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POWERS OF AN INFINITE DIMENSIONAL BROWNIAN MOTION ASSOCIATED WITH THE PRODUCT OF DISTRIBUTIONS

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Abstract. In this paper we define higher powers of an infinite dimensional Brownian motion on a space consisting of distributions without any renormalization, and give an extension of the Itô formula for the Brownian motion. Moreover we extend the Lévy and Voltera Laplacians to operators on a locally convex space taking the completion of the set of all distribution-coefficient polynomials on distributions with respect to some topology, and give a relationship between those Laplacians and the generator of the Brownian motion with realizing a divergent part in mathematics .

1. Introduction

Let $\mathcal{S}'(\mathbb{R})$ be the Schwartz space of tempered distributions. The infinite dimensional Brownian motion generated by the Gross Laplacian

$$\mathbb{B}(t) = \sum_{k=1}^{\infty} B_k(t) e_k$$

is a distribution in $\mathcal{S}'(\mathbb{R})$ with a sequence $\{B_k(t)\}_{k=1}^{\infty}$ of independent one dimensional Brownian motions and an orthonormal basis $\{e_k\}_{k=1}^{\infty}$ of $L^2(\mathbb{R})$ which is in $\mathcal{S}(\mathbb{R})$. We can not consider any power of $\mathbb{B}(t)$ in $\mathcal{S}'(\mathbb{R})$. Even if in the white noise analysis we have to take the renormalization of the power of that.

In this paper we introduce some closed subspace H of $L^2([0,1])$ based on functions

$$e_n(u) := e^{2\pi i n u}, \ n = 1, 2, 3, \ldots,$$

which does not include any constant except the zero function, and construct the Gel'fand triple

$$E \subset H \subset E^*$$

with a nuclear space E and its dual space E^*, of which we can compute the usual product of any elements as series expansions. Therefore in this space E^* we can discuss powers of the delta distribution and also an infinite dimensional Brownian motion with several interesting relationships between powers of the delta distribution and derivatives of that. We also discuss those relationships in the

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Cesàro Hilbert spaces and give an extension of the Itô formula for the power of the infinite dimensional Brownian motion.

Moreover, we introduce the set $\mathrm{Poly}(\widetilde{E}^*)$ of all polynomials on $\widetilde{E}^*$ with $\widetilde{E}^*$-coefficients for a closed subspace $\widetilde{E}$ of $L^2([0,1])$ adding constant functions in E. Then an interesting relationship is obtained between the generator of the infinite dimensional Brownian motion and the Lévy and Volterra Laplacians with realizing the divergent part $\delta(0)$ in mathematics, which appears in the functional derivative of a functional $F(\xi) = \xi(t)^2,\ \xi \in \mathcal{S}(\mathbb{R})$ for example as a formal expression. Taking the completion of $\mathrm{Poly}(\widetilde{E}^*)$ with respect to some topology we define a locally convex space on which those Laplacians are extended to operators. On the space the Brownian motion is generated by the extended Lévy Laplacian with $\delta(0)$.

The paper is organized as follows. In section 2 we introduce some space E^* of distributions, of which any product of elements is included in itself. In Section 3 we summarize the basic notions on the Cesàro Hilbert space which we introduced in [1] and also [3]. The Cesàro Hilbert space based on E^* also includes any product of elements in itself. In Section 4 we prove that any product of distributions in E^* is also elements in itself. In Section 5 we discuss powers of the delta distribution in E^* and give a representation of higher power of the delta distribution by derivatives of the delta distribution.

In Section 6 we discuss the product of distributions in the Cesàro Hilbert space. We also have some representations of higher powers of the delta distribution by derivatives of that in the Cesàro Hilbert space different from the representation in E^*. In Section 7 we introduce an E^*-valued Brownian motion and higher powers of that in E^*. Then an extension of the Itô formula for functionals of the power of the infinite dimensional Brownian motion is obtained. This formula is very useful to apply our calculus to the infinite dimensional stochastic analysis and expected to obtain fruitful results in this field.

In Section 8 adding the constant function to the basis in E we define a new space $\widetilde{E}$ and discuss the product of distributions on its dual space $\widetilde{E}^*$. Generalizing the definition of LV functionals as a domain of the Lévy and Volterra Laplacians in [7], we define those Laplacians on the generalized domain and realize $\delta(0)$ in mathematics. Then we obtain an important expression of the infinitesimal generator of the Brownian motion given in Section 7 by the Lévy and Volterra Laplacians and $\delta(0)$. Moreover we introduce a set $\mathrm{Poly}(\widetilde{E}^*)$ of all polynomials on $\widetilde{E}^*$ with $\widetilde{E}^*$-coefficients and taking the completion of $\mathrm{Poly}(\widetilde{E}^*)$ with respect to some topology we define a locally convex space $\mathbb{D}_{-\infty}$ on which the generator of the Brownian motion and the Lévy and Volterra Laplacians are extended to operators. Finally we give a stochastic expression of the semigroup associated with the extended generator on $\mathbb{D}_{-\infty}$ as the expectation of the translation operator by the Brownian motion. The generator is represented by the extended Lévy Laplacian with $\delta(0)$ on $\mathbb{D}_{-\infty}$.

The formulations obtained in this paper will generalize the results on the Itô formula in [10] and on the infnite dimensional Schrödinger equation including the Lévy and Volterra Laplacians in [11] without any renormalization by $\delta(0)$. We will discuss those elsewhere.

2. Basic Spaces H, E and E^*

Let $e_n(u) := e^{2\pi i n u}$ for $n \in \mathbb{N}^* := \{1, 2, 3, \ldots\}$. Then the set

$$\{e_n;\ n \in \mathbb{N}^*\}$$

is orthonormal in $L^2([0,1])$. Let H be a closed subspace of $L^2([0,1])$ generated by the functions $e_n, n \in \mathbb{N}^*$. Then for any sequence $\{\ell_k\}_{k=1}^{\infty}$ satisfying

$$1 < \ell_1 < \ell_2 < \ell_3 < \cdots, \text{ and } \sum_{k=1}^{\infty} \ell_k^{-2} < \infty$$

and introducing a densely defined selfadjoint operator A on H by

$$A\xi = \sum_{k=1}^{\infty} \ell_k \alpha_k e_k,\ \xi = \sum_{k=1}^{\infty} \alpha_k e_k \in H,$$

we can define norms $|\cdot|_p,\ p \in \mathbb{R}$ and spaces E_p by

$$\begin{aligned} |\xi|_p^2 &:= |A^p \xi|_0^2 = \sum_{k=1}^{\infty} \ell_k^{2p} |\langle \xi, e_k \rangle_*|^2; \\ E_p &:= \{\xi \in H;\ |\xi|_p < \infty\} \text{ for } p > 0; \\ E_p &:= \text{completion of } H \text{ with respect to } |\cdot|_p \text{ for } p < 0, \end{aligned}$$

where $|\cdot|_0$ is the norm of $L^2([0,1])$ and $\langle \cdot, \cdot \rangle_*$ is the conjugate bilinear form of E^* and E.

We also define spaces E and E^* by the projective limit space of $E_p,\ p \in \mathbb{R}$ and the inductive limit space of $E_p,\ p \in \mathbb{R}$, respectively. We also assume that for any $\alpha > 0$ there exists $\beta > 0$ such that

$$(2(k+1)\pi)^{\alpha} \le \ell_k^{\beta},\ k = 1, 2, 3, \ldots.$$

Lemma 2.1. ([2]) *For any $a \in \mathbb{N}$ and $t, s \in [0,1]$ with $t - s \notin \mathbb{Z} \setminus \{0\}$, we have*

$$\lim_{N \to \infty} \frac{1}{N^{2a+1}} \sum_{n=1}^{N} \overline{e_n^{(a)}(t)} e_n^{(a)}(s) = \frac{(2\pi)^a}{2a+1} \delta_{t,s},$$

where $\delta_{t,s}$ is the Kronecker delta function.

The differential operator $\partial_\alpha \equiv \partial^\alpha$ is defined by $\partial_\alpha \xi := \xi^{(\alpha)},\ \xi \in E$ for any $\alpha \in \mathbb{N}$. Then we have

$$\partial_\alpha \xi = \sum_{n=1}^{\infty} \overline{\langle \xi, e_n \rangle_*} e_n^{(\alpha)}$$

for any $\xi \in E$. we can extend this operator to that on E^* as follows:

$$\partial_\alpha x = \sum_{n=1}^{\infty} \overline{\langle x, e_n \rangle_*} e_n^{(\alpha)}$$

for any $x \in E^*$. We use same notation ∂_α as the extension. Then we have the following:

Theorem 2.2. ([2], [3]) *The operator ∂_α is an isomorphism from E onto itself, and is also an isomorphism from E^* onto itself.*

3. The Cesàro Hilbert Space $H_{c,a}$

The Cesàro semi–norm of order α of $x \in E^*$ is defined by

$$|x|_{c,\alpha}^2 := \lim_{N\to\infty} \frac{1}{N^\alpha} \sum_{n=1}^{N} \overline{\langle x, e_n\rangle_*}\langle x, e_n\rangle_*$$

in the sense that, when the limit exists, the semi–norm is defined by the above limit.

The Cesàro conjugate bilinear form of order α between $x, y \in E^*$ is defined, in the same sense, by

$$\langle x, y\rangle_{c,\alpha} := \lim_{N\to\infty} \frac{1}{N^\alpha} \sum_{n=1}^{N} \langle x, e_n\rangle_* \overline{\langle y, e_n\rangle_*}.$$

For all $\alpha \geq 1$ and $\lambda \in \mathbb{R}$, define

$$x_{\lambda,a} := \sum_{n=1}^{\infty} e^{-2\pi i n\lambda} e_n^{(a)}$$

based on the expansion of the a-th derivative of delta distribution δ_λ.

