. Applying Hölder's inequality in (2.4), we obtain

$$E\mathcal{E}(M)_T^a \leq \left(E \exp\left(\frac{y(y-1)}{2(r-1)}[M]_\infty\right)\right)^{1/s}. \tag{2.5}$$

Next, note that the mapping $y \mapsto y(y-1)$ is increasing for $y \geq 1$. Therefore, $\inf_{y>r>1} y(y-1)/(2(r-1)) = \inf_{r>1} r/2 = 1/2$, and so there exists $y > r > 1$ such that $y(y-1)/(2(r-1)) \leq (1+\varepsilon)/2$. Fixing such $y > r > 1$ and putting $a = y/r$, we obtain $a > 1$ and (2.5) allows us to conclude that with the supremum being over all bounded stopping times, we have

$$\sup_T E\mathcal{E}(M)_T^a \leq \left(E \exp\left((1+\varepsilon)\frac{1}{2}[M]_\infty\right)\right)^{1/s}, \tag{2.6}$$

where the right-hand side is finite by assumption. By Lemma 2.3, $\mathcal{E}(M)$ is a uniformly integrable martingale.

Next, we merely assume that $\liminf_{\varepsilon\to 0} \varepsilon \log E\exp(((1-\varepsilon)/2)[M]_\infty)$ is finite. In particular, for all $\varepsilon > 0$, $\exp(((1-\varepsilon)/2)[M]_\infty)$ is integrable. Therefore, $[M]_\infty$ is integrable, so M is a square-integrable martingale and the limit M_∞ exists. Fix $0 < \lambda < 1$. As $[\lambda M]_t = \lambda^2[M]_t$, we have by our earlier results that $\mathcal{E}(\lambda M)$ is a uniformly integrable martingale. Using (2.2) of Lemma 2.4, we have

$$\begin{aligned} 1 &= E\exp\left(\lambda M_\infty - \frac{\lambda^2}{2}[M^c]_\infty + \sum_{0<t} \log(1+\lambda\Delta M_t) - \lambda\Delta M_t\right) \\ &= E\mathcal{E}(M)_\infty^\lambda \exp\left(\frac{\lambda(1-\lambda)}{2}[M^c]_\infty + \sum_{0<t}\log\frac{1+\lambda\Delta M_t}{(1+\Delta M_t)^\lambda}\right) \\ &\leq E\mathcal{E}(M)_\infty^\lambda \exp\left(\frac{\lambda(1-\lambda)}{2}[M]_\infty\right). \end{aligned} \tag{2.7}$$

Now fix $\gamma \geq 0$. Applying Jensen's inequality in (2.7) with the concave function $x \mapsto x^\lambda$ as well as Hölder's inequality with the dual exponents $\frac{1}{\lambda}$ and $\frac{1}{1-\lambda}$, we obtain, with $F_\gamma = ([M]_\infty > \gamma)$, that

$$\begin{aligned} 1 &\leq E\mathcal{E}(M)_\infty^\lambda \exp\left(\frac{\lambda\gamma(1-\lambda)}{2}\right) + E\mathcal{E}(M)_\infty^\lambda \exp\left(\frac{\lambda(1-\lambda)}{2}[M]_\infty\right)1_{F_\gamma} \\ &\leq (E\mathcal{E}(M)_\infty)^\lambda \exp\left(\frac{\lambda\gamma(1-\lambda)}{2}\right) + (E\mathcal{E}(M)_\infty 1_{F_\gamma})^\lambda\left(E\exp\left(\frac{\lambda}{2}[M]_\infty\right)\right)^{1-\lambda}. \end{aligned}$$

By our assumptions, we have that $\liminf_{\lambda\to 1}(E\exp((\lambda/2)[M]_\infty))^{1-\lambda}$ is finite. Let c denote the value of the limes inferior. By the above, we then obtain

$$1 \leq E\mathcal{E}(M)_\infty + cE\mathcal{E}(M)_\infty 1_{([M]_\infty>\gamma)}. \tag{2.8}$$

Letting γ tend to infinity, we obtain $1 \leq E\mathcal{E}(M)_\infty$, which by Lemma 2.2 shows that $\mathcal{E}(M)$ is a uniformly integrable martingale. □

