Communications on Stochastic Analysis
Vol. 8, No. 4 (2014) 439-448

Serials Publications
www.serialspublications.com

ABSOLUTE CONTINUITY UNDER TIME SHIFT FOR ORNSTEIN-UHLENBECK TYPE PROCESSES WITH DELAY OR ANTICIPATION

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Abstract. The paper is concerned with one-dimensional two-sided Ornstein-Uhlenbeck type processes with delay or anticipation. We prove existence and uniqueness requiring almost sure boundedness on the left half-axis in case of delay and almost sure boundedness on the right half-axis in case of anticipation. For those stochastic processes $(X, P_{\boldsymbol{\mu}})$ we calculate the Radon-Nikodym density under time shift of trajectories, $P_{\boldsymbol{\mu}}(dX_{\cdot - t})/P_{\boldsymbol{\mu}}(dX)$, $t \in \mathbb{R}$.

1. Introduction, Basic Objects, and Main Result

1.1. Introduction. Let W be a one-dimensional two-sided Brownian motion with random initial value W_0 and $a \in (-1, 0)$ be non-random. The paper has two objectives. On the one hand, we are interested in solutions to a stochastic equation with delay of the form

$$dX_s = a\left(X_{s-1} + b_0\right) ds + dW_s\,, \quad s \in \mathbb{R}, \quad \text{where } \limsup_{v \to -\infty} |X_v| < \infty,$$

and solutions to a stochastic equation with anticipation of the form

$$dX_s = -a\left(X_{s+1} + b_0\right) ds + dW_s\,, \quad s \in \mathbb{R}, \quad \text{where } \limsup_{v \to +\infty} |X_v| < \infty,$$

for some $b_0 \equiv b_0(W)$, cf. Theorem 3.1 and Corollary 3.2 below. This part of the paper can be regarded as a supplement to the rich literature on stochastic delay differential equations. We wish to point to the many different topics of interest by referring to some of them, [4], [9], [11], and [12].

On the other hand, for the stochastic processes $(X, P_{\boldsymbol{\mu}})$ being solution to one of the two above equations, we are interested in quasi-invariance under time shift of trajectories. In Theorem 3.3 we provide the Radon-Nikodym density $P_{\boldsymbol{\mu}}(dX_{\cdot - t})/P_{\boldsymbol{\mu}}(dX)$, $t \in \mathbb{R}$, and relate this density to a recent result in the paper [8]. This type of quasi-invariance is particularly meaningful for two-sided stochastic processes. It adds to the well-established results on quasi-invariance in the non-anticipating or the non-gaussian case. For an overview on these, check for example, [1], Section 11.2, [2], and [13].

Received 2014-1-30; Communicated by the editors. Article is based on a lecture presented at the International Conference on Stochastic Analysis and Applications, Hammamet, Tunisia, October 14-19, 2013.
2010 *Mathematics Subject Classification.* Primary 60J65; Secondary 60H10.
Key words and phrases. Brownian motion, SDE with delay or anticipation, non-linear transformation of measures.

1.2. Definitions and notations. Two-sided Brownian motion with random initial value. Let $\boldsymbol{\nu}$ be a probability distribution on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let the probability space $(\Omega, \mathcal{F}, Q_{\boldsymbol{\nu}})$ be given by the following. For a similar definition see also [7], paragraph 1 of Section 2.

(i) $\Omega = C(\mathbb{R};\mathbb{R})$, the space of all continuous functions W from $\mathbb{R}$ to $\mathbb{R}$. Identify $C(\mathbb{R};\mathbb{R}) \equiv \{\omega \in C(\mathbb{R};\mathbb{R}) : \omega(0) = 0\} \times \mathbb{R}$.

(ii) $\mathcal{F}$, the σ-algebra of Borel sets with respect to uniform convergence on compact subsets of $\mathbb{R}$.

(iii) $Q_{\boldsymbol{\nu}}$, the probability measure on $(\Omega, \mathcal{F})$ for which, when given W_0, both $(W_s - W_0)_{s\ge 0}$ as well as $(W_{-s} - W_0)_{s\ge 0}$ are independent standard Brownian motions with state space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Assume W_0 to be independent of $(W_s - W_0)_{s\in\mathbb{R}}$ and distributed according to $\boldsymbol{\nu}$.

In addition, we will assume that the natural filtration $\{\mathcal{F}_u^v = \sigma(W_\alpha - W_\beta : u \le \alpha, \beta \le v) \times \sigma(W_0) : -\infty < u < v < \infty\}$ is completed by the $Q_{\boldsymbol{\nu}}$-completion of $\mathcal{F}$.

