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STOKES FORMULA ON THE SPACE OF TEMPERED DISTRIBUTIONS

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Abstract. In this paper, an integration theory is developed and a Stokes theorem is established on the dual space $S'(\mathbb{R})$ of the Schwartz space $S(\mathbb{R})$. We give some examples to illustrate this Stokes formula.

1. Introduction

In [3] the authors construct surface measures within the framework of white noise analysis for surfaces in the dual space of the Schwartz space $S(\mathbb{R})$ and give some examples to illustrate the precise description of such surface measures in terms of the Laplace transform. In section 2, we recall basic results used in this framework. In section 3, we define the space $\mathcal{DF}_\theta(\wedge^r\mathcal{L}^2(\mathbb{R}, dt))$ of differentials form α of degree r on $S'(\mathbb{R})$, which will be the main tools to prove the Stokes formula. In section 4, first we define the surface measure of a sub-manifold of finite codimension n. Second, we prove in Theorem 5.3 the stokes formula for the particular case where $n = 1$. Finally, we generalize in Theorem 5.8 the Stokes formula for a sub-manifold of finite codimension n, $n \in \mathbb{N}^*$. In section 5 we study the Stokes formula for some specific examples.

2. Basic Results

The infinite dimensional space on which we will construct surface measures is the dual space $S'(\mathbb{R})$ of the Schwartz space $S(\mathbb{R})$. The basic tool of our construction is the representation in Theorem 2.1 for positive generalized functions defined on the space $S'(\mathbb{R})$. We will briefly review these basic spaces below. For detail, see the paper [4] and the book [8].

2.1. The Schwartz space and its dual space. A function ξ on $\mathbb{R}$ is called *rapidly decreasing* if it is a smooth function such that $|t^n\xi^{(k)}(t)| \to 0$ as $|t| \to \infty$ for any positive integers n and k. The space $S(\mathbb{R})$ consisting of all rapidly decreasing functions is called the *Schwartz space* on $\mathbb{R}$. It is a Fréchet space with the following family of norms

$$|\xi|_{n,k} = \left(\int_{\mathbb{R}} |t^n\xi^{(k)}(t)|^2 dt\right)^{\frac{1}{2}},$$

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where $n, k \in \mathbb{N}$. On the other hand, we have the Hermite functions given by

$$e_n(x) = \frac{1}{\sqrt{\sqrt{\pi}\, 2^n n!}} H_n(x) e^{-x^2/2}, \quad n \geq 0, \tag{2.1}$$

where H_n is the Hermite polynomial of degree n. The collection $\{e_n; n \geq 0\}$ is an orthonormal basis for the Hilbert space $\mathcal{L}^2(\mathbb{R}, dt)$.

Now, let $f \in \mathcal{L}^2(\mathbb{R}, dt)$. For each $p \geq 0$, define

$$|f|_p = \left(\sum_{n=0}^{\infty} (2n+2)^{2p} (f, e_n)^2 \right)^{1/2}, \tag{2.2}$$

where $(\cdot\,,\cdot)_0$ is the inner product of $\mathcal{L}^2(\mathbb{R}, dt)$. Define

$$S_p(\mathbb{R}) = \big\{ f \in \mathcal{L}^2(\mathbb{R}, dt);\ |f|_p < \infty \big\}.$$

Then we have an increasing sequence $\{S_p(\mathbb{R}); p \in \mathbb{N}\}$ of Hilbert spaces such that the inclusion mapping $S_{p+1}(\mathbb{R}) \hookrightarrow S_p(\mathbb{R})$ is Hilbert Schmidt operator. It follows that the projective limit

$$\operatorname*{proj-lim}_{p\to\infty} S_p(\mathbb{R}) := \bigcap_{p \geq 0} S_p(\mathbb{R})$$

being equipped with the projective limit topology is a nuclear space.

It is well known that the families $\{|\cdot|_{n,k}\ ;\ n, k \geq 0\}$ and $\{|\cdot|_p\ ;\ p \geq 0\}$ are equivalent, i.e., they generate the same topology on $S(\mathbb{R})$. This implies that

$$S(\mathbb{R}) = \operatorname*{proj-lim}_{p\to\infty} S_p(\mathbb{R}).$$

Moreover, we have the following equality

$$S'(\mathbb{R}) = \operatorname*{ind-lim}_{p\to\infty} S_{-p}(\mathbb{R}),$$

where the inductive limit space is equipped with the inductive limit topology and $S_{-p}(\mathbb{R})$ is the completion of $\mathcal{L}^2(\mathbb{R}, dt)$ with respect to the norm

$$|f|_{-p} = \left(\sum_{n=0}^{\infty} (2n+2)^{-2p} (f, e_n)^2 \right)^{1/2}, \quad p \geq 0. \tag{2.3}$$

Using the Riesz representation theorem to identify the space $\mathcal{L}^2(\mathbb{R}, dt)$ with its dual space. Then we get a Gel'fand triple:

$$S(\mathbb{R}) = \operatorname*{proj-lim}_{p\to\infty} S_p(\mathbb{R}) \ \hookrightarrow\ \mathcal{L}^2(\mathbb{R}, dt) \ \hookrightarrow\ \operatorname*{ind-lim}_{p\to\infty} S_{-p}(\mathbb{R}) = S'(\mathbb{R}).$$

By the Minlos theorem (see, e.g., [6], [8], [10]), there exists a unique probability measure γ on $S'(\mathbb{R})$ such that

$$\int_{S'(\mathbb{R})} e^{i\langle y, \xi\rangle}\, d\gamma(y) = e^{-\frac{1}{2}|\xi|_0^2}, \quad \xi \in S(\mathbb{R}),$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $S'(\mathbb{R})$ and $S(\mathbb{R})$ and $|\cdot|_0$ is the norm on $\mathcal{L}^2(\mathbb{R}, dt)$.

The space $S'(\mathbb{R})$ is our infinite dimensional analogue of $\mathbb{R}^n$. Since there is no infinite dimensional Lebesgue measure, we will take the Gaussian measure γ on $S'(\mathbb{R})$ as the infinite dimensional replacement of the finite dimensional Lebesgue measure. Hence surface measures on subsets of $S'(\mathbb{R})$ will be defined with respect to measure γ.

2.2. Spaces of test and generalized functions. We first recall that a function $\theta : \mathbb{R}_+ \to \mathbb{R}$ is called a *Young function* if it is continuous, convex, strictly increasing, $\theta(0) = 0$, and

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = \infty.$$

Let B be a complex Banach space with norm $|\cdot|$. A holomorphic function f on B is said to have *exponential growth of order* θ *with finite type* $m > 0$ if it satisfies the following condition,

$$\|f\|_{\theta,m} := \sup_{z\in B} |f(z)|\, e^{-\theta(m|z|)} < \infty.$$

Let $\mathrm{Exp}(B, \theta, m)$ denote the collection of such functions, namely,

$$\mathrm{Exp}(B, \theta, m) = \Big\{ f \,:\, \|f\|_{\theta,m} < \infty \Big\}. \tag{2.4}$$

Then $\mathrm{Exp}(B, \theta, m)$ is a complex Banach space with norm $\|\cdot\|_{\theta,m}$.

