

LARGE DEVIATIONS ESTIMATES FOR SOME WHITE NOISE DISTRIBUTIONS

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ABSTRACT. We consider a positive distribution Φ which defines a probability measure $\mu = \mu_\Phi$ on X' the dual of some real nuclear Fréchet space. We consider the family $\{\mu_\epsilon, \epsilon > 0\}$, where μ_ϵ denotes the image measure of μ by the measurable map g_ϵ on X' given by $g_\epsilon(\lambda) = a(\epsilon)\lambda, \lambda \in X'$, where a is real positive valued function on \mathbb{R} such that $\lim_{\epsilon \rightarrow 0} a(\epsilon) = 0$. A large deviation principle is proved for the family $\{\mu_\epsilon, \epsilon > 0\}$, and application to stochastic differential equations is given.

1. Introduction

In recent years, there has been an enormous interest in the theory of large deviations. In fact, large deviations theory appears in several areas of mathematics and its applications in several problems.

H. Chernoff's motivation came from statistics. In particular, in [12] he was interested in questions about the asymptotic efficiency of statistical tests and initiated a program which has been carried further by several statisticians : Bahadur [1] and [2], Bahadur and Ranga Rao [3], Barndorff-Nielsen [6] and Dacunha-Castelle [14]. Schilder in [25] gives the first example of a large deviation result for measures on a function space. We cite also Borovkov [7] as one of first to study large deviations theory in a function space. Ventcel and Freidlin in [26], [27] and [28] use the same function-space introduced by Borovkov to analyse randomly perturbed dynamical systems.

Ellis, in his book [17] describes connections with statistical mechanics. We cite also Baldi Theorem [4] which is the extension of the Gärtner-Ellis Theorem [19], [16]. Amir Dembo and Ofer Zeitouni, in their book [15] emphasize applications related to electrical engineering. Cramér in [13] proved Cramér's theorem for distribution μ on \mathbb{R} which is not singular to the Lebesgue measure.

On the other hand, white noise analysis provides a lot of powerful tools as well for infinite dimensional calculus as for probability theory, a quite complete overview is given by [21] and [22]. So, the combination of these two subjects should inspire new results and give a feed-back to each of them.

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H. Ouerdiane and N. Privault in [23] give the first link between large deviations principle and white noise distributions.

S. Chaari, F. Cipriano, S. Gheryani and H. Ouerdiane in [9] prove Sanov theorem for white noise distributions and application to the Gibbs conditioning principle is given.

S. Chaari, F. Cipriano and H. Ouerdiane in [10], prove a large deviation principle for positive distributions introduced in [24]. In particular, a large deviation principle for white noise Gaussian measure is proved in [11].

In [10], the authors consider a positive distribution Φ as in [24], such that the Laplace transform $\widehat{\mu}$ of the associated positive Radon measure $\mu = \mu_\Phi$ satisfies the condition

$$\exists p, m > 0; \widehat{\mu}(\xi) = e^{\theta^*(m|\xi|_p)}, \xi \in X, \quad (1.1)$$

where θ^* is the Legendre transform of a Young function θ . Then, they prove that the family $\{\mu_\varepsilon, \varepsilon > 0\}$, where μ_ε denotes the image measure of μ by the map $g_\varepsilon(\lambda) = \sqrt{\varepsilon}\lambda$, $\lambda \in X'$, satisfies the upper bound condition, i.e., for all measurable subsets M in X' , we have

$$\liminf_{\varepsilon \rightarrow 0} \varepsilon \log(\mu_\varepsilon(M)) \leq \limsup_{\varepsilon \rightarrow 0} \varepsilon \log(\mu_\varepsilon(M)) \leq -\inf_M(\Lambda_\mu^*), \quad (1.2)$$

where Λ_μ^* is the Legendre transform of $\Lambda_\mu(\xi) := \lim_{\varepsilon \rightarrow 0} \varepsilon \log \left(\int_{X'} e^{\langle y, \frac{\xi}{\varepsilon} \rangle} d\mu_\varepsilon(y) \right)$.