Lemma 3.1. ([3]) For all $a \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, $x_{\lambda,a} \in E^*$.

Lemma 3.2. ([3]) For all $a \in \mathbb{N}$ and $\lambda, \mu \in \mathbb{R} \setminus \mathbb{Q}$, we have

$$\langle x_{\lambda,\alpha}, x_{\mu,\alpha}\rangle_{c,2a+1} = \frac{(2\pi)^a}{2a+1}\delta_{\lambda,\mu} \text{ (Kronecker's delta)}.$$

Let $\mathcal{C}$ be a countable set in $\mathbb{R} \setminus \mathbb{Q}$. Then, from Lemma 2.1 we know that, for each $a \in \mathbb{N}$, the set

$$\Big\{ e_{a,\lambda} := x_{\lambda,a}\sqrt{2a+1/(2\pi)^a} \ : \ \lambda \in \mathcal{C} \Big\}$$

is orthonormal for the scalar product $\langle \cdot, \cdot \rangle_{c,2a+1}$. Therefore, the space

$$H_{c,2a+1}^\circ := \text{linear span}\Big\{ e_{a,\lambda} : \ \lambda \in \mathcal{C} \Big\}$$

is a pre–Hilbert space with the inner product

$$\langle \xi, \eta\rangle_{c,2a+1} = \lim_{N\to\infty} \frac{1}{N^{2a+1}} \sum_{n=1}^{N} \langle \xi, e_n\rangle_* \overline{\langle \eta, e_n\rangle_*} \qquad ; \qquad \xi, \eta \in H_{c,2a+1}^\circ.$$

Let $H_{c,2a+1}$ be the completion of the pre–Hilbert space $H_{c,2a+1}^\circ$ with respect to the norm induced by the inner product $\langle \cdot, \cdot \rangle_{c,2a+1}$. In general, $H_{c,2a+1}$ will not be contained in E^*. Then $H_{c,2a+1}$ becomes an infinite dimensional separable Hilbert space whose inner product will still be denoted $\langle \cdot, \cdot \rangle_{c,2a+1}$, and the set $\{e_{a,\lambda}\}_{\lambda\in\mathcal{C}}$ is an orthonormal basis of $H_{c,2a+1}$.

4. The Product of Distributions in E^*

We can get the product of distributions in E^* as follows:

$$x \cdot y = \sum_{j=1}^{\infty} \overline{\langle x, e_j \rangle_*} e_j \cdot \sum_{j=1}^{\infty} \overline{\langle y, e_j \rangle_*} e_j = \sum_{j=2}^{\infty} \sum_{k=1}^{j-1} \overline{\langle x, e_k \rangle_* \langle y, e_{j-k} \rangle_*} e_j$$

for $x = \sum_{j=1}^{\infty} \overline{\langle x, e_j \rangle_*} e_j \in E^*$ and $y = \sum_{j=1}^{\infty} \overline{\langle y, e_j \rangle_*} e_j \in E^*$. We note that the usual product of distributions in E^* as series expansions coincides with the convolution product of those in E^*.

Theorem 4.1. *For any x and y in E^* the above product $x \cdot y$ is also in E^*. Moreover the product operator $\cdot: E^* \times E^* \to E^*$ is a continuous bilinear operator.*

Proof. Let x, y be elements of E^*. Then there exists $q > 0$ such that x and y are in E_{-q}. For some $r > 0$ and any $p > 2q + r + 1$ we can estimate the $(-p)$-norm $|x \cdot y|_{-p}$of $x \cdot y$ as follows:

$$\begin{aligned}
|x \cdot y|_{-p}^2 &= \sum_{\nu=1}^{\infty} \ell_\nu^{-2p} \left| \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} \overline{\langle x, e_k \rangle_* \langle y, e_{n-k} \rangle_*} \delta_{\nu,n} \right|^2 \\
&= \sum_{\nu=1}^{\infty} \ell_\nu^{-2p} \left| \sum_{k=1}^{\nu-1} \overline{\langle x, e_k \rangle_* \langle y, e_{\nu-k} \rangle_*} \right|^2 \\
&\le \sum_{\nu=1}^{\infty} \ell_\nu^{-2p} \sum_{k=1}^{\nu-1} |\langle x, e_k \rangle_*|^2 \sum_{k=1}^{\nu-1} |\langle y, e_{\nu-k} \rangle_*|^2 \\
&\le \sum_{\nu=1}^{\infty} \ell_\nu^{-2p} \sum_{k=1}^{\nu-1} |e_k|_q^2 |x|_{-q}^2 \sum_{k=1}^{\nu-1} |e_{\nu-k}|_q^2 |y|_{-q}^2 \\
&= \sum_{\nu=1}^{\infty} \ell_\nu^{-2p} (\nu-1)^2 \ell_{\nu-1}^{4q} |x|_{-q}^2 |y|_{-q}^2 \\
&\le \sum_{\nu=1}^{\infty} \ell_\nu^{-2p} \ell_{\nu-1}^{2r} \ell_{\nu-1}^{4q} |x|_{-q}^2 |y|_{-q}^2 \\
&= \sum_{\nu=1}^{\infty} \ell_\nu^{-2(p-2q-r)} |x|_{-q}^2 |y|_{-q}^2 \\
&\le \sum_{\nu=1}^{\infty} \ell_\nu^{-2} |x|_{-q}^2 |y|_{-q}^2.
\end{aligned}$$

Since $\sum_{\nu=1}^{\infty} \ell_\nu^{-2} < \infty$ this implies $x \cdot y \in E^*$. The bilinearity follows from properties of the conjugate bilinear form between E^* and E. Suppose that $x_\ell \to x_0,\ y_\ell \to y_0$ in E^* as $\ell \to \infty$. Then there exists $q > 0$ such that $|x_\ell - x_0|_{-q} \to 0,\ |y_\ell - y_0|_{-q} \to 0$. For $p > 2q + r + 1$ we can estimate the $(-p)$-norm $|x_\ell \cdot y_\ell - x_0 \cdot y_0|_{-p}$ for each

$\ell \in \mathbb{N}$ as follows:

$$\begin{aligned} |x_\ell \cdot y_\ell - x_0 \cdot y_0|_{-p} &= |(x_\ell - x_0)\cdot y_\ell - x_0 \cdot (y - y_0)|_{-p} \\ &\le |(x_\ell - x_0)\cdot y_\ell|_{-p} + |x_0 \cdot (y-y_0)|_{-p} \\ &\le C(|x_\ell - x_0|_{-q}|y_\ell|_{-q} + |x_0|_{-q}|y-y_0|_{-q}), \end{aligned}$$

where $C = \left(\sum_{\nu=1}^\infty \ell_\nu^{-2}\right)^{1/2}$. This implies that $x_\ell \cdot y_\ell \to x_0 \cdot y_0$ in E^* as $\ell \to \infty$. Thus we obtain the continuity of the product operator. □

5. Powers of the Delta Distribution in E^*

For $t \in [0,1]$ define $\delta_t \in E^*$ by $\delta_t(\xi) = \xi(t),\ \xi \in E$. Then the delta distribution δ_t has an expansion

$$\delta_t = \sum_{n=1}^\infty \overline{\langle \delta_t, e_n\rangle_*} e_n = \sum_{n=1}^\infty \overline{e_n(t)} e_n \tag{5.1}$$

in E^*. The m-th power of δ_t is given by

$$\begin{aligned} \delta_t^m &= \sum_{k_1,\ldots,k_m=1}^\infty \overline{e_{k_1+\cdots+k_m}(t)} e_{k_1+\cdots+k_m} \\ &= \sum_{k=m}^\infty \sum_{k_1+\cdots+k_m=k} \overline{e_k(t)} e_k \\ &= \sum_{k=m}^\infty \binom{k-1}{m-1} \overline{e_k(t)} e_k \end{aligned} \tag{5.2}$$

for $m \in \mathbb{N}$. Therefore the square of the delta distribution δ_t is given by

$$\delta_t^2 = \sum_{j,k=1}^\infty e^{-2\pi i(j+k)t} e_{j+k} = \sum_{\ell=2}^\infty \sum_{k=1}^{\ell-1} e^{-2\pi i \ell t} e_\ell = \sum_{\ell=1}^\infty (\ell-1) e^{-2\pi i \ell t} e_\ell$$

in E^*. Since

$$\delta_t' = 2\pi i \sum_{\ell=1}^\infty \ell e^{-2\pi i \ell t} e_\ell,$$

we have

$$\delta_t^2 = \frac{1}{2\pi i}\delta_t' - \delta_t$$

in E^*. In general we have the following:

Theorem 5.1. *For any $m \in \mathbb{N}^*$ and $t \in [0,1]$ we have $\delta_t \in E^*$ and the equality*

$$\delta_t^m = \frac{1}{(m-1)!} \sum_{j=0}^{m-1} \frac{1}{(2\pi i)^{m-j-1}} c_j \delta_t^{(m-j-1)} \tag{5.3}$$

holds in E^*, where constants $c_j,\ j = 0,1,2,\ldots,m$ are determined by the equality

$$(k-1)(k-2)\cdots(k-m+1) = \sum_{j=0}^{m-1} c_j k^{m-j-1}. \tag{5.4}$$

Proof. Since

$$\delta_t^{(j)} = (2\pi i)^j \sum_{k=1}^{\infty} k^j \overline{e_k(t)} e_k, \ \ j = 0, , 2, \ldots, m-1, \tag{5.5}$$

the right hand side of (5.3) equals to

$$\sum_{k=1}^{\infty} \frac{1}{(m-1)!} \sum_{j=0}^{m-1} k^{m-j-1} c_j \overline{e_k(t)} e_k.$$