Apply equation (2.4) to the case when B is the complexification $S_{-p,\mathbb{C}}(\mathbb{R})$ of the space $S_{-p}(\mathbb{R})$, $p \geq 0$. Then we have the spaces $\mathrm{Exp}(S_{-p,\mathbb{C}}(\mathbb{R}), \theta, m)$ for $m > 0$ and $p \geq 0$. The space of *test functions* on the infinite dimensional space $S'(\mathbb{R})$ is defined to be the space

$$\mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})) := \bigcap_{m,p} \mathrm{Exp}(S_{-p,\mathbb{C}}(\mathbb{R}), \theta, m),$$

where $S'_{\mathbb{C}}(\mathbb{R})$ is the complexification of $S'(\mathbb{R})$. The space of *generalized functions* on $S'(\mathbb{R})$ is defined to be the strong topological dual space $\mathcal{F}^*_\theta(S'_{\mathbb{C}}(\mathbb{R}))$ of $\mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R}))$.

We will assume that the Young function θ satisfies the condition

$$\lim_{t\to\infty} \frac{\theta(t)}{t^2} < \infty.$$

In that case, we have a Gel'fand triple by [4]:

$$\mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})) \subset L^2(S'(\mathbb{R}), \gamma) \subset \mathcal{F}^*_\theta(S'_{\mathbb{C}}(\mathbb{R})).$$

A test function $f \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R}))$ is called *positive* if it satisfies the condition

$$f(y + i0) \geq 0, \quad \forall\, y \in S'(\mathbb{R}).$$

We call a generalized function $\Psi \in \mathcal{F}^*_\theta(S'_{\mathbb{C}}(\mathbb{R}))$ *positive* if

$$\Psi(f) \geq 0, \quad \forall \text{ positive } f \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})).$$

The main tool of this paper is the following integral representation theorem from [11].

Theorem 2.1. *Let Ψ be a positive generalized function. Then there exists a unique finite positive Borel measure μ on $S'(\mathbb{R})$ such that*

$$\Psi(f) = \int_{S'(\mathbb{R})} f(y + i0)\, d\mu(y), \quad \forall\, f \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})). \tag{2.5}$$

We will also need the next theorem from [11]. For convenience, we will say that *μ represents a generalized function Ψ* in $\mathcal{F}^*_\theta(S'_{\mathbb{C}}(\mathbb{R}))$ when equation (2.5) holds.

Theorem 2.2. *Let μ be a finite Borel measure on $S'(\mathbb{R})$. Then μ represents a generalized function in $\mathcal{F}^*_\theta(S'_{\mathbb{C}}(\mathbb{R}))$ if and only if there exist $p \in \mathbb{N}$ and $m > 0$ such that μ is supported by $S_{-p}(\mathbb{R})$ and*

$$\int_{S_{-p}(\mathbb{R})} e^{\theta(m|y|_{-p})}\, d\mu(y) < \infty. \tag{2.6}$$

2.3. Surface Measures on $S'(\mathbb{R})$. We recall the divergence operator which we need for the definition of the space of differentials forms. For a real-valued function f on $S'(\mathbb{R})$, its *directional derivative* at x in the direction $h \in \mathcal{L}^2(\mathbb{R}, dt)$ is defined by

$$D_h f(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(f(x+\epsilon h) - f(x)\big),$$

where the limit is taken in $S'(\mathbb{R})$ almost everywhere with respect to the Gaussian measure γ on $S'(\mathbb{R})$. The *gradient* of f is a vector field $\nabla f : S'(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R}, dt)$ defined by

$$\langle \nabla f(x), h\rangle = D_h f(x), \quad \forall\, h \in \mathcal{L}^2(\mathbb{R}, dt).$$

The *divergence* of a vector field $U\colon S'(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R}, dt)$ is defined to be the function δU on $S'(\mathbb{R})$ satisfying the equality

$$\int_{S'(\mathbb{R})} f(x)(\delta U)(x)\, d\gamma(x) = \int_{S'(\mathbb{R})} \langle \nabla f(x), U(x)\rangle\, d\gamma(x) \tag{2.7}$$

for all smooth real-valued functions f on $S'(\mathbb{R})$.

For the rest of the paper we will use $\gamma\circ\Phi^{-1}$ to denote the distribution of measurable function $\Phi : S'(\mathbb{R}) \to \mathbb{R}$ with respect to γ, i.e., $(\gamma\circ\Phi^{-1})(A) = \gamma\big(\Phi^{-1}(A)\big)$, $A \in \mathcal{B}(\mathbb{R})$.

Now we recall the following fact from [3], [9] that if a real-valued function Φ on $S'(\mathbb{R})$ satisfies

(a) $\frac{1}{|\nabla\Phi(\cdot)|_0} \in \mathcal{L}^2(S'(\mathbb{R}), \gamma)$,

(b) $\delta \frac{\nabla\Phi(\cdot)}{|\nabla\Phi(\cdot)|_0^2} \in \mathcal{L}^1(S'(\mathbb{R}), \gamma)$,

then the probability measure $\gamma \circ \Phi^{-1}$ on $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure.

Let g be a γ-integrable function on $S'(\mathbb{R})$. For convenience, we will use γ_g to denote the signed measure $d\gamma_g = g\, d\gamma$. Suppose Φ a real-valued measurable function on $S'(\mathbb{R})$. Then it is obvious that the induced signed measure $\gamma_g \circ \Phi^{-1}$ is absolutely continuous with respect to the induced probability measure $\gamma \circ \Phi^{-1}$. As a consequence, the signed measure $\gamma_g \circ \Phi^{-1}$ on $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure.

In the following, we recall that γ denotes the standard Gaussian measure on $S'(\mathbb{R})$. And we will assume that Φ is a real-valued measurable function on $S'(\mathbb{R})$ such that $\gamma \circ \Phi^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, for instance, when Φ satisfies conditions (a) and (b).

The type of surfaces in $S'(\mathbb{R})$ on which we will construct surface measures is specified by

$$V^a = \big\{y \in S'_{\mathbb{C}}(\mathbb{R})\,;\ \Phi(y) = a\big\}, \tag{2.8}$$

where a is a real number.

Since $\gamma \circ \Phi^{-1}$ is assumed to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, the signed measure $\gamma_g \circ \Phi^{-1}$ for $g \in \mathcal{L}^2(S'(\mathbb{R}), \gamma)$ is also absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

For convenience, we introduce the following densities :

$$K(a) = \frac{d(\gamma \circ \Phi^{-1})}{dx}(a), \quad K_g(a) = \frac{d(\gamma_g \circ \Phi^{-1})}{dx}(a), \quad a \in \mathbb{R}. \tag{2.9}$$

and let $\mathcal{O}$ denote the set

$$\mathcal{O} = \{a \in \mathbb{R}\,;\, K(a) \neq 0\}.$$

Theorem 2.3. [3] *Assume that Φ is a real-valued measurable function on $S'(\mathbb{R})$ such that $\gamma \circ \Phi^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Then for each $a \in \mathcal{O}$, there exists a unique probability measure ν^a on $S'(\mathbb{R})$ satisfying the following equality:*

$$\int_{S'(\mathbb{R})} g(y)\, d\nu^a(y) = \frac{K_g(a)}{K(a)}, \quad \forall\, g \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})). \tag{2.10}$$

Moreover, there exist $p \in \mathbb{N}$ and $m > 0$ such that ν^a is supported by $S_{-p}(\mathbb{R})$ and

$$\int_{S_{-p}(\mathbb{R})} e^{\theta(m|y|_{-p})}\, d\nu^a(y) < \infty. \tag{2.11}$$

In addition, the probability measure ν^a given by equation (2.10) is supported by the surface V^a given by equation (2.8) *and the following equality*

$$\int_{S'(\mathbb{R})} v(\Phi(x)) g(x)\, d\gamma(x) = \int_{\mathbb{R}} v(a) K(a) \bigg(\int_{V^a} g(y)\, d\nu^a(y) \bigg) da \tag{2.12}$$

holds for any bounded smooth function v on $\mathbb{R}$ and any test function g in $\mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})$.