In this paper, we generalize large deviation results for a family of measures on an infinite dimensional space of analytic functions with growth condition. For this purpose, we consider a positive distribution $\Phi \in \mathcal{F}_\theta(N')^*_+$ where $\mathcal{F}_\theta(N')^*_+$ is the strong dual space of $\mathcal{F}_\theta(N')$ introduced in [18], such that Φ defines a positive Radon measure $\mu = \mu_\Phi$ on X' , i.e. the Laplace transform $\widehat{\mu}$ of μ satisfies the following growth condition:

$$\exists p, m, C > 0; \widehat{\mu}(\xi) \leq C e^{\theta^*(m|\xi|_p)}, \xi \in X.$$

We denote by μ_ε the distribution of $\lambda \mapsto a(\varepsilon)\lambda$ under μ , where a denotes some real positive valued function such that $\lim_{\varepsilon \rightarrow 0} a(\varepsilon) = 0$. The Logarithmic moment generating function for the measure μ_ε is given by

$$\Lambda_{\mu_\varepsilon}(\xi) := \log \widehat{\mu}_\varepsilon(\xi) = \log \widehat{\mu}(a(\varepsilon)\xi).$$

Let

$$\Lambda(\xi) = \limsup_{\varepsilon \rightarrow 0} b(\varepsilon) \Lambda_{\mu_\varepsilon} \left(\frac{\xi}{b(\varepsilon)} \right), \quad (1.3)$$

where b denotes some real valued function such that $\lim_{\varepsilon \rightarrow 0} b(\varepsilon) = 0$.

Then, under the following additional condition:

$$\limsup_{x \rightarrow 0} b(x) \theta^* \left(\frac{a(x)}{b(x)} \right) < \infty, \quad (1.4)$$

we prove the large deviation principle for the family $\{\mu_\varepsilon, \varepsilon > 0\}$ of white noise distributions with rate function Λ^* and normalization $b(\varepsilon)$.

2. Preliminaries

2.1. Nuclear algebras of entire functions. Let X be a real nuclear fréchet space with topology given by an increasing family $\{|\cdot|_p, p \in \mathbb{N}\}$ of Hilbertian norms. Then, X and its strong dual X' are represented by

$$X = \bigcap_{p \geq 0} X_p = \text{proj-lim}_{p \rightarrow \infty} X_p; \quad X' = \bigcup_{p \geq 0} X_{-p} = \text{ind-lim}_{p \rightarrow \infty} X_{-p}$$

where X_p is the completion of X with respect to the norm $|\cdot|_p$ and X_{-p} is its topological dual space. Let $N = X + iX$ and $N_p = X_p + iX_p, p \in \mathbb{Z}$ be respectively the complexions of X and X_p . Then, N and its strong dual space N' can be represented by

$$N = \text{proj-lim}_{p \rightarrow \infty} N_p, \quad \text{and} \quad N' = \text{ind-lim}_{p \rightarrow \infty} N_{-p}.$$

Let $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a Young function, i.e., θ is continuous, convex, strictly increasing and satisfies:

$$\theta(0) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} \frac{\theta(x)}{x} = +\infty.$$

The Legendre transform θ^* of θ given by

$$\theta^*(x) = \sup_{t \geq 0} \{tx - \theta(t)\}, x \geq 0,$$

which is also a Young function. Given a complex Banach space $(B, |\cdot|)$, let $H(B)$ be the space of entire functions on B , i.e., the space of continuous functions from B to \mathbb{C} , whose restriction to all affine lines of B are entire on \mathbb{C} .

Let $\text{Exp}(B, \theta, m)$ denotes the space of all entire functions on B with exponential growth of order θ and of finite type $m > 0$:

$$\text{Exp}(B, \theta, m) = \{f \in H(B); \|f\|_{\theta, m} = \sup_{x \in B} |f(x)| e^{-\theta(m\|x\|)} < \infty\}.$$

Let also

$$\|f\|_{\theta, m, p} = \sup_{u \in N_p} |f(u)| e^{-\theta(m\|u\|_p)}, \quad f \in \text{Exp}(N_p, \theta, m).$$

The intersection

$$\mathcal{F}_\theta(N') = \bigcap_{p \in \mathbb{N}^*, m > 0} \text{Exp}(N_{-p}, \theta, m),$$

equipped with the projective limit topology, is called the space of entire functions on N' of θ -exponential growth and minimal type. The union

$$\mathcal{G}_\theta(N) = \bigcup_{p \in \mathbb{N}^*, m > 0} \text{Exp}(N_p, \theta, m),$$

equipped with the inductive limit topology, is called the space of entire functions on N of θ -exponential growth and (arbitrarily) finite type. Denote by $\mathcal{F}_\theta^*(N')$ the strong dual of the test function space $\mathcal{F}_\theta(N')$.