For the measure $\boldsymbol{\nu}$ we shall assume the following throughout the paper.

(i) $\boldsymbol{\nu}$ admits a density m with respect to the one-dimensional Lebesgue measure.

(ii) $0 < m \in C^1(\mathbb{R};\mathbb{R})$.

It is known that

$$\frac{Q_{\boldsymbol{\nu}}(dW_{\cdot - t})}{Q_{\boldsymbol{\nu}}(dW)} = \frac{m\left(W_{-t}\right)}{m(W_0)}\,, \quad t \in \mathbb{R}. \tag{1.1}$$

In [8], Theorem 1.12 together with Proposition 1.16, this formula is generalized. Below we will outline the result of this reference in a simplified form for which we assume for technical reasons

(iii) that for all $y \in \mathbb{R}$,

$$\frac{d}{d\lambda}\bigg|_{\lambda=0} \frac{m(\,\cdot + \lambda y)}{m} = y \cdot \frac{\nabla m}{m} \quad \text{exists in } L^Q(\mathbb{R}, \boldsymbol{\nu})$$

for some $Q \in (1, \infty)$.

Parallel trajectories. Let $\xi \equiv \xi(W)$ be a random variable and let $\mathbb{1} \equiv \mathbb{1}(s)$, $s \in \mathbb{R}$, denote the constant function on $\mathbb{R}$ taking the value one. For fixed $W \in \Omega$ and $x \in \mathbb{R}$, denote

$$\nabla_x \xi(W + x\mathbb{1}) \equiv \frac{\partial}{\partial x} \xi(W + x\mathbb{1})$$

whenever this partial derivative exists. Set

$$\nabla_{W_0} \xi(W) := \nabla_x|_{x=W_0}\, \xi(W - W_0\mathbb{1} + x\mathbb{1})\,.$$

The process X. In Theorem 3.1 and Corollary 3.2 below, we will verify that the stochastic processes of Subsection 1.1 have a representation

$$X = W + A$$

where $A \equiv A(W)$ is a stochastic process with trajectories that belong to $C(\mathbb{R};\mathbb{R})$. This induces path wise a measurable map $X = X(W) : \Omega \equiv C(\mathbb{R};\mathbb{R}) \to C(\mathbb{R};\mathbb{R})$ which we show to be $Q_{\boldsymbol{\nu}}$-a.e. injective. The measure $P_{\boldsymbol{\mu}} := Q_{\boldsymbol{\nu}} \circ X^{-1}$ is the law of X. Here, the index $\boldsymbol{\mu}$ is just a part of a symbol added for compatibility of the notation with [8].

1.3. Change of measure under time shift. Following the ideas of [8], we shall verify in Theorem 3.1 and Corollary 3.2 below that $X \equiv X(W)$ is a *temporally homogeneous* function of $W \in \Omega$. In the present setting this means that for $W^u := W_{\cdot+u} + A_u(W)\mathbb{1}$, $u \in \mathbb{R}$, we have $A_0(W) = 0$ and

$$X_{\cdot+v}(W) = X(W^v)\,, \quad v \in \mathbb{R}.$$

In this case we have, among other things, $X_0 = W_0$. We may therefore also write ∇_{X_0} for ∇_{W_0}.

One purpose of the present paper is to show that the Radon-Nikodym densities

$$\omega_{-t}(W) := \frac{Q_{\boldsymbol{\nu}}(dW^{-t})}{Q_{\boldsymbol{\nu}}(dW)} \quad \text{and} \quad \rho_{-t}(X) := \frac{P_{\boldsymbol{\mu}}(dX_{\cdot-t})}{P_{\boldsymbol{\mu}}(dX)}\,, \quad t \in \mathbb{R}, \tag{1.2}$$

exist and to give a representation of these densities. In fact, in Section 3, we will use a particular property of the processes we are examining to establish the above Radon-Nikodym derivatives. However we will also discuss the relation to one the main results of [8], namely Theorem 1.12. There, for a much larger class of processes than those of Subsection 1.1, we derive $Q_{\boldsymbol{\nu}}$-a.e. the density

$$\omega_{-t}(W) = \frac{m\left(X_{-t}(W)\right)}{m(W_0)} \cdot \left|1 + \nabla_{W_0} A_{-t}(W)\right|$$

and $P_{\boldsymbol{\mu}}$-a.e. the density

$$\rho_{-t}(X) = \frac{m\left(X_{-t}\right)}{m\left(X_0\right)} \cdot \left|\nabla_{X_0} X_{-t}\right|\,, \quad t \in \mathbb{R}. \tag{1.3}$$