Note that, the probability measure ν^a given by equation (2.10) is supported by the set $V^a \cap S_{-p}(\mathbb{R})$ for some $p \in \mathbb{N}$ (which may depend on $a \in \mathbb{R}$). Note that equation (2.12) gives a decomposition of the standard Gaussian measure γ in terms of the surface measure ν^a on V^a and the Lebesgue measure on $\mathbb{R}$. Thus ν^a can be regarded as the conditional law of γ given that $\Phi = a$.

3. Exterior Product of Hilbert Spaces

Let H be a Hilbert space with inner product and norm denoted by $(\cdot, \cdot)_H$ and $|\cdot|_H$, respectively. For $\varphi_1, \varphi_2, \ldots, \varphi_m$ in H, we define the tensor product $\varphi_1 \otimes \ldots \otimes \varphi_m$ of $\varphi_1, \varphi_2, \ldots, \varphi_m$, as the m-linear form on H^m defined by

$$(\varphi_1 \otimes \ldots \otimes \varphi_m)(\psi_1, \psi_2, \ldots, \psi_m) = \prod_{i=1}^m (\varphi_i, \psi_i)_H, \quad \forall \psi_1, \psi_2, \ldots, \psi_m \in H. \tag{3.1}$$

We denote by $\bigotimes^m H$ the space of all m-linear forms spanned by such tensor products, and endowed with the inner product defined by

$$\begin{aligned}&(\varphi_1 \otimes \ldots \otimes \varphi_m, \psi_1 \otimes \ldots \otimes \psi_m)_{\bigotimes^m H} \\ &= \prod_{i=1}^m (\varphi_i, \psi_i)_H \quad \forall \varphi_1, \varphi_2, \ldots, \varphi_m, \psi_1, \psi_2, \ldots, \psi_m \in H.\end{aligned} \tag{3.2}$$

It is easy to prove that $(\eta,\xi)_{\bigotimes^m H}$ does note depend of the representation of $\eta,\xi \in \bigotimes^m H$, so this product is well defined and it is positively defined. As a consequence $\left((\xi,\xi)_{\bigotimes^m H}\right)^{\frac{1}{2}}$ is a norm on $\bigotimes^m H$. The Hilbert m-tensor product of H denoted by $\widehat{\bigotimes}^m H$ is the completion of $\bigotimes^m H$ under this norm. We denote the scalar product respectively the norm on $\widehat{\bigotimes}^m H$ by $(.,.)_{\widehat{\bigotimes}^m H}$, $|.|_{\widehat{\bigotimes}^m H}$.

If $\{e_i\}_{i\in I}$ is an orthonormal basis for H, then $\{e_{i_1}\otimes\ldots\otimes e_{i_m}, i_1,\ldots,i_m \in I\}$ constitutes an orthonormal basis for $\widehat{\bigotimes}^m H$.

In the following we consider $\widehat{\wedge}^m H$ the Hilbert m-exterior product of H. In order to define this space we first consider $\wedge^m H$ of H, the m-exterior product of H. This is the subspace of $\bigotimes^m H$ spanned by all the sums

$$\sum_{\sigma} \frac{1}{n!}\varepsilon(\sigma)\varphi_{\sigma(1)}\otimes\ldots\otimes\varphi_{\sigma(m)}, \tag{3.3}$$

where $\varphi_1,\ldots,\varphi_m \in H$, σ is a permutation of $\{1,\ldots,m\}$ and $\varepsilon(\sigma)\in\{-1,1\}$ is the signature of the permutation σ. The expression in equation (3.3) defines the exterior product of $\varphi_1,\ldots,\varphi_m$ and is denoted by $\varphi_1\wedge\varphi_2\wedge\ldots\wedge\varphi_m$. It is easy to see that the exterior product is multilinear.

We define on $\wedge^m H$ an inner product $(.,.)_{\wedge^m H}$ by first defining the inner product on two exterior products $\varphi_1\wedge\varphi_2\wedge\ldots\wedge\varphi_m$ and $\psi_1\wedge\psi_2\wedge\ldots\wedge\psi_m$;

$$\begin{aligned}&\big(\varphi_1\wedge\varphi_2\wedge\ldots\wedge\varphi_m, \psi_1\wedge\psi_2\wedge\ldots\wedge\psi_m\big)_{\wedge^m H}\\ &= \det\big((\varphi_i,\psi_j)_H\big)_{1\le i\le j\le m} \quad \forall\varphi_1,...,\varphi_m,\psi_1,...,\psi_m\end{aligned} \tag{3.4}$$

and then we extend this product to $\wedge^m H$ by linearity.

The space $\widehat{\wedge}^m H$ is the completion of $\wedge^m H$ under the norm $((\varphi,\psi)_{\wedge^m H})^{\frac{1}{2}}$. We denote by $(.,.)_{\widehat{\wedge}^m H}$ and $|.|_{\widehat{\wedge}^m H}$ the scalar product and the norm on $\widehat{\wedge}^m H$ induced by equation (3.4).

Let $\{e_i\}_{i\in I}$ be an orthonormal basis for H for which we assume that the family indices I is totally ordered. We recall that the family of exterior products :

$$e_{i_1}\wedge\cdots\wedge e_{i_m},\ i_1<\cdots<i_m;\ i_\alpha\in I \tag{3.5}$$

constitutes an orthonormal basis for $\wedge^m H$. For instance, if $\varphi_1,\cdots,\varphi_m \in H$, $\varphi_i = \sum_{j_k\in I} a_{ij_k}e_{j_k}$; $i=1,\cdots,m, I\subset\mathbb{N}$, then

$$\varphi_1\wedge\cdots\wedge\varphi_m = \sum_{i_1<\cdots<i_m} K_{i_1\cdots i_m} e_{i_1}\wedge\cdots\wedge e_{i_m}$$

where $K_{i_1\cdots i_m}$ is the determinant of the $m\times m$ matrix of elements :

$$a_{i,j},\ i=1,\cdots,m,\ j=i_1,\cdots,i_m.$$

4. Space of Differentials Forms

In this section, we define the space $\mathcal{DF}_\theta(\wedge^r\mathcal{L}^2(\mathbb{R},dt))$ of differentials forms α of degree r on $S'(\mathbb{R})$, which will be used in the proof of the Stokes formula.