Remark 2.1. By replacing N with $\mathbb{C} \times N$, we can use the same procedure to introduce the spaces $\mathcal{F}_\theta(\mathbb{C} \times N')$ and $\mathcal{F}_\theta^*(\mathbb{C} \times N')$.

On the other hand, it is easy to see from the condition $\lim_{x \rightarrow \infty} \frac{\theta(x)}{x} = +\infty$ that the exponential function defined, for $\xi \in N$, by

$$e^\xi : N' \longrightarrow \mathbb{C}$$

$$z \longmapsto e^\xi(z) = e^{\langle z, \xi \rangle},$$

belongs to $\mathcal{F}_\theta(N')$.

For every $\Phi \in \mathcal{F}_\theta^*(N')$ the Laplace transform of Φ is defined by

$$\widehat{\Phi}(\xi) = L(\Phi)(\xi) = \Phi(e^\xi), \quad \xi \in N.$$

Theorem 2.2. [18] *The Laplace transform of analytical functionals induces a topological isomorphism*

$$\mathcal{L} : \mathcal{F}_\theta^*(N') \longmapsto \mathcal{G}_{\theta^*}(N). \tag{2.1}$$

As a consequence, $\Phi \in \mathcal{F}_\theta^*(N')$ if and only if its Laplace transform satisfies the growth condition

$$|\widehat{\Phi}(\xi)| \leq C \exp(\theta^*(m|\xi|_p)), \quad \xi \in N, \tag{2.2}$$

for some $C, m > 0$ and $p \in \mathbb{N}^*$.

Let μ be the standard Gaussian measure on X' uniquely specified by its characteristic function

$$e^{-\frac{1}{2}|\xi|_0^2} = \int_{N'} e^{\langle x, \xi \rangle} \mu(dx), \quad \xi \in X.$$

Assume that the Young function θ satisfies the following additional condition:

$$\limsup_{x \rightarrow +\infty} \frac{\theta(x)}{x^2} < +\infty. \tag{2.3}$$

Then, we obtain the following complex nuclear Gel'fand triple

$$\mathcal{F}_\theta(N') \hookrightarrow \mathcal{L}^2(X', \mu, \mathbb{C}) \hookrightarrow \mathcal{F}_\theta^*(N'). \tag{2.4}$$

We denote by $\mathcal{F}_\theta(N')_+$ the cone of positive test functions, i.e., $f \in \mathcal{F}_\theta(N')_+$ if $f(y+i0) \geq 0$ for all y in the topological dual N' of N . We denote by $\mathcal{F}_\theta^*(N')_+$ the set of all positive generalized function, i.e. $\Phi \in \mathcal{F}_\theta^*(N')_+$ if $\langle\langle \Phi, \varphi \rangle\rangle > 0, \quad \forall \varphi \in \mathcal{F}_\theta(N')_+$.

Theorem 2.3. [24] *Let $\Phi \in \mathcal{F}_\theta^*(N')_+$. Then there exists a unique Radon measure μ_Φ on N' such that*

$$\Phi(f) = \int_{N'} f(y+i0) d\mu_\Phi(y); f \in \mathcal{F}_\theta(N').$$

Conversely, let μ be a finite, positive Borel measure on N' . Then, μ represents a positive distribution in $\mathcal{F}_\theta^(N')_+$ if and only if μ is supported by some $X_{-p}, p \in \mathbb{N}^*$ and there exists some $m > 0$ such that*

$$\int_{X_{-p}} e^{\theta(m|y|_p)} d\mu(y) < \infty. \tag{2.5}$$

Now, we recall the following estimate, which represents the Chernoff type inequality. For a given $\xi \in X$ and $a \in \mathbb{R}$, let $A_{\xi,a} = \{y \in N' : \langle y, \xi \rangle > a\}$, denote the half-plane in X' associated to ξ and x .