2. Ornstein-Uhlenbeck Type Process with Delay or Anticipation

In order to approach solutions to the stochastic equations of Subsection 1.1 we are first interested in a process $\tilde{X}$ that obeys a stochastic delay equation of the form

$$d\tilde{X}_s = a\tilde{X}_{s-1}\,ds + dW_s\,, \quad s \in \mathbb{R}, \quad \tilde{X}_u = \varphi(u, W)\,, \quad u \in [-1, 0], \tag{2.1}$$

or a stochastic equation with anticipation of the form

$$d\tilde{X}_s = -a\tilde{X}_{s+1}\,ds + dW_s\,, \quad s \in \mathbb{R}, \quad \tilde{X}_u = \varphi(u, W)\,, \quad u \in [0, 1], \tag{2.2}$$

for a non-random $a \in (-1, 0)$ and a certain continuous random function φ. Our approach will rely on a very specific choice of φ, cf. Theorem 2.5 and Corollary 2.6 below. In fact the construction of such a *two-sided* process $\tilde{X}$ will suggest to relate uniqueness to the asymptotic behavior rather than to an initial function.

Our first effort will be towards the $Q_{\boldsymbol{\nu}}$-a.e. pathwise existence and uniqueness of the solution to the equation (2.1). It will turn out that the $Q_{\boldsymbol{\nu}}$-a.e. pathwise existence and uniqueness of the solution to (2.2) is a consequence of the result for (2.1), simply by time reversal.

Our method to treat the equation (2.1) is adapted to the *method of steps* in the theory of delay differential equations. To begin with, we are going to specify the space of functions the initial function φ is related with.

Definition 2.1. (a) For $a \in \mathbb{R} \setminus \{0\}$, $\omega \in C((-\infty,0])$, and $k \in \mathbb{N}$, let $C^k_{a,\omega}([-1,0])$ denote the space of all real functions f on $[-1,0]$ satisfying the following. For all those f there is a function $p_k \in C([-1,0])$ and a collection $c_0, c_1, \ldots, c_k \in \mathbb{R}$ such that f has a representation in the form of a k-fold iterated integral

$$f(v_0) = \int_{-1}^{v_0} a \left(\ldots a \left(\int_{-1}^{v_{k-1}} a \left(p_k(v_k) + c_k + \omega(v_k - k) \right) dv_k + c_{k-1} \right.\right.$$
$$\left.\left. + \omega(v_{k-1} - (k-1)) \right) dv_{k-1} \ldots + c_1 + \omega(v_1 - 1) \right) dv_1 + c_0 + \omega(v_0) \,,$$

$v_0 \in [-1,0]$. In addition, let $p_0 \in C([-1,0])$ be defined by

$$p_0 := f - (c_0 + \omega) \,.$$

(b)

$$C^\infty_{a,\omega}([-1,0]) := \bigcap_{k \in \mathbb{N}} C^k_{a,\omega}([-1,0]) \,.$$

Obviously,

$$C^{k+1}_{a,\omega}([-1,0]) \subseteq C^k_{a,\omega}([-1,0]) \,, \quad k \in \mathbb{N}.$$

Furthermore, for $f \in C^k_{a,\omega}([-1,0])$ represented as in part (a) of this definition, let us use the notation $f = T^{(1)}(p_1; c_0, c_1) = T^{(2)}(p_2; c_0, c_1, c_2) = \ldots = T^{(k)}(p_k; c_0, \ldots, c_k)$ and $p_i \equiv p_i(\cdot, f)$, $i = 1, \ldots, k$, $c_j \equiv c_j(f)$, $j = 0, \ldots, i$. In the proof below we will also use the symbol $C^1_{a,\omega(\cdot - k)}([-1,0])$ for ω replaced by the restriction of $\omega(\cdot - k)$ to $(-\infty, 0]$, $k \in \mathbb{N}$.

Proposition 2.2. *Let $a \in \mathbb{R} \setminus \{0\}$ and $\omega \in C((-\infty,0])$. (a) For all $k \in \mathbb{N}$, $C^k_{a,\omega}([-1,0])$ is dense in $C([-1,0])$.*
(b) The set $C^\infty_{a,\omega}([-1,0])$ is dense in $C([-1,0])$.