Lemma 4.1. *Let $h \in \mathcal{L}^2(\mathbb{R}, dt)$ and for real-valued measurable functions f and g on $S'(\mathbb{R})$, we have*

$$\int_{S'(\mathbb{R})} (D_h f(x) - \langle x, h\rangle f(x)) g(x) d\gamma(x) = -\int_{S'(\mathbb{R})} f(x) D_h g(x) d\gamma(x). \qquad (4.1)$$

Proof. Using the quasi-invariance of the Gaussian measure γ on $S'(\mathbb{R})$, namely, for all $h \in L^2(S'(\mathbb{R}), \gamma)$;

$$\frac{\gamma(dx - h)}{\gamma(dx)} = e^{\langle x,h\rangle - \frac{1}{2}\langle h,h\rangle} = e^{\langle x,h\rangle - \frac{1}{2}\|h\|_0^2},\ x \in S'(\mathbb{R}),$$

where $\|.\|_0$ is the norm in $L^2(S'(\mathbb{R}), \gamma)$, we have

$$\int_{S'(\mathbb{R})} f(x + \varepsilon h) g(x) d\gamma(x) = \int_{S'(\mathbb{R})} f(x) g(x - \varepsilon h) e^{\langle x, \varepsilon h\rangle - \frac{1}{2}\varepsilon^2 \|h\|_0^2} d\gamma(x),$$

as a consequence,

$$\begin{aligned}
&\int_{S'(\mathbb{R})} D_h f(x) g(x) d\gamma(x) \\
&= \lim_{\varepsilon \mapsto 0} \frac{1}{\varepsilon} \int_{S'(\mathbb{R})} (f(x + \varepsilon h) - f(x)) g(x) d\gamma(x) \\
&= \int_{S'(\mathbb{R})} f(x) \lim_{\varepsilon \mapsto 0} \frac{g(x - \varepsilon h) e^{\langle x, \varepsilon h\rangle - \frac{1}{2}\varepsilon^2 \|h\|_0^2} - g(x)}{\varepsilon} d\gamma(x).
\end{aligned}$$

Then we use

$$e^{\langle x, \varepsilon h\rangle - \frac{1}{2}\varepsilon^2 \|h\|_0^2} = 1 + \varepsilon \langle x, h\rangle + \mathrm{O}(\varepsilon^2).$$

So we obtain,

$$\begin{aligned}
&\int_{S'(\mathbb{R})} D_h f(x) g(x) d\gamma(x) \\
&= \int_{S'(\mathbb{R})} f(x) \lim_{\varepsilon \mapsto 0} \frac{1}{\varepsilon} \Big(g(x - \varepsilon h)(1 + \varepsilon \langle x, h\rangle) - g(x) \Big) d\gamma(x) \\
&= \int_{S'(\mathbb{R})} f(x)(-D_h g(x)) d\gamma(x) + \int_{S'(\mathbb{R})} \langle x, h\rangle f(x) g(x) d\gamma(x),
\end{aligned}$$

which gives the desired result. □

Next, let $\wedge^r \mathcal{L}^2(\mathbb{R}, dt)$ be the r^{th} exterior product of the space $\mathcal{L}^2(\mathbb{R}, dt)$, and

$$\begin{array}{rcl} \alpha : S'(\mathbb{R}) & \longrightarrow & \wedge^r \mathcal{L}^2(\mathbb{R}, dt) \\ x & \longmapsto & \alpha(x) \end{array}$$

be a differential form of degree r on $S'(\mathbb{R})$.

Definition 4.2. We denoted by $\mathcal{DF}_\theta(\wedge^r \mathcal{L}^2(\mathbb{R}, dt))$ the set of differentials forms α of degree r on $S'(\mathbb{R})$, such that, for all $h_1, h_2, ..., h_r \in \mathcal{L}^2(\mathbb{R}, dt)$, the following application

$$\begin{array}{rcl} F(\alpha, h_1, \cdots, h_r) : S'(\mathbb{R}) & \longrightarrow & \mathbb{C} \\ x & \longmapsto & (\alpha(x), h_1 \wedge h_2 \wedge ... \wedge h_r)_{\wedge^r \mathcal{L}^2(\mathbb{R}, dt)} \end{array}$$

is in $\mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R}))$. In the following we denote the application $F(\alpha, h_1, \cdots, h_r)$ by $\alpha_{h_1 \wedge h_2 \wedge ... \wedge h_r}$.

In particular, for $r = 1$, $\alpha \in \mathcal{DF}_\theta(\mathcal{L}^2(\mathbb{R}, dt))$, and $h \in \mathcal{L}^2(\mathbb{R}, dt)$, the function

$$\begin{array}{rcl} \alpha_h : S'_{\mathbb{C}}(\mathbb{R}) & \longrightarrow & \mathbb{C} \\ x & \longmapsto & (\alpha(x), h)_0 \end{array}$$

is in $\mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R}))$.

For $\alpha \in \mathcal{DF}_\theta(\mathcal{L}^2(\mathbb{R}, dt))$, we associate the differential operator $d\alpha : S'(\mathbb{R})) \longrightarrow \wedge^2\mathcal{L}^2(\mathbb{R}, dt)$, such that

$$(d\alpha(x), h_1 \wedge h_2)_{\wedge^2\mathcal{L}^2(\mathbb{R},dt)} = D_{h_1}\alpha_{h_2}(x) - D_{h_2}\alpha_{h_1}(x)$$

for all $h_1, h_2 \in \mathcal{L}^2(\mathbb{R}, dt)$. More generally, if $\alpha \in \mathcal{DF}_\theta(\wedge^r\mathcal{L}^2(\mathbb{R}, dt))$, we associate the differential operator

$d\alpha : S'(\mathbb{R})) \longrightarrow \wedge^{r+1}\mathcal{L}^2(\mathbb{R}, dt)$, by:

$$\begin{aligned} &(d\alpha(x), h_1 \wedge h_2 \wedge ... \wedge h_{r+1})_{\wedge^{r+1}\mathcal{L}^2(\mathbb{R},dt)} \\ &= \sum_{j=1}^{r+1} (-1)^{j+1} D_{h_j}\alpha_{h_1\wedge h_2\wedge ...\wedge h_{j-1}\wedge h_{j+1}\wedge ... h_{r+1}}(x), \end{aligned}$$

for all $h_1, h_2, ..., h_{r+1} \in \mathcal{L}^2(\mathbb{R}, dt)$. For $f \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R}))$, we define the differential form $df \in \mathcal{DF}_\theta(\mathcal{L}^2(\mathbb{R}, dt))$ by $df = \nabla f : S'(\mathbb{R}) \longrightarrow \mathcal{L}^2(\mathbb{R}, dt)$ such that

$$\langle \nabla f(x), h\rangle = D_h f(x), \quad \forall\, h \in \mathcal{L}^2(\mathbb{R}, dt).$$

Since $(\wedge^r H) \wedge (\wedge^p H) \simeq \wedge^{r+p} H$, we define for $\alpha \in \mathcal{DF}_\theta(\wedge^r\mathcal{L}^2(\mathbb{R}, dt))$ and $\beta \in \mathcal{DF}_\theta(\wedge^p\mathcal{L}^2(\mathbb{R}, dt))$ the exterior product $\alpha \wedge \beta \in \mathcal{DF}_\theta(\wedge^{p+r}\mathcal{L}^2(\mathbb{R}, dt))$ by:

$$(\alpha \wedge \beta)(x) = \alpha(x) \wedge \beta(x).$$

If $f_1, ..., f_n \in \mathcal{F}_\theta(S'(\mathbb{R}))$ n, then $df_1 \wedge ... \wedge df_n \in \mathcal{DF}_\theta(\wedge^n\mathcal{L}^2(\mathbb{R}, dt))$ and for $1 \le i \le n$, $1 \le j \le n$

$$|df_1(x) \wedge ... \wedge df_n(x)|^2_{\wedge^n\mathcal{L}^2(\mathbb{R},dt)} = Det\Big(\big(\nabla f_i(x), \nabla f_j(x)\big)_0 \Big)_{i,j}.$$

If π is a differential form on $S'(\mathbb{R})$ of degree n, there exists a forme $\delta\pi$ on $S'(\mathbb{R})$ of degree $(n-1)$, such that

$$(\pi, d\lambda)_{L^2(S'(\mathbb{R})\to\wedge^n\mathcal{L}^2(\mathbb{R},dt),\gamma)} = (\delta\pi, \lambda)_{L^2(S'(\mathbb{R})\to\wedge^{n-1}\mathcal{L}^2(\mathbb{R},dt),\gamma)}$$