By (5.4) this equals to

$$\sum_{k=m}^{\infty} \frac{(k-1)(k-2)\cdots(k-m+1)}{(m-1)!} \overline{e_k(t)} e_k = \sum_{k=m}^{\infty} \begin{pmatrix} k-1 \\ m-1 \end{pmatrix} \overline{e_k(t)} e_k.$$

This is nothing but δ_t^m as (5.2). □

Proposition 5.2. *For any $s, t \in [0,1]$ with $s - t \notin \mathbb{Z}$ we have*

$$\delta_s \delta_t = \frac{1}{e^{-2\pi i(s-t)} - 1} \delta_s + \frac{1}{e^{-2\pi i(t-s)} - 1} \delta_t.$$

Proof. Since the series expansion of the delta distribution, it holds that

$$\begin{aligned}
\delta_s \delta_t &= \sum_{j=1}^{\infty} \sum_{j=1}^{\infty} \overline{e_j(s) e_k(t)} e_{j+k} \\
&= \sum_{\ell=2}^{\infty} \sum_{j=1}^{\ell-1} \overline{e_j(s) e_{\ell-j}(t)} e_\ell \\
&= \sum_{\ell=2}^{\infty} \sum_{j=1}^{\ell-1} e^{-2\pi i j s} e^{-2\pi i(\ell-j)t} e_\ell \\
&= \sum_{\ell=2}^{\infty} \sum_{j=1}^{\ell-1} e^{-2\pi i j(s-t)} e^{-2\pi i \ell t} e_\ell \\
&= \sum_{\ell=2}^{\infty} \frac{e^{-2\pi i(s-t)} - e^{-2\pi i \ell(s-t)}}{1 - e^{-2\pi i(s-t)}} e^{-2\pi i \ell t} e_\ell \\
&= \frac{e^{-2\pi i(s-t)}}{1 - e^{-2\pi i(s-t)}} \sum_{\ell=2}^{\infty} e^{-2\pi i \ell t} e_\ell - \frac{1}{1 - e^{-2\pi i(s-t)}} \sum_{\ell=2}^{\infty} e^{-2\pi i \ell s} e_\ell \\
&= \frac{1}{e^{-2\pi i(t-s)} - 1} \left(\sum_{\ell=1}^{\infty} e^{-2\pi i \ell t} e_\ell - e^{-2\pi i t} e_\ell \right) \\
&\quad + \frac{1}{e^{-2\pi i(s-t)} - 1} \left(\sum_{\ell=1}^{\infty} e^{-2\pi i \ell s} e_\ell - e^{-2\pi i s} e_\ell \right) \\
&= \frac{1}{e^{-2\pi i(t-s)} - 1} \delta_t + \frac{1}{e^{-2\pi i(s-t)} - 1} \delta_s.
\end{aligned}$$

□

6. Powers of the Delta Distribution in the Cesàro Hilbert Space

Proposition 6.1. *For any $t \in \mathcal{C}$ we have $\delta_t^2 \in H_{c,3}$.*

Proof. By the expansion of δ_t^2 we have

$$\langle \delta_t^2, e_n \rangle_* = \sum_{k_1+k_2=n} e^{2\pi int} = (n-1)e^{2\pi int}$$

for $n \geq 2$. Therefore we can easily calculate that

$$\begin{aligned} |\delta_t^2|_{c,3}^2 &= \lim_{N\to\infty} \frac{1}{N} \sum_{n=2}^{N} \left(\frac{n-1}{N} \right)^2 \\ &= \int_0^1 x^2 dx = \frac{1}{3} < \infty. \end{aligned}$$

Consequently we obtain $\delta_t^2 \in H_{c,3}$ for any $t \in \mathcal{C}$. □

The expansion of the derivative δ_t' of δ_t is given by

$$\delta_t' = \sum_{k=1}^{\infty} (2\pi ik)\overline{e_k(t)} e_k.$$

Then we can calculate that

$$\begin{aligned} \left| \delta_t^2 - \frac{1}{2\pi i}\delta_t' \right|_{c.3}^2 &= \lim_{N\to\infty} \frac{1}{N^3} \sum_{n=1}^{N} \left| \left\langle \delta_t^2 - \frac{1}{2\pi i}\delta_t', e_n \right\rangle_* \right|^2 \\ &= \lim_{N\to\infty} \frac{1}{N^3} \sum_{n=1}^{N} |(n-1)e_n(t) - ne_n(t)|^2 \\ &= \lim_{N\to\infty} \frac{1}{N^3} \sum_{n=1}^{N} \left| e^{-2\pi int} \right|^2 = \lim_{N\to\infty} \frac{1}{N^2} = 0. \end{aligned}$$

Thus we have the following formula.

Proposition 6.2. *For all $t \in \mathcal{C}$ the equality*

$$\delta_t^2 = \frac{1}{2\pi i}\delta_t'$$

holds in $H_{c,3}$.

Remark 6.3. The product δ_t^2 is also in E^* as Theorem 5.1. However the above formula in Proposition 6.2 does not hold in E^*.

An expansion of the product of δ_t and δ_s is given by

$$\delta_t \delta_s = \sum_{j,k=1}^{\infty} e^{-2\pi i(jt+ks)} e_{j+k}.$$

Since

$$|\langle \delta_t \delta_s, e_n \rangle_*|^2 = \sum_{j+k=n} e^{2\pi i(jt+ks)},$$

we can calculate that

$$\begin{aligned}
&\frac{1}{N}\sum_{n=1}^{N}|\langle\delta_t\delta_s, e_n\rangle_*|^2 \\
&= \frac{1}{N}\sum_{n=1}^{N}\left|\frac{e^{2\pi i(t-s)}(1-e^{2\pi in(t-s)})}{1-e^{2\pi i(t-s)}}\right|^2 \\
&= \frac{1}{N}\frac{1}{1-\cos(2\pi(t-s))}\sum_{n=1}^{N}(1-\cos(2\pi n(t-s))) \\
&= \frac{1}{N}\frac{1}{1-\cos(2\pi(t-s))}\left(N-\sum_{n=1}^{N}\cos(2\pi n(t-s))\right) \\
&= \frac{1}{1-\cos(2\pi(t-s))}\left(1-\frac{1}{N}\mathrm{Re}\sum_{n=1}^{N}\left(e^{2\pi i(t-s)}\right)^n\right) \\
&= \frac{1}{1-\cos(2\pi(t-s))}\left(1-\frac{1}{N}\mathrm{Re}\frac{e^{2\pi i(t-s)}(1-e^{2\pi iN(t-s)})}{1-e^{2\pi i(t-s)}}\right)
\end{aligned}$$

for any $t, s \in \mathcal{C}$ with $t-s \notin \mathbb{Z}\setminus\{0\}$. Therefore we have

$$\begin{aligned}
|\delta_t\delta_s|_{c,1}^2 &= \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}|\langle\delta_t\delta_s, e_n\rangle_*|^2 \\
&= \frac{1}{1-\cos(2\pi(t-s))}
\end{aligned}$$

for any $t, s \in \mathcal{C}$ with $t-s \notin \mathbb{Z}\setminus\{0\}$. Thus we obtain the following:

Proposition 6.4. *For any $t, s \in \mathcal{C}$ with $t-s \notin \mathbb{Z}\setminus\{0\}$, we have*

$$\delta_t\delta_s \in H_{c,1}.$$

In general we have the following:

Theorem 6.5. *For each $m \in \mathbb{N}^*$ and $t \in \mathcal{C}$ the equality*

$$\delta_t^m = \frac{1}{(m-1)!(2\pi i)^{m-1}}\delta_t^{(m-1)} \tag{6.1}$$

holds in $H_{c,2m-1}$.

Proof. By expansions (5.1) and (5.5) we can see that

$$\begin{aligned}
&\left|\delta_t^m - \frac{1}{(m-1)!(2\pi i)^{m-1}}\delta_t^{(m-1)}\right|_{c.2m-1}^2 \\
&= \lim_{N\to\infty}\frac{1}{N^{2m-1}}\sum_{n=1}^{N}\left|\left\langle \delta_t^m - \frac{1}{(m-1)!(2\pi i)^{m-1}}\delta_t^{(m-1)}, e_n\right\rangle_*\right|^2 \\
&= \lim_{N\to\infty}\frac{1}{N^{2m-1}}\sum_{n=1}^{N}\left|\frac{1}{(m-1)!}\sum_{j=1}^{m-1}\frac{1}{(2\pi i)^{m-j-1}}c_j\langle\delta_t^{(m-j-1)}, e_n\rangle_*\right|^2 \\
&= \lim_{N\to\infty}\frac{1}{N^{2m-1}}\sum_{n=1}^{N}\left|\frac{1}{(m-1)!}\sum_{j=1}^{m-1}c_j n^{m-j-1}e^{-2\pi i n t}\right|^2 \\
&\le \frac{1}{(m-1)!^2}\sum_{j=1}^{m-1}c_j^2\sum_{j=1}^{m-1}\lim_{N\to\infty}\frac{1}{N^{2m-1}}\sum_{n=1}^{N}n^{2m-2(j+1)} \\
&\le \frac{1}{(m-1)!^2}\sum_{j=1}^{m-1}c_j^2\sum_{j=1}^{m-1}\lim_{N\to\infty}\frac{1}{N^{2m-1}}N^{2m-2j-1} = 0.
\end{aligned}$$