Proof. (a) This follows immediately by induction.
(b) Let $\varphi \in C([-1,0])$, $0 < \varepsilon < \frac{1}{2}$, and $\varphi_1 \in C^1_{a,\omega}([-1,0])$ such that $\|\varphi - \varphi_1\| < \varepsilon$ where $\|\cdot\|$ denotes the norm in $C([-1,0])$ in this proof. Furthermore, choose ψ_1 with $\psi_1(-1) = 0$ and $\psi_1 + \omega(\cdot - 1)|_{[-1,0]} \in C^1_{a,\omega(\cdot-1)}([-1,0])$ such that $\|\psi_1 - p_1(\cdot, \varphi_1)\| < \varepsilon^2 \wedge \varepsilon^2/|a|$. Setting $\varphi_2 := T^{(1)}(\psi_1; c_0(\varphi_1), c_1(\varphi_1))$ it follows that

$$\varphi_2 \in C^2_{a,\omega}([-1,0])$$

and

$$\|p_1(\cdot, \varphi_1) - p_1(\cdot, \varphi_2)\| < \varepsilon^2 \quad \text{as well as} \quad \|\varphi_1 - \varphi_2\| < \varepsilon^2 \,.$$

We continue that way by choosing for all $k \in \mathbb{N}$ a ψ_k with $\psi_k(-1) = 0$ and $\psi_k + \omega(\cdot - k)|_{[-1,0]} \in C^1_{a,\omega(\cdot-k)}([-1,0])$ such that $\|\psi_k - p_k(\cdot, \varphi_k)\| < \varepsilon^{k+1} \wedge \varepsilon^{k+1}/|a|^k$. We put $\varphi_{k+1} := T^{(k)}(\psi_k; c_0(\varphi_k), \ldots, c_k(\varphi_k))$ and obtain

$$\varphi_{k+1} \in C^{k+1}_{a,\omega}([-1,0]) \quad \text{and} \quad \|\varphi_k - \varphi_{k+1}\| < \varepsilon^{k+1} \tag{2.3}$$

as well as

$$\|p_i(\cdot, \varphi_k) - p_i(\cdot, \varphi_{k+1})\| < \varepsilon^{k+1} \,, \quad i \in \{1, \ldots, k\} \,.$$

From here it follows that the sequences $(\varphi_k)_{k\in\mathbb{N}}$ and $(p_i(\cdot,\varphi_k))_{k\geq i}$, $i\in\mathbb{N}$, simultaneously converge in $C([-1,0])$ as $k\to\infty$. Furthermore, their limits $\varphi_\infty:=\lim_k\varphi_k$ and $\pi_i:=\lim_k p_i(\varphi_k)$ have the property $\pi_i=p_i(\cdot,\varphi_\infty)$, $i\in\mathbb{N}$. Thus, $\varphi_\infty\in C^\infty_{a,\omega}([-1,0])$ and because of (2.3), $\|\varphi-\varphi_\infty\|<2\varepsilon$. □

Let us recall that for any real number $a\neq 0$ there exists a unique solution r to the equation

$$r(s)\equiv r(s;a)=1\chi_{[0,\infty)}(s)+a\int_0^s r(u-1)\chi_{[1,\infty)}(u)\,du\,,\quad s\geq 0. \tag{2.4}$$

In fact, we have

$$r(s)=0\ \text{ if } s<0 \quad\text{and}\quad r(s)=\sum_{l=0}^{k-1}\frac{a^l}{l!}(s-l)^l\ \text{ if } s\in[k-1,k),\ k\in\mathbb{N}.$$

For $s<0$, we set $r(s):=0$. Let us moreover collect some facts about the problem

$$dX_s^{[-1,\infty)}=aX_{s-1}^{[-1,\infty)}+dW_s\,,\ s\geq 0,\quad X_u^{[-1,\infty)}=f(u),\ u\in[-1,0], \tag{2.5}$$

from [12]. For $s\geq 0$ and $u\in[-1,0]$ define $(I(s))(u)=0$ if $s+u<0$ and

$$\begin{aligned}(I(s,W))(u)\ &:=W_{s+u}+\int_0^{s+u}W_{s-v+u}\,dr(v)-r(s+u)\cdot W_0\\ &=\int_0^{s+u}r(s-v+u)\,dW_v\,,\quad s+u\geq 0.\end{aligned}$$

Also, there is a strongly continuous semigroup $(T_s)_{s\geq 0}$ in $C([-1,0])$ such that with $f\in C([-1,0])$, $T_sf(q)$ depends only on f and $q+s$. In particular, for $s+q\in[-1,0]$, $T_sf(q)$ is directly defined by $T_sf(q):=f(s+q)$. For all other $s\geq 0$ and $q\in[-1,0]$, $T_sf(q)$ is implicitly given by the following. The function $g(v):=T_sf(v-s)$ for $v\in[-1,\infty)$ and any $s\in[v,v+1]$ is continuous on $v\in[-1,\infty)$, continuously differentiable on $v>0$, and satisfies

$$g'(s)=ag(s-1)\,,\ s\geq 0,\quad g(u)=f(u)\,,\ u\in[-1,0].$$

There exists a pathwise unique solution $X^{[-1,\infty)}$ of (2.5). In particular,

$$X_{s+u}^{[-1,\infty)}\equiv X_{s+u}^{[-1,\infty)}(f,W)=(T_sf)(u)+(I(s,W))(u)\,,\quad s\geq 0,\ u\in[-1,0].$$

Let ω be the restriction of $W\in\Omega$ to the interval $(-\infty,0]$. Below we will pathwise use the symbol $C^\infty_{a,W}([-1,0])$ for $C^\infty_{a,\omega}([-1,0])$.