In particular, if π is a differential form of degree 1, then the $\delta\pi$ is a function and we have

$$\int_{S'(\mathbb{R})} \delta\pi d\gamma = 0$$

and in fact,

$$(\delta\pi, 1)_{L^2(S'(\mathbb{R})\to\mathbb{R},\gamma)} = (\pi, d1)_{L^2(S'(\mathbb{R})\to\mathcal{L}^2(\mathbb{R},dt),\gamma)} = 0.$$

Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{L}^2(\mathbb{R}, dt)$, and π a differential form of degree 1 such that

$$\pi(x) = \sum_k \pi_k(x)e_k.$$

Using Lemma 4.1, for $\varphi \in \mathcal{F}_\theta(S'(\mathbb{R}))$, we have

$$\int_{S'(\mathbb{R})} \pi_k(x)D_{e_k}\varphi(x)d\gamma(x) = -\int_{S'(\mathbb{R})} [D_{e_k}\pi_k(x) - \langle x, e_k\rangle\pi_k(x)]\varphi(x)d\gamma(x).$$

So

$$\int_{S'(\mathbb{R})} (\pi, \nabla\varphi)_{\mathcal{L}^2(\mathbb{R},dt)} d\gamma(x) = (\delta\pi, \varphi)_{L^2(S'(\mathbb{R}),d\gamma)}$$

with $\delta\pi(x) = -\sum_{k\in\mathbb{N}}\Big(D_{e_k}\pi_k(x) - \langle x, e_k\rangle \pi_k(x)\Big)$.

Let π be a differential form of degree $(n+1)$ and α a differential form of degree n, there exist a differential form of degree 1, interior product of α and π, denoted $\alpha \lrcorner \pi$ such that

$$(\pi, \lambda \wedge \alpha)_{L^2(S'(\mathbb{R})\to\wedge^{n+1}\mathcal{L}^2(\mathbb{R},dt),\gamma)} = (\alpha\lrcorner\pi, \lambda)_{L^2(S'(\mathbb{R})\to\mathcal{L}^2(\mathbb{R},dt),\gamma)} \tag{4.2}$$

for all differential forms λ of degree 1.

5. The Stokes Formula

Let V be a C^∞-differential real manifold of dimension n (in particular, $V = \mathbb{R}^n$). Suppose that $\Phi : S'(\mathbb{R}) \longrightarrow V$ is a measurable function on $S'(\mathbb{R})$ and let $\gamma \circ \Phi^{-1}$ be the image measure of γ under Φ.

First, we take $n = 1$. Assume that Φ is a real-valued measurable function on $S'(\mathbb{R})$ such that $\gamma\circ\Phi^{-1}$ is a measure on $\mathbb{R}$ and is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Moreover, $\gamma \circ \Phi^{-1} = K(x)dx$ where, $K : \mathbb{R} \to \mathbb{R}$ is C^∞-differentiable.

Definition 5.1. [3] Let Φ be a real-valued measurable function on $S'(\mathbb{R})$ such that $\gamma \circ \Phi^{-1}$ is a measure on $\mathbb{R}$ and is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, and

$$V^a = \big\{y \in S'_{\mathbb{C}}(\mathbb{R})\,;\, \Phi(y) = a\big\},$$

where a is a real number. We define the *surface measure* dA^Φ on V^a by:

$$\int_{V^a} g(x)dA^\Phi(x) = K(a)\int_{V^a} |\nabla\Phi|_0 g(x) d\nu^a(x),\ \forall g \in \mathcal{F}_\theta(S'(\mathbb{R})).$$

Theorem 5.2. (Coarea formula) *Let Φ be a real-valued measurable function on $S'(\mathbb{R})$ such that $\gamma\circ\Phi^{-1}$ is a measure on $\mathbb{R}$ and is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, $g \in \mathcal{F}_\theta(S'(\mathbb{R}))$ and $v : \mathbb{R} \longrightarrow \mathbb{R}$. Then we have*

$$\int_{S'(\mathbb{R})} v(\Phi(x))g(x)|\nabla\Phi|_0 d\gamma(x) = \int_{\mathbb{R}} v(a)\bigg(\int_{V^a} g(x)dA^\Phi(x)\bigg)da.$$

Proof. This theorem is a consequence of Theorem 2.3 by replacing $g(x)$ with $g(x)|\nabla\Phi|_0$. □

5.1. One dimensional Stokes formula.

Theorem 5.3. *Let Φ be a real-valued measurable function on $S'(\mathbb{R})$ such that $\gamma\circ\Phi^{-1}$ is a measure on $\mathbb{R}$ and is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Then for any $f \in \mathcal{DF}_\theta(\mathcal{L}^2(\mathbb{R}, dt))$, we have*

$$\int_{V^a}\bigg(\frac{\nabla\Phi(x)}{|\nabla\Phi|_0}, f(x)\bigg)_0 dA^\Phi(x) = \int_{\Phi\le a}(\delta f)(x)d\gamma(x). \tag{5.1}$$

Proof. We have

$$\int_{S'(\mathbb{R})} (\delta f)(x)h(x)\, d\gamma(x) = \int_{S'(\mathbb{R})} (\nabla h(x), f(x))_0\, d\gamma(x). \tag{5.2}$$

On the other hand, if we take $h(x) = v(\Phi(x))$, then $\nabla h(x) = v'(\Phi(x))\nabla\Phi(x)$. So equation (5.2) becomes

$$\begin{aligned}
\textstyle\int_{S'(\mathbb{R})} (\delta f)(x)h(x)\, d\gamma(x) &= \textstyle\int_{S'(\mathbb{R})} (\nabla h(x), f(x))_0\, d\gamma(x) \\
&= \textstyle\int_{S'(\mathbb{R})} v'(\Phi(x))(\nabla\Phi(x), f(x))_0\, d\gamma(x) \\
&= \textstyle\int_{\mathbb{R}} v'(a)K(a)\Big(\int_{V^a} (\nabla\Phi(x), f(x))_0 d\nu^a \Big) da.
\end{aligned}$$

Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions such that

$$v_n(x) = \begin{cases} 1, & \text{if } x \le a, \\ 0, & \text{if } x > a + \frac{1}{n}, \end{cases}$$

and let $h_n(x) = v_n(\Phi(x))$. Since $v_n'(a)da$ converges to δ_a, we have

$$\begin{aligned}
&\int_{S'(\mathbb{R})} (\delta f)(x)h(x)\, d\gamma(x) \\
&= \lim_{n\mapsto+\infty} \int_{S'(\mathbb{R})} (\delta f)(x)h_n(x)\, d\gamma(x) \\
&= \lim_{n\mapsto+\infty} \int_{\mathbb{R}} v_n'(a)K(a)\Big(\int_{V^a} (\nabla\Phi(x), f(x))_0 d\nu^a \Big) da.
\end{aligned}$$

Finally

$$\int_{V^a} \left(\frac{\nabla\Phi(x)}{\|\nabla\Phi\|_{\mathcal{L}^2(\mathbb{R},dt)}}, f(x) \right)_0 dA^\Phi(x) = \int_{\Phi\le a} (\delta f)(x) d\gamma(x).$$