This implies the assertion (6.1). □

Remark 6.6. For any $m \in \mathbb{N}^*$ and $t \in \mathcal{C}$ we can regard the m-th power δ_t^m of the delta distribution δ_t as an element of $\bigoplus_{n=0}^{\infty} H_{c,2n+1}$ by (5.3):

$$\delta_t^m = \left(\frac{1}{(m-1)!}c_{m-1}\delta_t, \dots, \frac{1}{(m-1)!}\frac{1}{(2\pi i)^{m-1}}c_0\delta_t^{(m-1)}, 0, 0, 0, \dots\right).$$

7. Higher Powers of an Infinite Dimensional Brownian Motion

Let $\{B_k(t);\ t \ge 0\},\ k = 1, 2, 3, \dots$ be a sequence of independent one dimensional Brownian motions and set

$$\mathbf{B}(t) = \sum_{k=1}^{\infty} B_k(t)e_k,\ \ t \ge 0.$$

From Theorem 4.1 we can define $\mathbf{B}(t)^m \in E^*$ for any $m \in \mathbb{N}^*$ and $t \ge 0$. The distribution $\mathbf{B}(t)^m$ is given by

$$\mathbf{B}(t)^m = \sum_{k_1,k_2,\dots,k_m=1}^{\infty} B_{k_1}(t)B_{k_2}(t)\cdots B_{k_m}(t)e_{k_1+k_2+\cdots+k_m}.$$

Define an operator $\mathcal{A}$ on $C^2(E^*)$ by

$$\mathcal{A}\varphi(x) = \sum_{k=1}^{\infty}\varphi''(x)(e_k, e_k)$$

for $\varphi \in C^2(E^*)$. Since $\sum_{k=1}^{\infty} e_k \otimes e_k \in E^* \otimes E^*$, for any $\varphi \in C^2(E^*)$, $\mathcal{A}\varphi$ exists and can be written by

$$\mathcal{A}\varphi(x) = \sum_{k=1}^{\infty} \varphi''(x)(e_k \otimes e_k) = \varphi''(x)\left(\sum_{k=1}^{\infty} e_k \otimes e_k\right), \; x \in E^*.$$

Example 7.1. [1] Let $\varphi(x) = x^p(f),\ f \in E,\ p \geq 2$. Then φ is in $C^2(E^*)$ and

$$\mathcal{A}\varphi(x) = p(p-1)x^{p-2}\sum_{k=1}^{\infty} e_{2k}(f).$$

[2] Let $\varphi(x) = e^{x(f)},\ f \in E$. Then φ is in $C^2(E^*)$ and

$$\mathcal{A}\varphi(x) = \left(\sum_{k=1}^{\infty} e_k(f)\right)^2 \varphi(x).$$

[3] Let $\varphi(x) = e^{x^2(f)},\ f \in E$. Then φ is in $C^2(E^*)$ and

$$\mathcal{A}\varphi(x) = \left(2\sum_{k=1}^{\infty} e_{2k}(f) + \left(2x\sum_{k=1}^{\infty} e_k(f)\right)^2\right)\varphi(x).$$

Let $\mathcal{L}(\mathcal{X};\mathcal{Y})$ denote the continuous linear operators from $\mathcal{X}$ into $\mathcal{Y}$ for locally convex linear topological spaces $\mathcal{X}$ and $\mathcal{Y}$. We denote $\mathcal{L}(\mathcal{X};\mathcal{X})$ by $\mathcal{L}(\mathcal{X})$ simply. We endow a base probability space $(\Omega, \mathcal{F}, P)$ with a reference family $\{\mathcal{F}_t;\ t \geq 0\}$ of σ-subalgebras of $\mathcal{F}$, and assume that $\{\mathbf{B}(t);\ t \geq 0\}$ is adapted to $(\mathcal{F}_t)$, i.e., for each $t \geq 0$, $\mathbf{B}(t)$ is $\mathcal{F}_t$-measurable.

Let $\{X(t);\ t \geq 0\}$ be an $\mathcal{L}(E^*;\mathbb{C})$-valued stochastic process. If a function $[0,\infty) \times \Omega \ni (t,\omega) \mapsto X(t)(\omega)(f) \in \mathbb{C}$ is a $(\mathcal{F}_t)$-adapted process for every $f \in E^*$, then $\{X(t);\ t \geq 0\}$ is called a $(\mathcal{F}_t)$-*adapted* $\mathcal{L}(E^*;\mathbb{C})$-*valued process*.

We define the stochastic integral $\int_0^t X(t)d\mathbf{B}(t)$ by

$$\int_0^t X(t)d\mathbf{B}(t) := \sum_{k=1}^{\infty}\int_0^t X(t)(e_k)dB_k(t)$$

under the following conditions:

(1): $\{X(t);\ t \geq 0\}$ is a $(\mathcal{F}_t)$-adapted $\mathcal{L}(E^*;\mathbb{C})$-valued process.
(2): $\sum_{k=1}^{\infty}\int_0^t E[|X(s)(e_k)|^2]ds < \infty,\ t \geq 0$.

For any $\varphi \in C^2(E^*)$ with $\sup_{x\in E^*}|\varphi''(x)|_{\mathcal{L}(E_{-p},\mathcal{L}(E_{-p}))} < \infty$ for some $p \geq 1$ and $t \geq 0$ the stochastic integral $\int_0^t \varphi'(\mathbf{B}(u))(d\mathbf{B}(u))$ is given by

$$\int_0^t \varphi'(\mathbf{B}(u))(d\mathbf{B}(u)) = \sum_{k=1}^{\infty}\int_0^t \varphi'(\mathbf{B}(u))(e_k)dB_k(u)$$

since

$$E\left[\left|\sum_{k=1}^{\infty}\int_0^t \varphi'(\mathbf{B}(u))(e_k)dB_k(u)\right|^2\right] = \sum_{k=1}^{\infty}\int_0^t E[|\varphi'(\mathbf{B}(u))(e_k)|^2]du$$
$$\leq const.\sum_{k=1}^{\infty}\ell_k^{-2p}\int_0^t E[|\mathbf{B}(u)|_{-p}^2]du$$
$$\leq const.\left(\sum_{k=1}^{\infty}\ell_k^{-2p}\right)^2 t^2.$$

With the first exit time

$$\sigma_{x,-p}^{(r)} := \inf\{t \geq 0;\ x + \mathbf{B}(t) \notin B_r(x)\}$$

for $x + \mathbf{B}(t)$ from the open ball $B_r(x) := \{y \in E_{-p};\ |y - x|_{-p} < r\}$ in E_{-p} for $p \geq 1$ we have an extension of the Itô formula using the similar method in [6] with changing the time parameter t by $t \wedge \sigma_{x,-p}^{(r)}$.

Theorem 7.2. *Let $\varphi \in C^2(E^*)$. Then the equality*

$$\varphi(\mathbf{B}(t)) - \varphi(\mathbf{B}(s)) = \sum_{k=1}^{\infty}\int_s^t \varphi'(\mathbf{B}(u))(e_k)dB_k(u) + \frac{1}{2}\int_s^t \mathcal{A}\varphi(\mathbf{B}(u))du$$

holds for $t \geq s \geq 0$, which is written by

$$d\varphi(\mathbf{B}(t)) = \sum_{k=1}^{\infty}\varphi'(\mathbf{B}(t))(e_k)dB_k(t) + \frac{1}{2}\mathcal{A}\varphi(\mathbf{B}(t))dt.$$

Remark 7.3. As in example 7.1 $C^2(E^*)$ includes functionals of polynomials of $x \in E^*$.

Since for any $\varphi \in C^2(E^*)$ and $m \in \mathbb{N}^*$ the equalities

$$d\varphi(\mathbf{B}(t)^m) = \varphi'(\mathbf{B}(t)^m)(d(\mathbf{B}(t)^m)) + \frac{1}{2}\varphi''(\mathbf{B}(t)^m)(d(\mathbf{B}(t)^m), d(\mathbf{B}(t)^m)) \tag{7.1}$$

and

$$d(\mathbf{B}(t)^m) = m\mathbf{B}(t)^{m-1}d\mathbf{B}(t) + \frac{m(m-1)}{2}\mathbf{B}(t)^{m-2}(d\mathbf{B}(t))^2$$
$$= m\mathbf{B}(t)^{m-1}d\mathbf{B}(t) + \frac{m(m-1)}{2}\mathbf{B}(t)^{m-2}\sum_{k=1}^{\infty}e_{2k}dt$$

hold, we have the following:

Theorem 7.4. *For any $\varphi \in C^2(E^*)$ and $m \in \mathbb{N}^*$, the equality*

$$\begin{aligned} d\varphi(\mathbf{B}(t)^m) &= m\sum_{k=1}^{\infty}\varphi'(\mathbf{B}(t)^m)(\mathbf{B}(t)^{m-1}e_k)dB_k(t) \\ &\quad + \frac{m(m-1)}{2}\sum_{k=1}^{\infty}\varphi'(\mathbf{B}(t)^m)(\mathbf{B}(t)^{m-2}e_{2k})dt \\ &\quad + \frac{m^2}{2}\sum_{k=1}^{\infty}\varphi''(\mathbf{B}(t)^m)(\mathbf{B}(t)^{m-1}e_k, \mathbf{B}(t)^{m-1}e_k)dt \end{aligned}$$

holds.