Theorem 2.4. [23] *Let $\Phi \in \mathcal{F}_\theta^*(N')_+$ be such that Φ defines a positive Radon measure μ_Φ on X' . Then, for all $\xi \in X$ and $a > 0$, there exists $m > 0$ and $p \in \mathbb{N}$ such that*

$$\mu_\Phi(A_{\xi,a}) \leq \|\widehat{\Phi}\|_{\theta,m,p} \exp\left(-\theta\left(\frac{a}{m|\xi|_p}\right)\right). \tag{2.6}$$

2.2. Some generalities of large deviation theory. Let X' be the dual of a real nuclear Fréchet space X . A function $I : X' \rightarrow [0, +\infty]$ is said to be a good rate function, if it is lower semi-continuous and the sets $\{x \in X', I(x) \leq L\}$ are compact for all $L \geq 0$.

We say that a family $\{\mu_\epsilon, \epsilon > 0\}$ of Borel probability measures on the space X' , satisfies a large deviation principle (LDP) with good rate function I and normalization $b(\epsilon)$, if the following conditions are satisfied

- (1) (UPPER BOUND) For all closed subsets F in X' ,

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(F) \leq - \inf_{x \in F} I(x). \tag{2.7}$$

- (2) (LOWER BOUND) For all open subset G in X' ,

$$\liminf_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(G) \geq - \inf_{x \in G} I(x). \tag{2.8}$$

A family of probability measures $\{\mu_\epsilon, \epsilon > 0\}$ on X' is exponentially tight with normalization $b(\epsilon)$, if for every $L > 0$, there exists a compact set $K_L \subset X'$ such that

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(K_L^c) \leq -L,$$

where $K_L^c = X' \setminus K_L$

3. Large Deviation Principle for White Noise Distributions

We consider $\Phi \in \mathcal{F}_\theta^*(N')_+$ such that Φ defines a positive Radon measure $\mu = \mu_\Phi$ on X' . The Laplace transform $\widehat{\mu}$ of μ satisfies the growth condition:

$$\exists p, m, C > 0; \quad \widehat{\mu}(\xi) \leq C e^{\theta^*(m|\xi|_p)}, \xi \in X.$$

Let Λ_μ be the Logarithmic moment generating function, given by

$$\Lambda_\mu(\xi) := \log \left(\int_{X'} e^{\langle x, \xi \rangle} d\mu(x) \right), \quad \xi \in X.$$

We denote by Λ_μ^* the Legendre transform of Λ_μ ,

$$\Lambda_\mu^*(x) = \sup_{\xi \in X} \left\{ \langle x, \xi \rangle - \Lambda_\mu(\xi) \right\}, \quad x \in X'.$$

We consider the family $\{\mu_\epsilon, \epsilon > 0\}$, where μ_ϵ is the image measure of μ by the map:

$$\begin{aligned} g_\epsilon : X' &\longrightarrow X' \\ \lambda &\longmapsto a(\epsilon)\lambda, \end{aligned}$$

where a denotes some real positive valued function such that $\lim_{\epsilon \rightarrow 0} a(\epsilon) = 0$. Then, the logarithmic moment generating function for the measure μ_ϵ is given by

$$\Lambda_{\mu_\epsilon}(\xi) := \log \widehat{\mu}_\epsilon(\xi) = \log \widehat{\mu}(a(\epsilon)\xi).$$

We denote by

$$\Lambda(\xi) := \limsup_{\epsilon \rightarrow 0} b(\epsilon) \Lambda_{\mu_\epsilon} \left(\frac{\xi}{b(\epsilon)} \right), \tag{3.1}$$

where $b(\epsilon)$ is a real function such that $\lim_{\epsilon \rightarrow 0} b(\epsilon) = 0$, and the limit exists. The Legendre transform Λ^* of Λ is given by

$$\Lambda^*(x) = \sup_{\xi \in X} \{ \langle x, \xi \rangle - \Lambda(\xi) \}. \tag{3.2}$$

Proposition 3.1. (1) For any given positive real function $a(\epsilon), \epsilon \geq 0$ and for any function $b(\epsilon)$ such that $\lim_{\epsilon \rightarrow 0} b(\epsilon) = 0$, if we have

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \theta^* \left(\frac{a(\epsilon)}{b(\epsilon)} \right) < \infty, \tag{3.3}$$

then, for all $\xi \in X, \Lambda(\xi) < \infty$.