□

Example 5.4. From [3] we recall the *Surface Measures on Hyperplanes:* Let $\Phi_1(x) = \langle x, \xi_1\rangle$ with $\xi_1 \in S(\mathbb{R})$, $\xi_1 \ne 0$. Then the corresponding function $K(y)$ in equation (2.9) is given by

$$K(y) = \frac{1}{\sqrt{2\pi}\,|\xi_1|_0}\, e^{-\frac{y^2}{2|\xi_1|_0^2}}, \quad y \in \mathbb{R}. \tag{5.3}$$

Moreover, for the test function $g(x) = e^{\langle x,\eta\rangle}$ with $\eta \in S(\mathbb{R})$, the corresponding function $K_g(y)$ in equation (2.9) is given by

$$K_g(y) = \frac{e^{\frac{|\eta|_0^2}{2}}}{\sqrt{2\pi}\,|\xi_1|_0}\, e^{-\frac{(y-\langle\xi_1,\eta\rangle)^2}{2|\xi_1|_0^2}}, \quad y \in \mathbb{R}. \tag{5.4}$$

As a consequence, the surface measure ν^a on $V^a = \{x \in S'_{\mathbb{C}}(\mathbb{R})\,;\, \langle x, \xi_1\rangle = a\}$ is given by its Laplace transform

$$(\mathcal{L}\nu^a)(\eta) = \exp\left(|\eta|_0^2 - \frac{\langle\xi_1,\eta\rangle^2}{2|\xi_1|_0^2} + a\frac{\langle\xi_1,\eta\rangle}{|\xi_1|_0^2} \right), \quad \eta \in S(\mathbb{R}). \tag{5.5}$$

Upon applying the Stokes formula given by equation (5.1), we get the equality

$$\int_{V^a} \frac{1}{\sqrt{2\pi}\,|\xi_1|_0}\, e^{-\frac{a^2}{2|\xi_1|_0^2}} \big(\xi_1, f(x)\big)_0 d\nu^a(x) = \int_{\{x\in S'_{\mathbb{C}}(\mathbb{R});\phi(x)\le a\}} (\delta f)(x) d\gamma(x),$$

for all $f \in \mathcal{DF}_\theta(\mathcal{L}^2(\mathbb{R}, dt))$.

5.2. n-dimensional Stokes formula. Let $(\Phi_i)_{i\ge 1}$ a sequence of real-valued functions on $S'(\mathbb{R})$. Next, let $\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_n) : S'(\mathbb{R}) \longrightarrow \mathbb{R}^n$, such that, the measure $\gamma \circ \Phi^{-1}$ is a measure on $\mathbb{R}^n$ and is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. Moreover, $\gamma \circ \Phi^{-1} = K(v)dx$ where, $K : \mathbb{R}^n \to \mathbb{R}$ is C^∞-differentiable.

Next, for φ a volume form on $V = \mathbb{R}^n$ denoted by $\sigma = \varphi \circ \Phi^{-1}$ the differential form on $S'(\mathbb{R})$. By definition:

$$\Big(\sigma(x), h_1 \wedge h_2 ... \wedge h_n\Big)_{\wedge^n \mathcal{L}^2(\mathbb{R},dt)} = \varphi\Big(\Phi(x)\Big)\Big(d\Phi(x)h_1 \wedge ... \wedge d\Phi(x)h_n\Big).$$

Lemma 5.5. *Let $\Phi = (\phi_1, \cdots, \phi_n) : S'(\mathbb{R}) \longrightarrow \mathbb{R}^n$ be such that the measure $\gamma\circ\Phi^{-1}$ is a measure on $\mathbb{R}^n$ and is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$. Let $k(\xi) = \frac{\gamma\circ\Phi^{-1}(\xi)}{d\xi}$ be the density of $\gamma \circ \Phi^{-1}$ with respect to the Lebesgue measure on $\mathbb{R}^n$. Then*

$$k(\Phi(x))|d\phi_1 \wedge ... \wedge d\phi_n|_{\wedge^n \mathcal{L}^2(\mathbb{R},dt)}$$

is invariant under all diffeomorphism of $\mathbb{R}^n$, i.e., if $\widetilde{\Phi} = \Psi \circ \Phi = (\widetilde{\phi}_1, ..., \widetilde{\phi}_n)$ where Ψ is a diffeomorphism of $\mathbb{R}^n$ and $\widetilde{k}(\xi) = \frac{\gamma\circ\widetilde{\Phi}^{-1}(\xi)}{d\xi}$, then we have

$$k(\Phi(x))|d\phi_1 \wedge ... \wedge d\phi_n|_{\wedge^n \mathcal{L}^2(\mathbb{R},dt)} = \widetilde{k}(\widetilde{\Phi}(x))|d\widetilde{\phi}_1 \wedge ... \wedge d\widetilde{\phi}_n|_{\wedge^n \mathcal{L}^2(\mathbb{R},dt)} \tag{5.6}$$

In the following we take $V = \mathbb{R}^n$, and $\Phi = (\phi_1, ..., \phi_n) : S'(\mathbb{R}) \longrightarrow \mathbb{R}^n$, $\Phi \in \mathcal{F}_\theta(S'(\mathbb{R}))$. A sub-set $\mathcal{C} \subseteq S'(\mathbb{R})$ is called a *sub-manifold of codimension n* if $\mathcal{C} = \Phi^{-1}(\xi) = V^\xi$, $\xi \in \mathbb{R}^n$.

We define the surface measure dA^Φ on $\mathcal{C} = V^\xi$ by

$$\int_{\mathcal{C}} f(x) dA^\Phi(x) = k(\xi) \int_{\mathcal{C}} f(x)[\det \Phi]^{\frac{1}{2}}(x) d\nu^\xi(x) \tag{5.7}$$

So $dA^\Phi = k(\xi)[\det \Phi]^{\frac{1}{2}}(x)d\nu^\xi(x)$, where $[\det \Phi](x) = |d\phi_1 \wedge ... \wedge d\phi_n|_{\wedge^n \mathcal{L}^2(\mathbb{R},dt)}$, and f is a continuous function with compact support in $S'(\mathbb{R})$.

As a deduction of Lemma 5.5 we have the following

Corollary 5.6. *If $\widetilde{\Phi} = \Psi \circ \Phi$ where Ψ is a diffeomorphism of $\mathbb{R}^n$, then the two measures $da^{\widetilde{\Phi}}$ and da^Φ are equal on $\mathcal{C} = \widetilde{\Phi}^{-1}(\xi) = \Phi^{-1}(\Psi^{-1}(\xi))$.*

Proof. Using equations (5.6) and (5.7) and the fact that

$$\nu^{\widetilde{\Phi}}(dx) := \gamma(dx|\widetilde{\Phi}(x) = \xi) = \gamma(dx|\Phi(x) = \Psi^{-1}(\xi)),$$

we have the desired result. ☐

Theorem 5.7. (Coarea formula on the space of tempered distributions) *For any function $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ with compact support and for any function $g : S'(\mathbb{R}) \longrightarrow \mathbb{R}$ and $g \in \mathcal{F}_\theta(S'(\mathbb{R}))$, we have*

$$\int_{S'(\mathbb{R})} V(\Phi(x))\Big(g(x)[\det \Phi]^{\frac{1}{2}}(x)\Big)d\gamma(x) = \int_{\mathbb{R}^n} V(\xi)\Big(\int_{\Phi^{-1}(\xi)} g(x)dA^{\Phi}(x)\Big)d\xi. \tag{5.8}$$

Proof. The proof follows from Theorem 2.3 and Theorem 5.2. □

For any $1 \leq i \leq n$, let $\Phi_i \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R}))$. In the following, for any $1 \leq m \leq n+1$, we define

$$V_m = \big\{x \in S'(\mathbb{R}); \Phi_i(x) = \xi_i, i \in \{1, 2, ..., m\}\big\}$$

$$W_m = V_m \cup \big\{x \in S'(\mathbb{R}); \Phi_{m+1}(x) \leq \xi_{m+1}\big\}$$

and let $dA^{\phi_1,\dots,\phi_n}$ denote the surface measure on V_n.