Remark 7.5. By the equation 7.1 we have a recursion formula:

$$E[\mathbf{B}(t)^m] = \frac{m(m-1)}{2}\int_0^t E[\mathbf{B}(t_1)^{m-2}]dt_1 \sum_{k=1}^{\infty} e_{2k}.$$

Therefore we can calculate that

$$E[\mathbf{B}(t)^{2n}] = (2n-1)!!t^n \left(\sum_{k=1}^{\infty} e_{2k}\right)^n, \; E[\mathbf{B}(t)^{2n-1}] = 0$$

for any $n \in \mathbb{N}^*$.

Let $\mathcal{P}_k(x) := x^k,\; x \in E^*$ for $k \in \mathbb{N}$ and denote the set of all elements Φ expressed in the form

$$\Phi = \sum_{k=0}^{n} c_k \mathcal{P}_k,\; c_k \in E^* \oplus \mathbb{C},\; k = 0, 1, 2, \ldots, n,\; n \in \mathbb{N}$$

by $\mathrm{Poly}(E^*)$. This is the set of all polynomials on E^* with $E^* \oplus \mathbb{C}$-coefficients and is in the space $C^\infty(E^*, E^*)$ of C^∞-functions from E^* into itself. Let

$$\mathcal{D} := LS\{\Phi(\cdot)(f) |\; f \in E,\; \Phi \in \mathrm{Poly}(E^*)\}.$$

Then $\mathcal{D}$ is included in $C^2(E^*)$ and we have the following.

Theorem 7.6. *For any $\varphi \in \mathcal{D}$ and $t \geq 0$ we have*

$$e^{\frac{t}{2}\mathcal{A}}\varphi(x) = E[\varphi(x + \mathbf{B}(t))],\; x \in E^*.$$

Proof. We may prove the theorem for functionals $\varphi \in \mathcal{D}$ given in the form

$$\varphi(x) = x^p(f),\; p \in \mathbb{N}^*,\; f \in E.$$

It is easily calculate that

$$\mathcal{A}^n\varphi(x) = p(p-1)\cdots(p-2n+1)\left(x^{p-2n}\left(\sum_{k=1}^{\infty} e_{2k}\right)^n\right)(f)$$

for any $n \in \mathbb{N}$. Therefore we have

$$e^{\frac{t}{2}\mathcal{A}}\varphi(x) = \sum_{n=0}^{[p/2]} \left(\frac{t}{2}\right)^n \begin{pmatrix} p \\ 2n \end{pmatrix} \frac{(2n)!}{n!} \left(x^{p-2n}\left(\sum_{k=1}^{\infty} e_{2k}\right)^n\right)(f).$$

On the other hand, by Lemma 7.5, we can calculate that

$$\begin{aligned} E[\varphi(x+\mathbf{B}(t))] &= E[(x+\mathbf{B}(t))^p(f)] \\ &= \sum_{k=0}^{\infty} \begin{pmatrix} p \\ k \end{pmatrix} \left(x^{p-k}E[\mathbf{B}(t)^k]\right)(f) \\ &= \sum_{n=0}^{[p/2]} \left(\frac{t}{2}\right)^n \begin{pmatrix} p \\ 2n \end{pmatrix} \frac{(2n)!}{n!} \left(x^{p-2n}\left(\sum_{k=1}^{\infty} e_{2k}\right)^n\right)(f). \end{aligned}$$

Thus the assertion is obtained. □

8. A Relationship to the Lévy and Volterra Laplacians

Adding a constant function $e_0(u) := 1$ to $e_n(u) := e^{2\pi i n u}, n \in \mathbb{N}^*$ we consider the set

$$\{e_n;\ n \in \mathbb{N}\},\ \mathbb{N} := \{0, 1, 2, 3, \ldots\}.$$

This set is also orthonormal in $L^2([0,1])$. Let H_0 be a closed subspace of $L^2([0,1])$ generated by the functions $e_n, n \in \mathbb{N}$. Then for any sequence $\{\ell_k\}_{k=1}^{\infty}$ satisfying

$$1 < \ell_0 < \ell_1 < \ell_2 < \ell_3 < \cdots, \text{ and } \sum_{k=0}^{\infty} \ell_k^{-2} < \infty,$$

and introducing a densely defined selfadjoint operator A on H_0 by

$$A\xi = \sum_{k=0}^{\infty} \ell_k \alpha_k e_k,\ \xi = \sum_{k=0}^{\infty} \alpha_k e_k \in H_0,$$

we can define spaces E and E^* similarly in Section 2. We denote these spaces E and E^* by $\widetilde{E}$ and $\widetilde{E}^*$, respectively. We can also compute the product of distributions in $\widetilde{E}^*$ as follows:

$$x \cdot y = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \overline{\langle x, e_k\rangle_* \langle y, e_{j-k}\rangle_*} e_j$$

for $x = \sum_{j=0}^{\infty} \overline{\langle x, e_j\rangle_*} e_j \in \widetilde{E}^*$ and $y = \sum_{j=0}^{\infty} \overline{\langle y, e_j\rangle_*} e_j \in \widetilde{E}^*$. Then, assuming that for any $\alpha > 0$ there exists $\beta > 0$ such that

$$(2(k+1)\pi)^{\alpha} \leq \ell_k^{\beta},\ k = 1, 2, 3, \ldots,$$

we also have the following as in Section 2.

Theorem 8.1. *For any x and y in $\widetilde{E}^*$ the above product $x \cdot y$ is also in $\widetilde{E}^*$. Moreover the product operator $\cdot :\ \widetilde{E}^* \times \widetilde{E}^* \to \widetilde{E}^*$ is a continuous bilinear operator.*

The Lévy Laplacian and the Volterra Laplacian are introduced in [8] and [12], respectively. By the Theorem 8.1 we can define the *LV functional* φ by a functional in $C^2(\widetilde{E}^*)$ such that for any $x \in \widetilde{E}^*$ its second derivation $\varphi''(x)(y,z), y, z \in \widetilde{E}^*$ is given by the form

$$\varphi''(x)(y,z) = \varphi''_L(x)(yz) + \varphi''_V(x)(y,z),\ y, z \in \widetilde{E}^*,$$

where $\varphi''_L(x) \in \mathcal{L}(\widetilde{E}^*)$ and $\varphi''_V(x) \in \mathcal{L}(\widetilde{E}^* \times \widetilde{E}^*)$ that is the trace class operator of H_0.

Here we define the *Lévy Laplacian* $\Delta_L\varphi$ of $\varphi \in C(\widetilde{E}^*)$ by

$$\Delta_L\varphi(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \varphi''(x)(J(e_n), e_n),\ x \in \widetilde{E}^*,$$

if the limit exists, where J is the conjugate operator. Then for an LV functional φ we have $\Delta_L\varphi(x) = \varphi''_L(x)(e_0)$. Since $\sum_{k=0}^{\infty} e_{2k} \in \widetilde{E}^*$ we set this distribution by $\delta(0)$. This means the divergent part in usual infinite dimensional analysis. However we can get the part by a distribution in $\widetilde{E}^*$. Therefore we can introduce an operator $\tau_{\delta(0)}$ defined on the range of the Lévy Laplacian by

$$\tau_{\delta(0)}\Delta_L\varphi(x) = \varphi''_L(x)(\delta(0)e_0).$$

The Volterra Laplacian Δ_V for an LV functional is defined as a trace of $\varphi''_V(x)$ for $x \in \widetilde{E}^*$:

$$\Delta_V\varphi(x) := \text{trace } \varphi''_V(x) \left[:= \sum_{k=0}^{\infty} \varphi''_V(x)(e_k, e_k)\right].$$

Then we have an interesting formula:

$$\mathcal{A}\varphi = \tau_{\delta(0)}\Delta_L\varphi + \Delta_V\varphi$$

for any LV functional φ. It is important that $\mathcal{A}\varphi, \tau_{\delta(0)}\Delta_L\varphi$ and $\Delta_V\varphi$ can be defined explicitly for any LV functional φ, and this formula holds without any divergent part.

For each example in Example 7.1 we can calculate the Lévy Laplacian and the Volterra Laplacian as follows.

Example 8.2. [4] Let $\varphi(x) = x^p(f),\ f \in \widetilde{E},\ p \geq 2$. Then φ is in $C^2(\widetilde{E}^*)$ and

$$\Delta_L\varphi(x) = p(p-1)x^{p-2}(f),\ \Delta_V\varphi(x) = 0.$$

[5] Let $\varphi(x) = x^p(f)x^q(g),\ f, g \in \widetilde{E},\ p, q \geq 2$. Then φ is in $C^2(\widetilde{E}^*)$ and

$$\begin{aligned}\Delta_L\varphi(x) &= p(p-1)x^{p-2}(f) + q(q-1)x^{q-2}(g),\\ \Delta_V\varphi(x) &= 2pq\sum_{k=0}^{\infty}(x^{p-1}e_k)(f)(x^{q-1}e_k)(g).\end{aligned}$$

[6] Let $\varphi(x) = e^{x(f)},\ f \in \widetilde{E}$. Then φ is in $C^2(\widetilde{E}^*)$ and

$$\Delta_L\varphi(x) = 0,\ \Delta_V\varphi(x) = \sum_{k=0}^{\infty} e_k(f)^2\varphi(x).$$

[7] Let $\varphi(x) = e^{x^2(f)},\ f \in \widetilde{E}$. Then φ is in $C^2(\widetilde{E}^*)$ and

$$\Delta_L\varphi(x) = 2e_0(f)\varphi(x),\ \Delta_V\varphi(x) = 4\sum_{k=0}^{\infty}(xe_k)(f)^2\varphi(x).$$