For the remaining case of $0 \leq \alpha < 1$, we need the following further inequalities. Here, the upper inequality in Lemma 2.4 is not obvious. However, an indication that the constant $\alpha\frac{\lambda(1-\lambda)}{2}$ on the right-hand side of (2.12) is the right one may be obtained since

$$\lim_{x\to 0}\frac{1}{x^2}\log\frac{1+\lambda x+(1+\sqrt{1-\alpha}x)^\lambda-(1+\lambda\sqrt{1-\alpha}x)}{(1+x)^\lambda}=\alpha\frac{\lambda(1-\lambda)}{2}, \qquad (2.9)$$

by the l'Hôpital rule.

Lemma 2.5. *Let* $x \geq 0$. *It then holds that*

$$0 \leq (1+\lambda x)-(1+x)^\lambda \leq \frac{\lambda(1-\lambda)}{2}x^2 \quad \textit{and} \qquad (2.10)$$

$$0 \leq (1+x)^a-(1+ax) \leq \frac{a(a-1)}{2}x^2, \qquad (2.11)$$

for $0 \leq \lambda \leq 1$ *and* $1 \leq a \leq 2$.

Lemma 2.6. *Let* $x \geq 0$. *It then holds that*

$$0 \leq \log\frac{1+\lambda x+(1+\sqrt{1-\alpha}x)^\lambda-(1+\lambda\sqrt{1-\alpha}x)}{(1+x)^\lambda} \leq \alpha\frac{\lambda(1-\lambda)}{2}x^2 \qquad (2.12)$$

for $\alpha, \lambda \in [0,1]$.

Proof of Theorem 2.1 for the case $0 \leq \alpha < 1$. We consider the case $0 < \alpha < 1$, the remaining case of $\alpha = 0$ follows by a similar method.

Fix $\varepsilon > 0$. We first prove that $\mathcal{E}(M)$ is a uniformly integrable martingale under the stronger condition that $\exp((1+\varepsilon)\frac{1}{2}(\alpha[M]_\infty+(1-\alpha)\langle M\rangle_\infty))$ is integrable. Let $a, r > 1$ be given with $ar \leq 2$. Defining $U_t = ar\sum_{0<s\leq t}\log(1+\Delta M_s)-\Delta M_s$, we have

$$\mathcal{E}(M)_t^a = \exp\left(arM_t-\frac{1}{2}[arM]_t+U_t\right)^{1/r}\exp\left(\frac{a(ar-1)}{2}[M^c]_t\right). \qquad (2.13)$$

We wish to decompose the first factor in the right-hand side of (2.13) in two ways, one involving an optional increasing factor and one involving a predictable increasing factor. Put $N_t^o = arM_t$. For the optional decomposition, we note that

$$U_t = \left(\sum_{0<s\leq t}\log(1+\Delta N_s^o)-\Delta N_s^o\right)+\sum_{0<s\leq t}\log\frac{(1+\Delta M_s)^{ar}}{1+ar\Delta M_s}, \qquad (2.14)$$

which yields

$$\exp\left(arM_t-\frac{1}{2}[arM]_t+U_t\right)^{\alpha/r} = \mathcal{E}(N^o)_t^{\alpha/r}\exp\left(\frac{\alpha}{r}\sum_{0<s\leq t}\log\frac{(1+\Delta M_s)^{ar}}{1+ar\Delta M_s}\right).$$

Next, for $1 \leq \beta \leq 2$, we define $W_t^\beta = \sum_{0<s\leq t}(1+\Delta M_s)^\beta-(1+\beta\Delta M_s)$. Note that the sum is well-defined, increasing and locally integrable by (2.11) of Lemma 2.5, as $[M]$ is locally integrable by our assumptions. Therefore, the compensator V^β of W^β is well-defined, and is increasing and locally integrable as well. Define two local martingales by putting $N_t^p = arM_t+W_t^{ar}-V_t^{ar}$ and $\bar{N}_t^p = \int_0^t(1+\Delta V_s^{ar})^{-1}\,\mathrm{d}N_s^p$,

where $\bar{N}^p$ is well-defined as $\Delta V^{ar} \geq 0$ and $(1+\Delta V^{ar}_s)^{-1}$ is predictable and locally bounded.