Lemma 2.3. *Let $a\in(-1,0)$ and $v_0\in[0,1]$. (a) For $Q_{\boldsymbol{\nu}}$-a.e. W there exists a unique $f\equiv f(W,\cdot)\in C^\infty_{a,W}([-1,0])$ satisfying the following.*

(i) *For all $k\in\mathbb{N}$, the function f, considered as an element belonging to $C^k_{a,W}([-1,0])$, has the representation*

$$\begin{aligned}f(v_0)\ =&\int_{-1}^{v_0}a\left(\ldots a\left(\int_{-1}^{v_{k-1}}a\left(p_k(v_k)-W_{-k-1}+W_{v_k-k}\right)dv_k-W_{-k}\right.\right.\\ &\left.\left.+W_{v_{k-1}-(k-1)}\right)dv_{k-1}\ldots-W_{-2}+W_{v_1-1}\right)dv_1-W_{-1}+W_{v_0}\end{aligned}$$

for some sequence p_k uniformly bounded in the sup*-norm with respect to $k \in \mathbb{Z}_+$ with $p_k(\cdot) - W_{-k-1} + W_{\cdot-k} \in C^\infty_{a,W_{\cdot-k}}([-1,0])$.*

(ii) *The function f has the representation*

$$f(v_0) \;= \sum_{k=1}^{\infty} a^k \int_{-1}^{v_0} \dots \int_{-1}^{v_{k-1}} \left(-W_{-k-1} + W_{v_k - k}\right) \, dv_k \dots dv_1 \\ - W_{-1} + W_{v_0} \, ,$$

where the infinite sum converges absolutely.

Proof. We use

$$\limsup_{k\to\infty} \sup_{-1\le v\le 0} \frac{-W_{-k-1} + W_{v-k}}{\sqrt{2\log k}} = 1 \quad Q_{\boldsymbol{\nu}}\text{-a.e.} \tag{2.6}$$

cf. [10], Exercise 5.1, from which it follows that the expression in (ii) is $Q_{\boldsymbol{\nu}}$-a.e. well-defined and absolutely converging for every $v_0 \in [-1,0]$. From (ii) we get (i). □

Proposition 2.4. *Let $a \in (-1,0)$ and let $p_0, p_1, \dots$ be the functions given by Lemma 2.3 and $p_0 := f - (-W_{-1} + W)$.*
(a) The infinite sum

$$q(v) \equiv q(W,v) := \sum_{k=0}^{\infty} \left(p_k(0) - W_{-k-1} + W_{-k}\right) \cdot r(v+k)$$

converges absolutely $Q_{\boldsymbol{\nu}}$-a.e. for any $v \in (-\infty,0]$. The random function $q : (-\infty,0] \to \mathbb{R}$ is $Q_{\boldsymbol{\nu}}$-a.e. bounded on $(-\infty,0]$.
(b) The random function $X^{(-\infty,0)} \equiv X^{(-\infty,0)}(W) : (-\infty,0) \to \mathbb{R}$ given by

$$X_v^{(-\infty,0)} = \sum_{k=0}^{\infty} \left(p_k(v+k) - W_{-k-1} + W_v\right) \cdot \chi_{[-k-1,-k)}(v) + q(v) \, ,$$

$v \in (-\infty,0)$, is the unique random function that satisfies $Q_{\boldsymbol{\nu}}$-a.e.

(i) *$dX_s = aX_{s-1} + dW_s$, $s \in (-\infty,0)$, and*
(ii) *$\limsup_{v\to-\infty} |X_v^{(-\infty,0)}| < \infty$.*

Moreover, we have

(iii) *$X^{(-\infty,0)}(W) = X^{(-\infty,0)}(W + x\mathbb{1})$, $x \in \mathbb{R}$.*

Proof. Part (a) follows from (2.6) and that by $a \in (-1,0)$ the solution $r(u)$ of (2.4) tends to 0 as $u \to \infty$ exponentially fast. For this, recall that all roots of the characteristic equation $\lambda = ae^{-\lambda}$ have a negative real part bounded away from zero. For a detailed description of the behavior of $r(u)$ as $u \to \infty$ we refer to [3], [5], and [6].