Let $e_{\mathbb{R},0},\ e_{\mathbb{R},2k},\ e_{\mathbb{R},2k-1},\ k = 1, 2, 3, \ldots$ be functions on $[0, 1]$ given by

$$e_{\mathbb{R},0}(u) = 1,\ e_{\mathbb{R},2k}(u) := \sqrt{2}\sin(2k\pi u),\ e_{\mathbb{R},2k-1}(u) := \sqrt{2}\cos(2k\pi u),\ u \in [0,1]$$
$$k = 1, 2, 3, \ldots,$$

which form an orthonormal basis of $L^2_{\mathbb{R}}([0,1])$. Let $H_{\mathbb{R}}$ be a closed subspace of $L^2_{\mathbb{R}}([0,1])$ generated by functions

$$e_{\mathbb{R},2k},\ e_{\mathbb{R},2k-1};\ k=1,2,\ldots.$$

Then we can construct spaces $E_{\mathbb{R}}, E^*_{\mathbb{R}}$ using the sequence $\{\ell_k\}$ and an operator $A_{\mathbb{R}}$ given by

$$A_{\mathbb{R}}\xi=\sum_{n=1}^{\infty}\ell_n\alpha_n e_{\mathbb{R},n},\ \xi=\sum_{n=1}^{\infty}\alpha_n e_{\mathbb{R},n}\in H_{\mathbb{R}}$$

as in Section 2, and have an expression of $\widetilde{E}^*$ as follows:

$$\begin{aligned}\widetilde{E}^* \quad = \quad & \Bigg\{c+x_r+i\widetilde{x_r};\ c\in\mathbb{C},\ x_r=\sum_{n=1}^{\infty}\alpha_n e_{\mathbb{R},n}\in E^*_{\mathbb{R}},\\ & \qquad \widetilde{x_r}=\sum_{k=1}^{\infty}(\alpha_{2k-1}e_{\mathbb{R},2k}-\alpha_{2k}e_{\mathbb{R},2k-1})\Bigg\}. \qquad (8.1)\end{aligned}$$

Remark 8.3. For any $x=\sum_{n=0}^{\infty}\beta_n e_n\in\widetilde{E}^*$ we have

$$\begin{aligned}x \quad &= \quad \beta_0+\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}}(\beta_{1,k}+i\beta_{2,k})(e_{\mathbb{R},2k-1}+ie_{\mathbb{R},2k})\\ &= \quad \beta_0+\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}}(\beta_{1,k}e_{\mathbb{R},2k-1}-\beta_{2,k}e_{\mathbb{R},2k})+i\sum_{k=1}^{\infty}\frac{1}{\sqrt{2}}(\beta_{1,k}e_{\mathbb{R},2k}+\beta_{2,k}e_{\mathbb{R},2k-1}),\end{aligned}$$

where $\beta_{1,n}=\mathrm{Re}\beta_n$ and $\beta_{2,n}=\mathrm{Im}\beta_n$. Then, putting

$$\alpha_0=\beta_0,\ \alpha_{2k-1}=\frac{1}{\sqrt{2}}\beta_{1,k},\ \alpha_{2k}=-\frac{1}{\sqrt{2}}\beta_{2,k},\ k=1,2,3,\ldots,$$

we obtain the expression (8.1) of $\widetilde{E}^*$.

By the Bochner-Minlos theorem there exists a probability measure μ_r on the Borel field $\mathcal{B}(E^*_{\mathbb{R}})$ of $E^*_{\mathbb{R}}$ such that the characteristic functional is given by

$$\int_{E^*_{\mathbb{R}}}e^{i\langle x,\xi\rangle}d\mu_r(x)=e^{-\frac{1}{2}\int_{[0,1]}\xi(u)^2du},\ \xi\in E_{\mathbb{R}},$$

where $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $E^*_{\mathbb{R}}\times E_{\mathbb{R}}$. We also denote the canonical bilinear form on $E^*_{\mathbb{C}}\times E_{\mathbb{C}}$ by the same notation $\langle\cdot,\cdot\rangle$. Here $E_{\mathbb{C}}$ and $E^*_{\mathbb{C}}$ are the complexifications of $E_{\mathbb{R}}$ and $E^*_{\mathbb{R}}$, respectively. Note that $\widetilde{E}^*$ is purely included in $E^*_{\mathbb{C}}$.

We introduce a probability measure μ on the Borel field $\mathcal{B}(\widetilde{E}^*)$ of $\widetilde{E}^*$ by

$$\mu(A)=\mu_0(B_0)\mu_r(B)$$

for $A=\{c+x_r+i\widetilde{x_r}\in\widetilde{E}^*;c\in B_0\ x_r\in B\}\in\mathcal{B}(\widetilde{E}^*),\ B_0\in\mathcal{B}(\mathbb{C}),\ B\in\mathcal{B}(E^*_{\mathbb{R}})$, where μ_0 is the standard Gaussian measure on the Borel field $\mathcal{B}(\mathbb{C})\equiv\mathcal{B}(\mathbb{R}^2)$ of $\mathbb{C}$ and $\mathcal{B}(E^*_{\mathbb{R}})$ is the Borel field of $E^*_{\mathbb{R}}$.

Remark 8.4. For $B_j \in \mathcal{B}(E^*_{\mathbb{R}}),\ j = 1, 2$, set

$$A_j = \{c + x_r + i\widetilde{x_r} \in E^*; c \in C_j,\ x_r \in B_j\} \in \mathcal{B}(\widetilde{E}^*).$$

Since a mapping $E^*_{\mathbb{R}} \ni x_r \mapsto \widetilde{x_r} \in E^*_{\mathbb{R}}$ is a bijection, we have

$$A_1 \cap A_2 = \phi \Leftrightarrow (C_1 \times B_1) \cap (C_2 \times B_2) = \phi.$$

Then $\mathcal{D}$ is in $L^2(\widetilde{E}^*, \mu)$ with norm $||\cdot||_0$ given by

$$||\varphi||_0 = \left(\int_{\widetilde{E}^*} |\varphi(x)|^2 d\mu(x)\right)^{1/2},\ \varphi \in L^2(\widetilde{E}^*, \mu).$$

Let $L^2(\widetilde{E}^*, \mu; \widetilde{E}_{-p})$ be the Hilbert space consisting of $\widetilde{E}_{-p}$-valued L^2-functions on $(\widetilde{E}^*, \mu)$ with a Hilbertian norm $||\cdot||_{*,-p}$ for any $p \geq 0$, given by

$$||\Phi||_{*,-p} = \left(\int_{\widetilde{E}^*} |\Phi(x)|^2_{-p} d\mu(x)\right)^{1/2},\ \Phi \in L^2(\widetilde{E}^*, \mu; \widetilde{E}_{-p}).$$

Lemma 8.5. *For any* $\Phi \in \mathrm{Poly}(\widetilde{E}^*)$ *there exists* $\alpha > 0$ *such that* Φ *is in* $L^2(\widetilde{E}^*, \mu; \widetilde{E}_{-\alpha})$.

Proof. By the Minkovskii and Hörder inequalities, for any $m \in \mathbb{N}$ and $\beta > 1$ it holds that

$$\begin{aligned}
&\int_{\widetilde{E}^*} |x|^{2m}_{-\beta} d\mu(x) = \int_{\mathbb{C}\oplus E^*_{\mathbb{R}}} |y|^{2m}_{-\beta} d\mu_0 \otimes \mu_r(y) \\
=\ & \int_{\mathbb{C}\oplus E^*_{\mathbb{R}}} \left(\sum_{j=0}^{\infty} \ell_j^{-2\beta} |\langle y, e_{\mathbb{R},2j+1}\rangle|^2 + \sum_{j=0}^{\infty} \ell_j^{-2\beta} |\langle y, e_{\mathbb{R},2j}\rangle|^2\right)^m d\mu_0 \otimes \mu_r(y) \\
\leq\ & \left[\left\{ \int_{\mathbb{C}\oplus E^*_{\mathbb{R}}} \left(\sum_{j=0}^{\infty} \ell_j^{-2\beta} |\langle y, e_{\mathbb{R},2j+1}\rangle|^2\right)^m d\mu_0 \otimes \mu_r(y) \right\}^{1/m} \right. \\
& \left. + \left\{ \int_{\mathbb{C}\oplus E^*_{\mathbb{R}}} \left(\sum_{j=0}^{\infty} \ell_j^{-2\beta} |\langle y, e_{\mathbb{R},2j}\rangle|^2\right)^m d\mu_0 \otimes \mu_r(y) \right\}^{1/m} \right]^m \\
\leq\ & \left[C^{1-\frac{1}{m}} \left\{ \int_{\mathbb{C}\oplus E^*_{\mathbb{R}}} \left(\sum_{j=0}^{\infty} \ell_j^{-2\beta} |\langle y, e_{\mathbb{R},2j+1}\rangle|^{2m}\right) d\mu_0 \otimes \mu_r(y) \right\}^{1/m} \right. \\
& \left. + C^{1-\frac{1}{m}} \left\{ \int_{\mathbb{C}\oplus E^*_{\mathbb{R}}} \left(\sum_{j=0}^{\infty} \ell_j^{-2\beta} |\langle y, e_{\mathbb{R},2j}\rangle|^{2m}\right) d\mu_0 \otimes \mu_r(y) \right\}^{1/m} \right]^m \\
\leq\ & (2C)^m \int_{\mathbb{R}} u^{2m} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du = (2C)^m (2m-1)!! < \infty,
\end{aligned} \tag{8.2}$$

where $C = \sum_{j=0}^{\infty} \ell_j^{-2}$. Let $\Phi(x) = \sum_{m=0}^{n} a_m x^m,\ n \in \mathbb{N},\ a_m \in \widetilde{E}^*$. Then by Theorem 4.1 there exists $\alpha > \beta$ such that $\Phi(x) \in \widetilde{E}_{-\alpha}$ for all $x \in \widetilde{E}^*$. The same theorem and (8.2) imply that

$$\begin{aligned}||\Phi||_{*,-\alpha}^2 &= \int_{\widetilde{E}^*} \left|\sum_{m=0}^n a_m x^m\right|_{-\alpha}^2 d\mu(x) \\ &\le \sum_{m=0}^n C^m |a_m|_{-\beta}^2 \int_{\widetilde{E}^*} |x|_{-\beta}^{2m} d\mu(x) < \infty.\end{aligned}$$