(2) The function Λ defined in (3.1) is convex on X and Λ^* is a convex rate function.

Proof. (1) We have

$$\begin{aligned} b(\epsilon) \Lambda_{\mu_\epsilon} \left(\frac{\xi}{b(\epsilon)} \right) &= b(\epsilon) \log \widehat{\mu} \left(\frac{a(\epsilon)}{b(\epsilon)} \xi \right) \\ &\leq b(\epsilon) \log(C) + b(\epsilon) \theta^* \left(\frac{a(\epsilon)}{b(\epsilon)} m |\xi|_p \right), \end{aligned}$$

which implies that for all $\xi \in X, \Lambda(\xi) < \infty$.

(2) It is easy to see that the function Λ defined in (3.1) is convex on X and Λ^* is a convex rate function. □

Now, we illustrate some relation between $a(\epsilon)$ and $b(\epsilon)$, such that the condition (3.3) is satisfied.

Example 3.2. (1) Let $\theta(x) = e^x - 1$ so $\theta^*(y) = y \log(y) - y + 1$. We have

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \theta^* \left(\frac{a(\epsilon)}{b(\epsilon)} \right) = \limsup_{\epsilon \rightarrow 0} a(\epsilon) \log \left(\frac{a(\epsilon)}{b(\epsilon)} \right) - a(\epsilon).$$

In particular, if $b(\epsilon) \underset{0}{\sim} a(\epsilon)$ then, we obtain that $\Lambda(\xi) < \infty$, for all $\xi \in X$.

(2) Let $\theta(x) = \frac{x^2}{2}$ so $\theta^*(y) = \frac{y^2}{2}$. We have

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \theta^* \left(\frac{a(\epsilon)}{b(\epsilon)} \right) = \limsup_{\epsilon \rightarrow 0} \frac{a^2(\epsilon)}{b(\epsilon)}.$$

In particular, if $b(\epsilon) \underset{0}{\sim} a^2(\epsilon)$ then, we obtain that $\Lambda(\xi) < \infty$, for all $\xi \in X$.

(3) Let $\theta(x) = \frac{x^p}{p}, p \geq 1$ and $\theta^*(y) = \frac{y^q}{q}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $p \in [1, 2]$ then $\lim_{x \rightarrow +\infty} \frac{\theta(x)}{x^2} = l < \infty$ and $q \in [2, +\infty[$. We have

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \theta^* \left(\frac{a(\epsilon)}{b(\epsilon)} \right) = \limsup_{\epsilon \rightarrow 0} a^q(\epsilon) b^{1-q}(\epsilon)$$

In particular, if $b(\epsilon) \underset{0}{\sim} a^p(\epsilon)$ then, we obtain that $\Lambda(\xi) < \infty$, for all $\xi \in X$.

(4) If $\lim_{x \rightarrow 0} \frac{a^2(x)}{b(x)} < \infty$ then the condition (3.3) is satisfied. In fact, under the condition $\lim_{x \rightarrow +\infty} \frac{\theta(x)}{x^2} = l < \infty$, for all $\epsilon > 0$ there exists $x_0 > 0$ such that, for $x \geq x_0$, we have $(l - \epsilon)x^2 \leq \theta(x)$. Then,

$$\theta^*(y) \leq \max \left\{ \frac{y^2}{4(l - \epsilon)}, x_0 y - \theta(x_0) \right\}.$$

Hence,

$$\lim_{x \rightarrow 0} b(x)\theta^*\left(\frac{a(x)}{b(x)}\right) \leq \lim_{x \rightarrow 0} \frac{a(x)^2}{4b(x)(l - \epsilon)}.$$

We conclude that if $\lim_{x \rightarrow 0} \frac{a(x)^2}{b(x)} < \infty$ then $\Lambda(\xi) < \infty$, for all $\xi \in X$.