For part (b) we first check (i) on the open intervals $(-k-1,-k)$ stepwise relative to $k \in \mathbb{Z}_+$. To verify (the integrated version of) (i) at $-k$, $k \in \mathbb{N}$ we use (2.4). Property (ii) of part (b) is a consequence of the $Q_{\boldsymbol{\nu}}$-a.e. uniform boundedness of the p_k, $k \in \mathbb{Z}_+$, in the sup-norm, cf. Lemma 2.3 (a), relation (2.6), and the $Q_{\boldsymbol{\nu}}$-a.e. boundedness of q, cf. part (a) of this proposition. Property (iii) is an immediate consequence of the definition of $X^{(-\infty,0)}$. □

By definition there is a continuous extension of $X^{(-\infty,0)}$ to a random function $(-\infty,0] \to \mathbb{R}$ which we denote by $X^{(-\infty,0]}$.

Theorem 2.5. *Let $a \in (-1,0)$. For $Q_{\boldsymbol{\nu}}$-a.e. $W \in \Omega$ there exists a pathwise unique solution to the equation*

$$d\tilde{X}_s = a\tilde{X}_{s-1}\,ds + dW_s\,, \quad s \in \mathbb{R},$$

satisfying $\limsup_{v\to-\infty} |\tilde{X}_v| < \infty$. *The following holds.*

(i) *The process $\tilde{X}$ is the solution to (2.1) where*

$$\varphi \equiv \varphi(\,\cdot\,,W) := X^{(-\infty,0]}(W)\Big|_{[-1,0]}$$

and $X^{(-\infty,0]}$ is the process of Proposition 2.4 (b).

(ii) $\tilde{X}_s = X_s^{(-\infty,0]}(W)$, $s \in (-\infty,0]$, *and* $\tilde{X}_s = X_s^{[-1,\infty)}(\varphi,W)$, $s \in [-1,\infty)$.

(iii) $\tilde{X}_{\cdot+v}(W) = \tilde{X}(W_{\cdot+v})$, $v \in \mathbb{R}$, *and* $\tilde{X}(W) = \tilde{X}(W + x\mathbb{1})$, $x \in \mathbb{R}$.

Proof. We first recall that any non-zero solution of $dx_s = ax_{s-1}\,ds$, $s \in \mathbb{R}$, satisfies $\limsup_{v\to-\infty} |x_v| = +\infty$. This is also a consequence of the fact that all roots of the characteristic equation $\lambda = ae^{-\lambda}$ have a negative real part bounded away from zero. Now the main statement of the theorem along with the properties (i) as well as (ii) follow from Proposition 2.4 (b) and (2.5). Here we also use the $Q_{\boldsymbol{\nu}}$-a.e. uniqueness of the solution to (2.5). The first part of property (iii) is obtained from the fact that uniqueness of $\tilde{X}$ is determined by $\limsup_{v\to-\infty} |\tilde{X}_v| < \infty$, the second part is obvious by the stochastic equation. $\square$

By time reversal we obtain the corresponding assertions relative to equation (2.2).

Corollary 2.6. *Let $a \in (-1,0)$. For $Q_{\boldsymbol{\nu}}$-a.e. $W \in \Omega$ there exists a pathwise unique solution to the equation*

$$d\tilde{X}_s = -a\tilde{X}_{s+1}\,ds + dW_s\,, \quad s \in \mathbb{R},$$

satisfying $\limsup_{v\to\infty} |\tilde{X}^{(-\infty,0)}(v)| < \infty$. *The following holds.*

(i) *The process $\tilde{X}$ is the solution to (2.2) where*

$$\varphi \equiv \varphi(u,W) := X_{-u}^{(-\infty,0]}(W_{-\,\cdot})\,, \quad u \in [0,1],$$

and $X^{(-\infty,0]}$ is the process of Proposition 2.4 (b).

(ii) *$\tilde{X}$ is the time reversal of the stochastic process constructed in Theorem 2.5 for $W_{-\,\cdot}$ instead of W. It satisfies (iii) of Theorem 2.5.*