We begin by considering some properties of $\bar{N}^p$. By elementary calculations, $\Delta \bar{N}^p > -1$. Defining $A^{ar}_t = \sum_{0<s\leq t} \Delta V^{ar}_s(1+\Delta V^{ar}_s)^{-1}$, A^{ar} is predictable, increasing and locally bounded, and it holds that $[A^{ar}, N^p]_t = \sum_{0<s\leq t} \Delta A^{ar}_s \Delta N^p_s$. By Proposition I.4.49 of [4], $[A^{ar}, N^p]$ is a local martingale. As the two local martingales $\int_0^t A^{ar}_s \,\mathrm{d}N^p_s$ and $[A^{ar}, N^p]$ are purely discontinuous and have the same jumps, they are equal by the uniqueness part of Theorem I.4.18 of [4], and we thus obtain

$$\bar{N}^p_t = arM_t + W^{ar}_t - V^{ar}_t - \sum_{0<s\leq t} (1+\Delta V^{ar}_s)^{-1} \Delta V^{ar}_s \Delta N^p_s. \tag{2.15}$$

Also, as the function $x \mapsto \log(1+x) - x$ is nonpositive for $x \geq 0$ and V^{ar} is increasing, we obtain $\log(1+\Delta V^{ar}) - \Delta V^{ar} \leq 0$. Combining our observations, we get

$$\begin{aligned} &ar\log(1+\Delta M_s) - ar\Delta M_s - (\log(1+\Delta \bar{N}^p_s) - \Delta \bar{N}^p_s) \\ =&\Delta W^{ar}_s - \frac{\Delta V^{ar}_s \Delta N^p_s}{1+\Delta V^{ar}_s} + \log(1+\Delta V^{ar}_s) - \Delta V^{ar}_s \leq \Delta W^{ar}_s - \frac{\Delta V^{ar}_s \Delta N^p_s}{1+\Delta V^{ar}_s}, \end{aligned} \tag{2.16}$$

where the logarithm in first expression is well-defined as $\Delta \bar{N}^p > -1$. This implies

$$\begin{aligned} U_t &\leq \left(\sum_{0<s\leq t} \log(1+\Delta \bar{N}^p_s) - \Delta \bar{N}^p_s \right) + \sum_{0<s\leq t} \Delta W^{ar}_s - \frac{\Delta V^{ar}_s \Delta N^p_s}{1+\Delta V^{ar}_s} \\ &= \bar{N}^p_t - arM_t + \left(\sum_{0<s\leq t} \log(1+\Delta \bar{N}^p_s) - \Delta \bar{N}^p_s \right) + V^{ar}_t. \end{aligned} \tag{2.17}$$

Also noting that $[\bar{N}^p]_t = [N^p]_t = [arM]_t$, we obtain the relationship

$$\exp\left(arM_t - \frac{1}{2}[arM]_t + U_t \right)^{(1-\alpha)/r} \leq \mathcal{E}(N^p)^{(1-\alpha)/r}_t \exp\left(\frac{1-\alpha}{r} V^{ar}_t \right).$$

Combining our results with (2.13), we obtain $\mathcal{E}(M)^a_t \leq \mathcal{E}(N^o)^{\alpha/r}_t \mathcal{E}(N^p)^{(1-\alpha)/r} X_t$, where the process X is defined by

$$X_t = \exp\left(\frac{a(ar-1)}{2}[M^c]_t + \frac{\alpha}{r} \sum_{0<s\leq t} \log\frac{(1+\Delta M_s)^{ar}}{1+ar\Delta M_s} + \frac{1-\alpha}{r} V_t \right). \tag{2.18}$$

Here, note that by (2.3) of Lemma 2.4 and (2.11) of Lemma 2.5, we have, as $1 \leq ar \leq 2$, that

$$\sum_{0<s\leq t} \log\frac{(1+\Delta M_s)^{ar}}{1+ar\Delta M_s} \leq \frac{ar(ar-1)}{2}[M^d]_t \quad \text{and} \tag{2.19}$$

$$V^{ar}_t \leq \frac{ar(ar-1)}{2}\langle M^d \rangle_t, \tag{2.20}$$

leading to the inequality

$$\mathcal{E}(M)_t^a \le \mathcal{E}(N^o)_t^{\alpha/r} \mathcal{E}(N^p)^{(1-\alpha)/r} \exp\left(\frac{a(ar-1)}{2}(\alpha[M]_t + (1-\alpha)\langle M\rangle_t\right). \quad (2.21)$$