Theorem 5.8. *For any $f \in \mathcal{DF}_\theta(\wedge^{n+1}\mathcal{L}^2(\mathbb{R}, dt))$, we have*

$$\int_{V_{n+1}} \big(f, \frac{d\Phi_1 \wedge d\Phi_2 \wedge ... \wedge d\Phi_n \wedge d\Phi_{n+1}}{|d\Phi_1 \wedge d\Phi_2 \wedge ... \wedge d\Phi_n \wedge d\Phi_{n+1}|_{\wedge^{n+1}\mathcal{L}^2(\mathbb{R},dt)}}\big)_{\wedge^{n+1}\mathcal{L}^2(\mathbb{R},dt)} dA^{\phi_1,\dots,\phi_{n+1}}(x)$$
$$= \int_{W_n} \big(\delta f, \frac{d\Phi_1 \wedge d\Phi_2 \wedge ... \wedge d\Phi_n}{|d\Phi_1 \wedge d\Phi_2 \wedge ... \wedge d\Phi_n|_{\wedge^{n}\mathcal{L}^2(\mathbb{R},dt)}}\big)_{\wedge^{n}\mathcal{L}^2(\mathbb{R},dt)} dA^{\phi_1,\dots,\phi_n}(x) \tag{5.9}$$

Proof. **Step 1.** Let $V : \mathbb{R}^n \longrightarrow \mathbb{R}$, $V \in \mathcal{C}^\infty$, and $f \in \mathcal{DF}_\theta(\wedge^{n+1}\mathcal{L}^2(\mathbb{R}, dt))$. We denote

$$A = d\Phi_1 \wedge d\Phi_2 \wedge \cdots \wedge d\Phi_n \lrcorner f.$$

We shall prove that

$$\delta((V \circ d\Phi)A)(x) = V(\Phi(x))\big(\delta f(x), d\Phi_1 \wedge d\Phi_2 \wedge \cdots \wedge d\Phi_n\big)_{\wedge^n\mathcal{L}^2(\mathbb{R},dt)}. \tag{5.10}$$

Using equations (2.7) and (4.2), we obtain for any $\lambda : S'(\mathbb{R}) \longrightarrow \mathbb{R}$, $\lambda \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R}))$

$$\begin{aligned}
&\langle \delta((V \circ d\Phi)d\Phi_1 \wedge d\Phi_2 \wedge \cdots \wedge d\Phi_n \lrcorner f), \lambda\rangle_{L^2(S'(\mathbb{R}),\gamma)} \\
&= \langle (V \circ d\Phi)d\Phi_1 \wedge d\Phi_2 \wedge \cdots \wedge d\Phi_n \lrcorner f, d\lambda\rangle_{L^2(\gamma,\mathcal{L}^2(\mathbb{R},dt))} \\
&= \langle f, (V \circ \Phi)d\lambda \wedge d\Phi_1 \wedge d\Phi_2 \wedge \cdots \wedge d\Phi_n\rangle_{L^2(\gamma,\wedge^{n+1}\mathcal{L}^2(\mathbb{R},dt))} \\
&= \int_{S'(\mathbb{R})} \big(f(x), (V \circ d\Phi)(x)d\lambda \wedge d\Phi_1 \wedge d\Phi_2 \wedge \cdots \wedge d\Phi_n\big)_{\wedge^{n+1}\mathcal{L}^2(\mathbb{R},dt)} d\gamma(x).
\end{aligned}$$

Since $dx \wedge dx = 0$, we obtain

$$d(\lambda(V \circ \Phi)d\Phi_1 \wedge d\Phi_2 \wedge \cdots \wedge d\Phi_n) = (V \circ \Phi)d\lambda \wedge d\Phi_1 \wedge d\Phi_2 \wedge \cdots \wedge d\Phi_n.$$

Using again equation (2.7), we obtain

$$\begin{aligned}
&\langle (V\circ d\Phi)d\Phi_1\wedge d\phi_2\wedge\cdots\wedge d\Phi_n\lrcorner f, d\lambda\rangle_{L^2(\gamma,\mathcal{L}^2(\mathbb{R},dt))}\\
&= \int_{S'(\mathbb{R})}\big(f(x), d(\lambda(V\circ\Phi)d\Phi_1\wedge d\phi_2\wedge\cdots\wedge d\Phi_n)\big)_{\wedge^{n+1}\mathcal{L}^2(\mathbb{R},dt)}d\gamma(x)\\
&= \int_{S'(\mathbb{R})}\big(\delta f, \lambda(x)(V\circ\Phi)(x)d\Phi_1\wedge d\phi_2\wedge\cdots\wedge d\Phi_n\big)_{\wedge^{n}\mathcal{L}^2(\mathbb{R},dt)}d\gamma(x),
\end{aligned}$$

and so equation (5.10) is proved.

Step 2. Let $G\in C^\infty$ and $f\in\mathcal{DF}_\theta(\wedge^{n+1}\mathcal{L}^2(\mathbb{R},dt))$, and let

$$\begin{aligned}
&J\\
&\equiv \int_{\mathbb{R}^{n+1}}G(\xi)\int_{V_{n+1}}\big(f, \frac{d\Phi_{n+1}\wedge d\Phi_1\wedge d\phi_2\wedge\cdots\wedge d\Phi_n}{|d\Phi_{n+1}\wedge d\Phi_1\wedge d\phi_2\wedge\cdots\wedge d\Phi_n|}\big)_{\wedge^n\mathcal{L}^2(\mathbb{R},dt)}dA^{\phi_1,\dots,\phi_{n+1}}d\xi \qquad (5.11)\\
&= \langle (G\circ\Phi)f, d\Phi_{n+1}\wedge d\Phi_1\wedge d\phi_2\wedge\cdots\wedge d\Phi_n\rangle_{L^2(\gamma,\wedge^{n+1}\mathcal{L}^2(\mathbb{R},dt))}\\
&= \langle (G\circ\Phi)d\Phi_1\wedge d\phi_2\wedge\cdots\wedge d\Phi_n\lrcorner f, d\Phi_{n+1}\rangle_{L^2(\gamma,\mathcal{L}^2(\mathbb{R},dt))}.
\end{aligned}$$