3. Absolute Continuity under Time Shift

Theorem 3.1. *Let*

$$b_0 \equiv b_0(W) := f(0) + q(0) - W_0,$$

where f and q are defined in Lemma 2.3 and Proposition 2.4. Then $Q_{\boldsymbol{\nu}}$-a.e. there exists a pathwise unique solution to the equation

$$dX_s = a\left(X_{s-1} + b_0\right)\,ds + dW_s\,, \quad s \in \mathbb{R}, \tag{3.1}$$

such that $\limsup_{v\to-\infty}|X_v|<\infty$. *This solution is temporally homogeneous, i. e. for* $A\equiv A(W):=X(W)-W$ *and* $W^u:=W_{\cdot+u}+A_u(W)\mathbb{1}$, $u\in\mathbb{R}$, *it satisfies* $A_0(W)=0$ *and*

$$X_{\cdot+v}(W)=X(W^v)\,,\quad v\in\mathbb{R}.$$

Proof. Let $\tilde{X}$ be the process introduced in Theorem 2.5. By $\limsup_{v\to-\infty}|\tilde{X}_v|<\infty$ there, the process

$$X=\tilde{X}-b_0\mathbb{1} \tag{3.2}$$

is the unique solution to (3.1) such that $\limsup_{v\to-\infty}|X_v|<\infty$. The definitions of b_0 and A yield $b_0(W)=\tilde{X}_0(W)-W_0$ and $A_0(W)=0$ for $Q_{\boldsymbol{\nu}}$-a.e. $W\in\Omega$. The definition of b_0 and Theorem 2.5 (iii) imply furthermore

$$\begin{aligned} b_0(W) &= \big(W_v+A_v(W)+b_0(0)\big)-\big(W_v+A_v(W)\big)\\ &= X_v(W)+b_0(W)-W_0^v=\tilde{X}_v(W)-W_0^v\\ &= \tilde{X}_0\left(W_{\cdot+v}\right)-W_0^v=\tilde{X}_0\left(W_{\cdot+v}+A_v(W)\mathbb{1}\right)-W_0^v\\ &= \tilde{X}_0(W^v)-W_0^v=b_0(W^v)\,,\quad v\in\mathbb{R}. \end{aligned}$$

Now we get from Theorem 2.5 (iii)

$$\begin{aligned} X_{\cdot+v}(W) &= \tilde{X}_{\cdot+v}(W)-b_0(W)\mathbb{1}\\ &= \tilde{X}\left(W_{\cdot+v}\right)-b_0(W)\mathbb{1}\\ &= \tilde{X}\left(W_{\cdot+v}+A_v(W)\mathbb{1}\right)-b_0(W^v)\\ &= X(W^v)\,,\quad v\in\mathbb{R}. \end{aligned}$$

□

An immediate consequence of time reversal is the following.

Corollary 3.2. *Let* $a\in(-1,0)$ *and let* $b_0\equiv b_0(W)$ *be the random variable defined in Theorem 3.1. Then* $Q_{\boldsymbol{\nu}}$*-a.e. there exists a pathwise unique solution to the equation*

$$dX_s=-a\left(X_{s+1}+b_0\right)\,ds+dW_s\,,\quad s\in\mathbb{R},$$

such that $\limsup_{v\to+\infty}|X_v|<\infty$. *This solution is temporally homogeneous.*

We obtain the following quasi-invariance under time shift of trajectories.

Theorem 3.3. *Let* m *denote the density of* W_0. *Let* X *be the Ornstein-Uhlenbeck type process established in Theorem 3.1 or the process of Corollary 3.2. Denoting by* $P_{\boldsymbol{\mu}}$ *the law of* X *we have* $P_{\boldsymbol{\mu}}$*-a.e.*

$$\rho_{-t}(X)=\frac{m(X_{-t})}{m(X_0)}\,,\quad t\in\mathbb{R}.$$

Proof. Step 1 Let us first focus on the case of the process with delay established in Theorem 3.1. Let the measure Q_x be obtained from $Q_{\boldsymbol{\nu}}$ by conditioning on $W_0=x$, $x\in\mathbb{R}$. We have $Q_{\boldsymbol{\nu}}=\int Q_x\,m(x)\,dx$. Next we use property (iii) of

Theorem 2.5 and relation (3.2). We have for $B = B_i \times B_p$ with $B_i \in \mathcal{B}(\mathbb{R})$ and $B_p \in \mathcal{F} \cap \{X \in \Omega : X_0 = 0\}$,

$$\begin{aligned} Q_{\boldsymbol{\nu}}\left(W : X(W) \in B\right) &= Q_{\boldsymbol{\nu}}\left(W : \tilde{X}(W) - f(0)\mathbb{1} - q(0)\mathbb{1} + W_0\mathbb{1} \in B\right) \\ &= \int_{\{W_0 \in B_i\}} Q_{W_0}\left(W : \tilde{X}(W) - f(0)\mathbb{1} - q(0)\mathbb{1} \in B_p\right) \boldsymbol{\nu}(dW_0) \end{aligned} \tag{3.3}$$