Next, as $\Delta N_t^o \ge 0 > -1$ and $\Delta \bar{N}_t^p > -1$, $\mathcal{E}(N^o)$ and $\mathcal{E}(\bar{N}^p)$ are nonnegative supermartingales, and so for all bounded stopping times T, $0 \le E\mathcal{E}(N^o)_T \le 1$ and $0 \le E\mathcal{E}(N^p)_T \le 1$. Now let s be the dual exponent of r, such that $s = r/(r-1)$. Noting that $\frac{1}{r/\alpha} + \frac{1}{r/(1-\alpha)} + \frac{1}{s} = \frac{1}{r} + \frac{1}{s} = 1$,we may then apply Hölder's inequality for triples of functions to the inequality (2.21), yielding for any bounded stopping time T that

$$E\mathcal{E}(M)_T^a \le \left(E\exp\left(\frac{y(y-1)}{2(r-1)}(\alpha[M]_\infty + (1-\alpha)\langle M\rangle_\infty\right)\right)^{1/s}, \quad (2.22)$$

where $y = ar$. This holds for all $a, r > 1$ such that $ar \le 2$, and is a bound similar to (2.5). Proceeding as in the proof of the case $\alpha = 1$, we then obtain as a consequence of Lemma 2.3 that $\mathcal{E}(M)$ is a uniformly integrable martingale.

Next, assume that $\liminf_{\varepsilon\to 0} \varepsilon \log E\exp(((1-\varepsilon)/2)\langle M\rangle_\infty)$ is finite. In particular, for $\varepsilon > 0$, $\exp(((1-\varepsilon)/2)\langle M\rangle_\infty)$ is integrable. In particular, $\langle M\rangle_\infty$ is integrable, which implies that $[M]_\infty$ is integrable. Thus, M is a square-integrable martingale, in particular the limit M_∞ exists. Now fix $0 < \lambda < 1$ and define

$$W^\lambda(\alpha)_t = \sum_{0<s\le t} (1 + \sqrt{1-\alpha}\Delta M_s)^\lambda - (1 + \lambda\sqrt{1-\alpha}\Delta M_s). \quad (2.23)$$

Note that by Lemma 2.5, the terms in the sum in (2.23) are nonpositive and bounded from below by $-(1-\alpha)\frac{1}{2}\lambda(1-\lambda)(\Delta M_s)^2$. In particular, we find that $W^\lambda(\alpha)$ is well-defined, decreasing and integrable. Let $V^\lambda(\alpha)$ be the compensator of $W^\lambda(\alpha)$, $V^\lambda(\alpha)$ is then decreasing and integrable as well, and it holds that $W^\lambda(\alpha) - V^\lambda(\alpha)$ is a uniformly integrable martingale. We show that $V^\lambda(\alpha)$ is continuous. To this end, let T be some predictable stopping time. By Theorem VI.12.6 of [20] and its proof, we have $E\Delta M_T = 0$ and $E\Delta(W^\lambda(\alpha) - V^\lambda(\alpha))_T = 0$, so that

$$EV^\lambda(\alpha)_T = EW^\lambda(\alpha)_T = E(1 + \sqrt{1-\alpha}\Delta M_T)^\lambda - 1 \ge 0, \quad (2.24)$$

because of our assumption that $\Delta M \ge 0$. We conclude that $\Delta V_T^\lambda = 0$ for all predictable stopping times. Lemma VI.19.2 of [20] then shows that $V^\lambda(\alpha)$ is continuous.