In particular, if we take

$$G(\xi) = v(\xi_1,\xi_2,\dots,\xi_n)u(\xi_{n+1}),$$

where $v:\mathbb{R}^n\longrightarrow\mathbb{R}$; $v\in C^\infty$ and $u:\mathbb{R}\longrightarrow\mathbb{R}$, $u\in C^\infty$, then Theorem 5.8 yields

$$\begin{aligned}
J &= \int_{S'(\mathbb{R})}(u\circ\Phi_{n+1})\big(v(\Phi_1,\dots,\Phi_n)d\Phi_1\wedge d\phi_2\wedge\cdots\wedge d\Phi_n\lrcorner f, d\Phi_{n+1}\big)_0 d\gamma(x)\\
&= \int_{\mathbb{R}}u(\xi_{n+1})\int_{\Phi_{n+1}^{-1}(\xi_{n+1})}\big(v(\Phi_1,\dots,\Phi_n)d\Phi_1\wedge d\phi_2\wedge\cdots\wedge d\Phi_n\lrcorner f, d\Phi_{n+1}\big)_0\\
&\qquad\qquad \frac{dA^{\Phi_{n+1}}}{|\nabla\Phi_{n+1}|_0}d\xi_{n+1},
\end{aligned}$$

where $dA^{\Phi_{n+1}}$ is the surface measure on $V^{\xi_{n+1}}$. Then use Theorem 5.2 and equation (5.10) to get

$$\begin{aligned}
J &= \int_{\mathbb{R}}u(\xi_{n+1})\bigg(\int_{\Phi_{n+1}\le\xi_{n+1}}\delta(v(\Phi_1,\dots,\Phi_n)d\Phi_1\wedge d\phi_2\wedge\cdots\\
&\qquad\qquad \wedge d\Phi_n\lrcorner f)d\gamma(x)\bigg)d\xi_{n+1}\\
&= \int_{\mathbb{R}}u(\xi_{n+1})\bigg(\int_{\Phi_{n+1}\le\xi_{n+1}}v(\Phi_1,\dots,\Phi_n)\big(\delta f, d\Phi_1\wedge d\phi_2\wedge\cdots\\
&\qquad\qquad \wedge d\Phi_n\big)_0 d\gamma(x)\bigg)d\xi_{n+1}.
\end{aligned}$$

Then by Theorem 5.8

$$\begin{aligned} J &= \int_{\mathbb{R}} u(\xi_{n+1})d\xi_{n+1} \int_{\mathbb{R}^n} v(\xi_1, \dots, \xi_n)d\xi_1 \dots d\xi_n \\ &\qquad \times \int_{W_n} \langle \delta f, d\Phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\Phi_n \rangle \frac{dA^{\Phi_n}}{|\nabla \Phi_n|_0} \\ &= \int_{\mathbb{R}^{n+1}} G(\xi)d\xi \int_{W_n} \langle \delta f, d\Phi_1 \wedge d\phi_2 \wedge \cdots \wedge d\Phi_n \rangle \frac{dA^{\Phi_n}}{|\nabla \Phi_n|_0} d\xi_n. \end{aligned} \tag{5.12}$$

Obviously, equations (5.11) and (5.12) lead to equation (5.9). □

Finally, we give an example. Let $\xi_1, \xi_2, ..., \xi_n \in S(\mathbb{R})$ be an orthogonal system for $\mathcal{L}^2(\mathbb{R}, dt)$ and consider the special function

$$\Phi(x) = \Big(\langle x, \xi_1 \rangle, \langle x, \xi_2 \rangle, ..., \langle x, \xi_n \rangle\Big). \tag{5.13}$$

Proposition 5.9. *Let Φ be the function given by equation (5.13). Then the measure $\gamma \circ \Phi^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$ and its density is given by*

$$K(y) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}|\xi_i|_0} e^{-\frac{y_i^2}{2|\xi_i|_0^2}}, \quad y = (y_1, ..., y_n) \in \mathbb{R}^n.$$

Moreover, for the test function $g(x) = e^{\langle x, \eta \rangle}$ with $\eta \in S(\mathbb{R})$, the measure $\gamma_g \circ \Phi^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$ and the corresponding density $K_g(y)$ is given by

$$K_g(y) = \prod_{i=1}^{n} \frac{e^{\frac{|\eta|_0^2}{2}}}{\sqrt{2\pi}|\xi_i|_0} e^{-\frac{(y_i - \langle \xi_i, \eta \rangle)^2}{2|\xi_i|_0^2}}, \quad y = (y_1, ..., y_n) \in \mathbb{R}^n.$$

In particular, for $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$, the surface measure ν^a on $V^a = \{x \in S'(\mathbb{R})\,;\, \Phi(x) = a\}$ is given by

$$(\mathcal{L}\nu^a)(\eta) = \prod_{i=1}^{n} \exp\left(|\eta|_0^2 - \frac{\langle \xi, \eta \rangle^2}{2|\xi_i|_0^2} + a_i \frac{\langle \xi_i, \eta \rangle}{|\xi_i|_0^2} \right), \quad \eta \in S(\mathbb{R}). \tag{5.14}$$

Proof. Using the fact that, for $\xi_1, \xi_2, \dots, \xi_n \in S(\mathbb{R})$ an orthogonal system for $\mathcal{L}^2(\mathbb{R}, dt)$ and f_1, f_2,...,f_n integrable functions on $\mathbb{R}$ with respect to the one dimensional standard Gaussian measure,

$$\int_{S'(\mathbb{R})} f_1(\langle x, \xi_1 \rangle) ... f_n(\langle x, \xi_n \rangle) d\gamma(x) = \prod_{i=1}^{n} \int_{S'(\mathbb{R})} f_i(\langle x, \xi_i \rangle) d\gamma(x),$$

we obtain for $t = (t_1, ..., t_n) \in \mathbb{R}^n$,

$$\begin{aligned}
\mathcal{L}(\gamma \circ \Phi^{-1})(t) &= \int_{\mathbb{R}^n} e^{\langle x,t\rangle} d(\gamma \circ \Phi^{-1})(x) \\
&= \int_{S'(\mathbb{R})} e^{\langle \Phi(x),t\rangle} d\gamma(x) \\
&= \int_{S'(\mathbb{R})} \prod_{i=1}^{n} e^{\Phi_i(x).t_i} d\gamma(x) \\
&= \prod_{i=1}^{n} \int_{S'(\mathbb{R})} e^{\Phi_i(x).t_i} d\gamma(x) \\
&= \prod_{i=1}^{n} \mathcal{L}(\gamma)(t_i x_i) \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}|\xi_i|_0} e^{-\frac{y^2}{2|\xi_i|_0^2}} dy \\
&= \int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}|\xi_i|_0} e^{-\frac{y_i^2}{2|\xi_i|_0^2}} dy_i.
\end{aligned}$$

So the corresponding function $K(y)$ in equation (2.9) is given by:

$$K(y) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}|\xi_i|_0} e^{-\frac{y_i^2}{2|\xi_i|_0^2}}, \quad y = (y_1, ..., y_n) \in \mathbb{R}^n.$$

□

Finally, we apply the Stokes formula given by equation (5.9) for $n = 2$ to obtain the following corollary.

Corollary 5.10. *For any $f \in \mathcal{DF}_\theta(\wedge^2 \mathcal{L}^2(\mathbb{R}, dt))$, we have the equality:*

$$\begin{aligned}
&\int_{V_2} \big(f, \frac{d\Phi_1 \wedge d\Phi_2}{|d\Phi_1 \wedge d\Phi_2|_{\wedge^2 \mathcal{L}^2(\mathbb{R},dt)}}\big) dA^{\Phi_1,\Phi_2}(x) \\
&= \int_{W_1} \big(\delta f, \frac{\xi_1}{|\xi_1|_0}\big)_0 dA^{\Phi_1}(x) = \int_{W_1} \big(\delta f, \xi_1\big)_0 d\nu_1^a(x),
\end{aligned}$$

where ν_1^a is the unique probability measure on $S'(\mathbb{R})$ defined in Theorem 2.3 for $\Phi(x) = \langle x, \xi_1\rangle$. Moreover,

$$V_2 = \big\{x \in S'(\mathbb{R}); \Phi_i(x) = \xi_i, i \in \{1, 2\}\big\},$$

$$W_1 = V_1 \cup \big\{x \in S'(\mathbb{R}); \Phi_2(x) \le \xi_2\big\}.$$

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