Thus we obtain the assertion. □

Define an operator $\overline{\mathcal{A}}$ on $\mathrm{Poly}(\widetilde{E}^*)$ by

$$(\overline{\mathcal{A}}\Phi(\cdot))(f) = \mathcal{A}(\Phi(\cdot)(f)),\ \ f \in \widetilde{E}.$$

We use the same notation $\mathcal{A}$ instead of $\overline{\mathcal{A}}$. Let D_y be an operator on $\mathrm{Poly}(\widetilde{E}^*)$ defined by

$$(D_y\Phi)(\cdot) = \Phi'(\cdot)(y),\ \ \Phi \in \mathrm{Poly}(\widetilde{E}^*)$$

for $y \in \widetilde{E}^*$. For $k \in \mathbb{N}^*$, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ and $\mathbf{j} = (j_1, \ldots, j_k) \in \mathbb{N}^k$ we denote

$$D_{e_{\mathbf{j}}}^{\alpha} = D_{e_{j_1}}^{\alpha_1} \cdots D_{e_{j_k}}^{\alpha_k}$$

Since for any $\Phi \in \mathrm{Poly}(\widetilde{E}^*)$ there exists $p > 0$ such that

$$\sum_{k=1}^{\infty} \sum_{\alpha_1,\ldots,\alpha_k=0}^{\infty} \sum_{j_1,\ldots,j_k=0}^{\infty} \ell_{j_1}^{2p} \cdots \ell_{j_k}^{2p} ||D_{e_{\mathbf{j}}}^{\alpha}\Phi||_{*,-p}^2 < \infty$$

for any $p > 0$ we define a norm $|||\cdot|||_{-p}$ on $\mathrm{Poly}(\widetilde{E}^*)$ by

$$|||\Phi|||_{-p} := \left(\sum_{k=1}^{\infty} \sum_{\alpha_1,\ldots,\alpha_k=0}^{\infty} \sum_{j_1,\ldots,j_k=0}^{\infty} \ell_{j_1}^{2p} \cdots \ell_{j_k}^{2p} ||D_{e_{\mathbf{j}}}^{\alpha}\Phi||_{*,-p}^2\right)^{1/2} \quad (\in [0,\infty]),$$

for $\Phi \in \mathrm{Poly}(\widetilde{E}^*)$ and also define spaces $\mathbb{D}_{-p}$ and $\mathbb{D}_{-\infty}$ by the completion of

$$\{\Phi \in \mathrm{Poly}(\widetilde{E}^*);\ |||\Phi|||_{-p} < \infty\}$$

with respect to $|||\cdot|||_{-p}$, and the inductive limit space of $\mathbb{D}_{-p},\ p > 0$, respectively.

Lemma 8.6. *Let $p \ge 1$. Then for any $y \in \widetilde{E}^*$ the inequality*

$$|||D_y\Phi|||_{-p} \le |y|_{-p}|||\Phi|||_{-p},\ \ \Phi \in \mathrm{Poly}(\widetilde{E}^*)$$

holds.

Proof. Let $p \geq 1$. Then for any $y \in \widetilde{E}^*$ and $\Phi \in \mathrm{Poly}(\widetilde{E}^*)$ we can check that

$$\begin{aligned}
|||D_y\Phi|||_{-p}^2 &= \sum_{k=1}^{\infty} \sum_{\alpha_1,\ldots,\alpha_k=0}^{\infty} \sum_{j_1,\ldots,j_k=0}^{\infty} \ell_{j_1}^{2p}\cdots\ell_{j_k}^{2p} ||D_{e_{\mathbf{j}}}^{\alpha} D_y\Phi||_{*,-p}^2 \\
&= \sum_{k=1}^{\infty} \sum_{\alpha_1,\ldots,\alpha_k=0}^{\infty} \sum_{j_1,\ldots,j_k=0}^{\infty} \ell_{j_1}^{2p}\cdots\ell_{j_k}^{2p} \left\| D_{e_{\mathbf{j}}}^{\alpha} \sum_{\nu=0}^{\infty} \overline{\langle y, e_\nu\rangle_*} D_{e_\nu}\Phi \right\|_{*,-p}^2 \\
&\leq \sum_{k=1}^{\infty} \sum_{\alpha_1,\ldots,\alpha_k=0}^{\infty} \sum_{j_1,\ldots,j_k=0}^{\infty} \ell_{j_1}^{2p}\cdots\ell_{j_k}^{2p} \cdot \\
&\qquad \cdot \left(\sum_{\nu=0}^{\infty} |\overline{\langle y, e_\nu\rangle_*}| \left\| D_{e_{\mathbf{j}}}^{\alpha} D_{e_\nu}\Phi \right\|_{*,-p} \right)^2 \\
&\leq \sum_{k=1}^{\infty} \sum_{\alpha_1,\ldots,\alpha_k=0}^{\infty} \sum_{j_1,\ldots,j_k=0}^{\infty} \ell_{j_1}^{2p}\cdots\ell_{j_k}^{2p} \sum_{\nu=0}^{\infty} \ell_\nu^{-2p} |\langle y, e_\nu\rangle_*|^2 \cdot \\
&\qquad \cdot \sum_{\nu=0}^{\infty} \ell_\nu^{2p} \left\| D_{e_{\mathbf{j}}}^{\alpha} D_{e_\nu}\Phi \right\|_{*,-p}^2 \\
&\leq |y|_{-p}^2 |||\Phi|||_{-p}^2.
\end{aligned}$$

Thus the assertion holds. □

By Lemma 8.6 for any $y \in \widetilde{E}^*$ the operator D_y can be extended to a continuous linear operator, denoted by the same notation, from $\mathbb{D}_{-p}$ into itself by $D_y\Phi = \lim_{n\to\infty} D_y\Phi_n$ in $\mathbb{D}_{-p}$, where $(\Phi_n)_{n=1}^{\infty}$ is a sequence of elements of $\mathrm{Poly}(\widetilde{E}^*)$ approximating Φ. The operator can also be extended to a continuous linear operator from $\mathbb{D}_{-\infty}$ into itself.

Lemma 8.7. *There exists a constant $C > 0$ such that*

$$|||\mathcal{A}\Phi|||_{-p} \leq C|||\Phi|||_{-p}$$

holds for any $p \geq 1$ and $\Phi \in \mathrm{Poly}(\widetilde{E}^*)$.

Proof. Let $p > 1$. Then it holds that

$$\begin{aligned}
|||\mathcal{A}\Phi|||_{-p}^2 &= \sum_{k=1}^{\infty} \sum_{\alpha_1,\ldots,\alpha_k=0}^{\infty} \sum_{j_1,\ldots,j_k=0}^{\infty} \ell_{j_1}^{2p}\cdots\ell_{j_k}^{2p} \left\| D_{e_{\mathbf{j}}}^{\alpha} \sum_{\nu=0}^{\infty} D_{e_\nu}^2\Phi \right\|_{*,-p}^2 \\
&\leq \sum_{k=1}^{\infty} \sum_{\alpha_1,\ldots,\alpha_k=0}^{\infty} \sum_{j_1,\ldots,j_k=0}^{\infty} \ell_{j_1}^{2p}\cdots\ell_{j_k}^{2p} \left(\sum_{\nu=0}^{\infty} \left\| D_{e_{\mathbf{j}}}^{\alpha} D_{e_\nu}^2\Phi \right\|_{*,-p} \right)^2 \\
&\leq \sum_{k=1}^{\infty} \sum_{\alpha_1,\ldots,\alpha_k=0}^{\infty} \sum_{j_1,\ldots,j_k=0}^{\infty} \ell_{j_1}^{2p}\cdots\ell_{j_k}^{2p} \sum_{\nu=0}^{\infty} \ell_\nu^{-2p} \sum_{\nu=0}^{\infty} \ell_\nu^{2p} \left\| D_{e_{\mathbf{j}}}^{\alpha} D_{e_\nu}^2\Phi \right\|_{*,-p}^2
\end{aligned}$$

for any $\Phi \in \mathrm{Poly}(\widetilde{E}^*)$. Putting $C = \left(\sum_{\nu=0}^{\infty} \ell_\nu^{-2}\right)^{1/2}$ we obtain the assertion. □

By Lemma 8.7 the operator $\mathcal{A}$ can also be extended to a continuous linear operator, denoted by the same notation, from $\mathbb{D}_{-p}$ into itself as follows. The extension is given by $\mathcal{A} = \sum_{k=0}^{\infty} D_{e_k}^2$. The operator can also be extended to a continuous linear operator from $\mathbb{D}_{-\infty}$ into itself.

By Lemma 8.7, for any $t \geq 0$, $e^{\frac{t}{2}\mathcal{A}}$ is also naturally extended to a continuous linear operator on $\mathbb{D}_{-\infty}$ into itself.