3.1. Upper bound. In this subsection, we prove the upper bound inequality for the family of white noise distributions. We consider the family $\{\mu_\epsilon, \epsilon > 0\}$, where μ_ϵ defined in (3.1). The Laplace transform $\widehat{\mu}$ of μ satisfies the growth condition:

$$\exists p, m, C > 0; \quad \widehat{\mu}(\xi) \leq C e^{\theta^*(m|\xi|_p)}, \quad \xi \in X,$$

where θ^* is the Legendre transform of a Young function θ .

To prove the upper bound inequality for the family of measures $\{\mu_\epsilon, \epsilon > 0\}$, i.e., for all closed subset F of X' ,

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(F) \leq -\inf_F (\Lambda^*). \tag{3.4}$$

For this purpose, we need the following two technical lemmas.

Lemma 3.3. *For any compact set $\Gamma \subset X'$, we have*

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(\Gamma) \leq -\inf_{x \in \Gamma} \Lambda^*(x). \tag{3.5}$$

Proof. Let be $\Gamma \subset X'$ a compact set and $\delta > 0$. Let I^δ be the δ -rate function associated with Λ^* , i.e.,

$$I^\delta(x) := \min(\Lambda^*(x) - \delta, \frac{1}{\delta}).$$

Then, for any $\eta \in \Gamma$ there exists $\varphi_\eta \in X$, such that

$$\langle \eta, \varphi_\eta \rangle - \Lambda(\varphi_\eta) \geq I^\delta(\eta).$$

For a given $\eta \in X'$, let $B_q(\eta, r) = \{y \in X', |\eta - y|_{-q} < r\}$. We have

$$\begin{aligned}
\mu_\epsilon(\overline{B}_q(\eta, r)) &= \mu\left(\overline{B}_q\left(\frac{\eta}{a(\epsilon)}, \frac{r}{a(\epsilon)}\right)\right) \\
&= \int_{y \in \overline{B}_q\left(\frac{\eta}{a(\epsilon)}, \frac{r}{a(\epsilon)}\right)} d\mu(y) \\
&\leq \sup_{y \in \overline{B}_q\left(\frac{\eta}{a(\epsilon)}, \frac{r}{a(\epsilon)}\right)} \left[\exp - \left\langle y, \frac{a(\epsilon)}{b(\epsilon)} \varphi_\eta \right\rangle \right] \\
&\quad \times \int_{y \in \overline{B}_q\left(\frac{\eta}{a(\epsilon)}, \frac{r}{a(\epsilon)}\right)} \exp\left[\left\langle y, \frac{a(\epsilon)}{b(\epsilon)} \varphi_\eta \right\rangle\right] d\mu(y) \\
&= \sup_{y \in \overline{B}_q(\eta, r)} \left[\exp - \left\langle \frac{y}{b(\epsilon)}, \varphi_\eta \right\rangle \right] \\
&\quad \times \int_{y \in \overline{B}_q\left(\frac{\eta}{a(\epsilon)}, \frac{r}{a(\epsilon)}\right)} \exp\left[\left\langle y, \frac{a(\epsilon)}{b(\epsilon)} \varphi_\eta \right\rangle\right] d\mu(y) \\
&\leq \sup_{y \in \overline{B}_q(\eta, r)} \left[\exp\langle \eta - y, \frac{\varphi_\eta}{b(\epsilon)} \rangle \right] \exp\left(-\langle \eta, \frac{\varphi_\eta}{b(\epsilon)} \rangle\right) \\
&\quad \times \int_{y \in \overline{B}_q\left(\frac{\eta}{a(\epsilon)}, \frac{r}{a(\epsilon)}\right)} \exp\left[\left\langle y, \frac{a(\epsilon)}{b(\epsilon)} \varphi_\eta \right\rangle\right] d\mu(y) \\
&\leq \exp\left[\frac{1}{b(\epsilon)} r |\varphi_\eta|_q - \frac{1}{b(\epsilon)} \langle \eta, \varphi_\eta \rangle + \Lambda_{\mu_\epsilon}\left(\frac{\varphi_\eta}{b(\epsilon)}\right)\right].
\end{aligned}$$

This shows that

$$b(\epsilon) \log \mu_\epsilon(\overline{B}_q(\eta, r)) \leq r |\varphi_\eta|_q - \langle \eta, \varphi_\eta \rangle + b(\epsilon) \Lambda_{\mu_\epsilon}\left(\frac{\varphi_\eta}{b(\epsilon)}\right).$$