where we have used $X_0 = W_0$ and $X = \tilde{X} - b_0\mathbb{1} = \tilde{X} - f(0)\mathbb{1} - q(0)\mathbb{1} + W_0\mathbb{1}$. Furthermore, it holds that $\tilde{X}(W) = \tilde{X}(W + x\mathbb{1})$ according to Theorem 2.5 (iii) and, by definition, $f(0) \equiv f(W, 0) = f(W + x\mathbb{1}, 0)$ as well as $q(0) \equiv q(W, 0) = q(W + x\mathbb{1}, 0)$, $x \in \mathbb{R}$. This says that $Q_x(W : \tilde{X}(W) - f(0)\mathbb{1} - q(0)\mathbb{1} \in B_p)$ is independent of x. For the next chain of equations we recall that $X_{\cdot - t}(W) = X\left(W^{-t}\right)$, cf. Theorem 3.1, $t \in \mathbb{R}$. We obtain

$$\begin{aligned} &Q_{\boldsymbol{\nu}}\left(W : X_{\cdot - t}(W) \in B\right) = Q_{\boldsymbol{\nu}}\left(W : X\left(W^{-t}\right) \in B\right) \\ &\quad = Q_{\boldsymbol{\nu}}\left(W : \tilde{X}(W^{-t}) - f(W^{-t}, 0)\mathbb{1} - q(W^{-t}, 0)\mathbb{1} + W_0^{-t}\mathbb{1} \in B\right) \\ &\quad = Q_{\boldsymbol{\nu}}\left(W : \tilde{X}(W^{-t}) - f(W^{-t}, 0)\mathbb{1} - q(W^{-t}, 0)\mathbb{1} \in B_p,\, W_0^{-t}\mathbb{1} \in B_i\right) \\ &\quad = \int_{x \in B_i} Q_{\boldsymbol{\nu}}\left(W : \tilde{X}(W^{-t}) - f(W^{-t}, 0)\mathbb{1} - q(W^{-t}, 0)\mathbb{1} \in B_p \,\middle|\, W_0^{-t} = x\right) \\ &\qquad\qquad Q_{\boldsymbol{\nu}}\left(W : W_0^{-t} \in dx\right) \\ &\quad = \int_{x \in B_i} Q_{\boldsymbol{\nu}}\left(W : \tilde{X}(W_{\cdot - t}) - f(W_{\cdot - t}, 0)\mathbb{1} - q(W_{\cdot - t}, 0)\mathbb{1} \in B_p \,\middle|\, W_0^{-t} = x\right) \\ &\qquad\qquad Q_{\boldsymbol{\nu}}\left(W : W_0^{-t} \in dx\right) \\ &\quad = \int_{x \in B_i} Q_x\left(W : \tilde{X}(W) - f(W, 0)\mathbb{1} - q(W, 0)\mathbb{1} \in B_p\right) Q_{\boldsymbol{\nu}}\left(W : W_0^{-t} \in dx\right) \end{aligned} \tag{3.4}$$

$t \in \mathbb{R}$. Recalling (1.2), the claim follows now from (3.3) and (3.4) where use we that $Q_x(W : \tilde{X}(W) - f(0)\mathbb{1} - q(0)\mathbb{1} \in B_p)$ is independent of x. We note also that $W_0 = X_0$, $W_0^{-t} = X_{-t}(W)$, and that the distribution of W_0 is $\boldsymbol{\nu}$.

Step 2 Again, an immediate consequence of time reversal is the claim for he process with anticipation of Corollary 3.2. □

Remark 3.4. (1) It is also reasonable to verify the conditions of [8], Theorem 1.12 and Proposition 1.16. This would lead to (1.3),

$$\begin{aligned} \rho_{-t}(X) &= \frac{m\left(X_{-t}\right)}{m\left(X_0\right)} \cdot \left|\nabla_{X_0} X_{-t}\right| \\ &= \frac{m\left(X_{-t}\right)}{m\left(X_0\right)} \cdot \left|\nabla_{W_0}\left(\tilde{X} - f(0)\mathbb{1} - q(0)\mathbb{1} + W_0\mathbb{1}\right)\right| \\ &= \frac{m\left(X_{-t}\right)}{m\left(X_0\right)}, \quad t \in \mathbb{R}, \end{aligned}$$

where again we have used $\tilde{X}(W) = \tilde{X}(W + x\mathbb{1})$, $f(W,0) = f(W + x\mathbb{1}, 0)$, and $q(W,0) = q(W + x\mathbb{1}, 0)$, $x \in \mathbb{R}$. The above proof demonstrates that, in comparably simple situations, the result can be obtained independently by direct and short calculations.
(2) We observe the same simple form of the Radon-Nikodym derivative as in (1.1).

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