Lemma 8.8. *For any $t \geq 0$ $e^{\frac{t}{2}\mathcal{A}}$ is a C_0-semigroup of continuous linear operators from $\mathbb{D}_{-\infty}$ into itself.*

Proof. The linearity of $\mathcal{A}$ is obvious. It is easily checked that

$$e^{\frac{t}{2}\mathcal{A}} e^{\frac{s}{2}\mathcal{A}} = e^{\frac{t+s}{2}\mathcal{A}}$$

for any $t, s \geq 0$ and $e^{0\mathcal{A}}$ is an identity operator from $\mathbb{D}_{-\infty}$ into itself. Let $\Phi \in \mathbb{D}_{-\infty}$. Then there exists $p > 0$ such that $\Phi \in \mathbb{D}_{-p}$. By Lemma 8.7 it holds that

$$|||e^{\frac{t}{2}\mathcal{A}}\Phi|||_{-p} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{t}{2}\right)^k |||\mathcal{A}^k \Phi|||_{-p} \leq e^{C\frac{t}{2}} |||\Phi|||_{-p}.$$

Since

$$|||e^{\frac{t}{2}\mathcal{A}}\Phi - \Phi|||_{-p} = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{t}{2}\right)^k |||\mathcal{A}^k \Phi|||_{-p} \leq \left(e^{C\frac{t}{2}} - 1\right) |||\Phi|||_{-p} \to 0$$

as $t \to 0$ and Lemma 8.7 we have

$$|||e^{\frac{t}{2}\mathcal{A}}\Phi - e^{\frac{s}{2}\mathcal{A}}\Phi|||_{-p} = |||e^{\frac{s}{2}\mathcal{A}}(e^{\frac{t-s}{2}\mathcal{A}} - 1)\Phi|||_{-p} \leq e^{C\frac{s}{2}} |||(e^{\frac{t-s}{2}\mathcal{A}} - 1)\Phi|||_{-p} \to 0$$

as $t \searrow s$. Thus we obtain the assertion. □

Let T_y be a translation operator on $\mathrm{Poly}(\widetilde{E}^*)$ defined by $T_y\Phi(x) = \Phi(x+y)$ for $y \in \widetilde{E}^*$. This operator is given by

$$T_y\Phi = \sum_{j=0}^{\infty} \frac{1}{j!} D_y^j \Phi, \ \Phi \in \mathrm{Poly}(\widetilde{E}^*).$$

Then we have the following:

Lemma 8.9. *Let $p \geq 1$ and $\Phi \in \mathrm{Poly}(\widetilde{E}^*)$. Then for any $y \in \widetilde{E}^*$ it holds that*

$$|||T_y\Phi|||_{-p} \leq e^{|y|_{-p}} |||\Phi|||_{-p}.$$

Proof. Let $p \geq 1$ and $\Phi \in \mathrm{Poly}(\widetilde{E}^*)$. Then for any $y \in \widetilde{E}^*$ we can check that

$$\begin{aligned} |||T_y\Phi|||_{-p} &= \left|\left|\left| \sum_{j=0}^{\infty} \frac{1}{j!} D_y^j \Phi \right|\right|\right|_{-p} \leq \sum_{j=0}^{\infty} \frac{1}{j!} |||D_y^j \Phi|||_{-p} \\ &\leq \sum_{j=0}^{\infty} \frac{1}{j!} |y|_{-p}^j |||\Phi|||_{-p} = e^{|y|_{-p}} |||\Phi|||_{-p}. \end{aligned}$$

□

By Lemma 8.9 for any $y \in \widetilde{E}^*$ the operator T_y can also be extended to a continuous linear operator, denoted by the same notation, from $\mathbb{D}_{-p}$ into itself. The extension is also given by $T_y = \sum_{k=0}^{\infty} \frac{1}{k!} D_y^k$ with the extension D_y on $\mathbb{D}_{-p}$. The operator is also extended to a continuous linear operator from $\mathbb{D}_{-\infty}$ into itself. We also use the same notation for the operator.

It is easily checked that $\mathcal{A}T_y = T_y\mathcal{A}$ on $\mathrm{Poly}(E^*)$ for any $y \in E^*$ by direct calculations. By the continuity of operators $\mathcal{A}$ and $T_y,\ y \in E^*$ on $\mathbb{D}_{-\infty}$ due to Lemmas 8.7 and 8.9, we obtain $\mathcal{A}T_y = T_y\mathcal{A}$ on $\mathbb{D}_{-\infty}$.

Theorem 8.10. *Let $\Phi \in \mathbb{D}_{-\infty}$. Then the equality*

$$e^{\frac{t}{2}\mathcal{A}}\Phi = E[T_{\mathbf{B}(t)}\Phi]$$

holds for $t \geq 0$.

Proof. Let $\Phi \in \mathbb{D}_{-\infty}$. Then there exists a sequence $(\Phi_n)_{n=1}^{\infty} \subset \mathrm{Poly}(\widetilde{E}^*)$ such that $\Phi_n \to \Phi$ in $\mathbb{D}_{-\infty}$ as $n \to \infty$. By Lemma 8.8 we have

$$e^{\frac{t}{2}\mathcal{A}}\Phi_n \to e^{\frac{t}{2}\mathcal{A}}\Phi \text{ in } \mathbb{D}_{-\infty},$$

as $n \to \infty$ for all $t \geq 0$. This means that there exists $p > 0$ such that

$$|||e^{\frac{t}{2}\mathcal{A}}\Phi_n - e^{\frac{t}{2}\mathcal{A}}\Phi|||_{-p} \to 0,$$

as $n \to \infty$ for all $t \geq 0$. By Lemma 8.7 and $\mathcal{A}T_y = T_y\mathcal{A}$ on $\mathbb{D}_{-\infty}$ we can check that

$$\begin{aligned}
&|||E[T_{\mathbf{B}(t)}\Phi_n] - E[T_{\mathbf{B}(t)}\Phi]|||_{-p}^2 \\
&= \sum_{k=0}^{\infty} \int_{\widetilde{E}^*} \left| E[\mathcal{A}^k T_{\mathbf{B}(t)}\Phi_n(x) - \mathcal{A}^k T_{\mathbf{B}(t)}\Phi(x)] \right|_{-p}^2 d\mu(x) \\
&= \sum_{k=0}^{\infty} \int_{\widetilde{E}^*} \left| E\left[\left(T_{\mathbf{B}(t)}\Phi_n^{(2k)}(x) - T_{\mathbf{B}(t)}\Phi^{(2k)}(x) \right) \left(\mathbf{e}^{\otimes k} \right) \right] \right|_{-p}^2 d\mu(x) \\
&= \sum_{k=0}^{\infty} \int_{\widetilde{E}^*} \left| E\left[\sum_{j=0}^{\infty} \frac{1}{j!} \left(\Phi_n^{(2k+j)}(x) - \Phi^{(2k+j)}(x) \right) \left(\mathbf{e}^{\otimes k} \otimes \mathbf{B}(t)^{\otimes j} \right) \right] \right|_{-p}^2 d\mu(x) \\
&= \sum_{k=0}^{\infty} \int_{\widetilde{E}^*} \left| \sum_{j=0}^{\infty} \frac{1}{j!} \left(\Phi_n^{(2k+j)}(x) - \Phi^{(2k+j)}(x) \right) \left(\mathbf{e}^{\otimes k} \otimes E\left[\mathbf{B}(t)^{\otimes j} \right] \right) \right|_{-p}^2 d\mu(x) \\
&= \sum_{k=0}^{\infty} \int_{\widetilde{E}^*} \left| \sum_{m=0}^{\infty} \frac{(t/2)^m}{m!} \left(\Phi_n^{(2(k+m))}(x) - \Phi^{(2(k+m))}(x) \right) \left(\mathbf{e}^{\otimes (k+m)} \right) \right|_{-p}^2 d\mu(x) \\
&= \sum_{k=0}^{\infty} \int_{\widetilde{E}^*} \left| \sum_{m=0}^{\infty} \frac{(t/2)^m}{m!} \mathcal{A}^{k+m} \left(\Phi_n(x) - \Phi(x) \right) \right|_{-p}^2 d\mu(x) \\
&= \sum_{k=0}^{\infty} \int_{\widetilde{E}^*} \left| \mathcal{A}^k e^{\frac{t}{2}\mathcal{A}} \left(\Phi_n(x) - \Phi(x) \right) \right|_{-p}^2 d\mu(x) = |||e^{\frac{t}{2}\mathcal{A}}\Phi_n - e^{\frac{t}{2}\mathcal{A}}\Phi|||_{-p}^2 \to 0,
\end{aligned}$$

as $n \to \infty$ for all $t \geq 0$, where $\mathbf{e} = \sum_{j=0}^{\infty} e_j^{\otimes 2}$. Since the equality

$$e^{\frac{t}{2}\mathcal{A}}\Phi = E[T_{\mathbf{B}(t)}\Phi]$$

holds for $\Phi \in \mathrm{Poly}(\widetilde{E}^*)$ and $t \geq 0$ by Theorem 7.6, we obtain the assertion. □

Remark 8.11. It holds that $\mathcal{A} = \tau_{\delta(0)}\Delta_L$ on $\mathbb{D}_{-\infty}$, where Δ_L is an extended Lévy Laplacian on $\mathbb{D}_{-\infty}$ defined by

$$(\Delta_L \Phi)(\cdot)(f) = \Delta_L(\Phi(\cdot)(f)), \ f \in \widetilde{E}.$$

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