Let $L_t^\lambda = \lambda M_t + W^\lambda(\alpha) - V^\lambda(\alpha)$. By our previous observations, L^λ is a uniformly integrable martingale, in particular the limit L_∞^λ exists. Note that $(L^\lambda)^c = \lambda M^c$, so it holds that $[(L^\lambda)^c]_t = \lambda^2[M^c]_t$. Also note that by continuity of $V^\lambda(\alpha)$, we have

$$\begin{aligned}\Delta L_t^\lambda &= \lambda\Delta M_t + (1 + \sqrt{1-\alpha}\Delta M_t)^\lambda - (1 + \lambda\sqrt{1-\alpha}\Delta M_t) \\ &\ge \lambda(1 - \sqrt{1-\alpha})\Delta M_t \ge 0, \quad (2.25)\end{aligned}$$

and as $W^\lambda(\alpha)$ has nonpositive jumps, we also have $\Delta L_t^\lambda \le \lambda\Delta M_t$. Combining these observations, we obtain $[L^\lambda]_t \le \lambda^2[M]_t$, yielding that L^λ is square-integrable.

We also obtain $\langle L^\lambda\rangle_t \leq \lambda^2\langle M\rangle_t$. This implies

$$\alpha[L^\lambda]_\infty + (1-\alpha)\langle L^\lambda\rangle_\infty \leq \lambda^2(\alpha[M]_\infty + (1-\alpha)\langle M\rangle_\infty), \tag{2.26}$$

so by what we already have shown, $\mathcal{E}(L^\lambda)$ is a uniformly integrable martingale. By elementary calculations, we obtain

$$\mathcal{E}(L^\lambda)_\infty = \mathcal{E}(M)_\infty^\lambda \exp\left(\frac{\lambda(1-\lambda)}{2}[M^c]_\infty + \sum_{0<t}\log\frac{1+\Delta N_t}{(1+\Delta M_t)^\lambda} - V^\lambda(\alpha)_\infty\right).$$

By (2.10) of Lemma 2.5 and Lemma 2.6, we obtain the two inequalities

$$-V^\lambda(\alpha)_\infty \leq (1-\alpha)\frac{\lambda(1-\lambda)}{2}\langle M^d\rangle_\infty \tag{2.27}$$

$$\sum_{0<t}\log\frac{1+\Delta L_t^\lambda}{(1+\Delta M_t)^\lambda} \leq \alpha\frac{\lambda(1-\lambda)}{2}[M^d]_\infty, \tag{2.28}$$

so that combining our conclusions, we have

$$\begin{aligned} 1 &= E\mathcal{E}(L^\lambda)_\infty \\ &\leq E\mathcal{E}(M)_\infty^\lambda \exp\left(\frac{\lambda(1-\lambda)}{2}([M^c]_\infty + \alpha[M^d]_\infty + (1-\alpha)\langle M^d\rangle_\infty)\right) \\ &= E\mathcal{E}(M)_\infty^\lambda \exp\left(\frac{\lambda(1-\lambda)}{2}(\alpha[M]_\infty + (1-\alpha)\langle M\rangle_\infty)\right), \end{aligned} \tag{2.29}$$

which is a bound similar to (2.7). Therefore, proceeding as in the proof of the case $\alpha = 1$, we obtain as a consequence of Lemma 2.2 that $\mathcal{E}(M)$ is a uniformly integrable martingale. □

We take a moment to reflect on the methods applied in the above proof, and make the following observations. First, while the proof of the case $0 \leq \alpha < 1$ is more complicated than the proof of the case $\alpha = 1$, both proofs follow very much the same plan: Use Hölder's inequality to argue that the result holds in a simple case where $\frac{1}{2}$ is exchanged with $(1+\varepsilon)\frac{1}{2}$ in the exponent, then use Hölder's inequality again to obtain the general proof. Also, note that the local martingale $\bar{N}^p$ used in the first part of the proof of the case $0 \leq \alpha < 1$ is related to general decompositions of exponential martingales, see Lemma II.1 of [13].

The comparatively simple structure of the proof is made possible by three main factors: The factor $\lambda(1-\lambda)$ present in the real analysis inequalities allows us to apply Hölder's inequality in the second parts of the proofs. Some of these inequalities have been noted earlier with a factor $1-\lambda$ instead of $\lambda(1-\lambda)$, compare for example (2.10) with (1.2) and (1.3) of [12]. The more advanced triple-parameter inequality (2.12) allows us to obtain a criterion combining the quadratic variation and the predictable quadratic variation. Finally, the assumption $\Delta M \geq 0$, apart from making most of the real analysis inequalities applicable, also ensures that the compensator $V^\lambda(\alpha)$ in the second part of the proof of the case $0 \leq \alpha < 1$ is continuous.

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