We choose $r \leq \frac{\delta}{1+|\varphi|_q}$, then we have

$$b(\epsilon) \log \mu_\epsilon(\overline{B}_q(\eta, r)) \leq \delta - \left[\langle \eta, \varphi_\eta \rangle - b(\epsilon) \Lambda_{\mu_\epsilon}\left(\frac{\varphi_\eta}{b(\epsilon)}\right) \right].$$

Since Γ is a compact set of X' , there exist $\eta_1, \eta_2, \dots, \eta_n \in \Gamma$, $r_1, r_2, \dots, r_n \in \mathbb{R}_+^*$ and $q_1, \dots, q_n \in \mathbb{N}$ such that $\Gamma \subset \bigcup_{k=1}^n \overline{B}_{q_k}(\eta_k, r_k)$. Then, we have

$$\begin{aligned}
\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(\Gamma) &\leq \limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \left(\sum_{k=1}^n \mu_\epsilon(\overline{B}_{q_k}(\eta_k, r_k)) \right) \\
&= \max_{k=1}^n \left(\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(\overline{B}_{q_k}(\eta_k, r_k)) \right) \\
&= \delta - \min_{k=1}^n \{ \langle \eta_k, \varphi_{\eta_k} \rangle - \Lambda(\varphi_{\eta_k}) \} \\
&\leq \delta - \min_{k=1}^n I^\delta(\eta_k).
\end{aligned}$$

Moreover, $\eta_k \in \Gamma$ for each k , yielding the inequality

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(\Gamma) \leq \delta - \inf_{\eta \in \Gamma} I^\delta(\eta).$$

The proof of the theorem is complete by taking $\delta \rightarrow 0$. □

From now on, we denote by

$$l(\xi) = \limsup_{\epsilon \rightarrow 0} b(\epsilon) \theta^* \left(\frac{a(\epsilon)}{b(\epsilon)} m | \xi |_p \right), \quad (3.6)$$

for all $\xi \in X$, such that the limit exists.

In the following Lemma, we will show that the family $\{\mu_\epsilon = a(\epsilon)\mu, \epsilon > 0\}$ is exponentially tight.

Lemma 3.4. *For each $L > 0$ and $\xi \in X$, the set*

$$K_L := \{y \in X'; | \langle y, \xi \rangle | \leq L + 2l(\xi)\} \quad (3.7)$$

is a compact set of X' and we have

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(K_L^c) \leq -L. \quad (3.8)$$

Proof. For all $\epsilon > 0$, we have

$$\begin{aligned} \mu_\epsilon(K_L^c) &= \mu_\epsilon \left(\left\{ y \in X'; \left| \langle y, \frac{\xi}{b(\epsilon)} \rangle \right| > \frac{L}{b(\epsilon)} + \frac{2l(\xi)}{b(\epsilon)} \right\} \right) \\ &\leq e^{-\frac{L}{b(\epsilon)} - \frac{2l(\xi)}{b(\epsilon)}} \int_{X'} e^{\langle y, \frac{\xi}{b(\epsilon)} \rangle} \mu_\epsilon(dy) + e^{-\frac{L}{b(\epsilon)} - \frac{2l(\xi)}{b(\epsilon)}} \int_{X'} e^{\langle y, -\frac{\xi}{b(\epsilon)} \rangle} \mu_\epsilon(dy). \end{aligned}$$

Thus,

$$b(\epsilon) \log \mu_\epsilon(K_L^c) \leq -L - 2l(\xi) + 2b(\epsilon) \theta^* \left(\frac{a(\epsilon)}{b(\epsilon)} m | \xi |_p \right).$$

Hence, by the condition (3.3), we obtain the inequality (3.8). \square

Lemmas (3.3) and (3.4) lead to the following results.

Theorem 3.5. *Let F be closed subset of X' . Then, we have*

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(F) \leq -\inf_F \Lambda^*. \quad (3.9)$$

Proof.

$$\begin{aligned} \limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(F) &\leq \limsup_{\epsilon \rightarrow 0} b(\epsilon) \log [\mu_\epsilon(F \cap K_L) + \mu_\epsilon(K_L^c)] \\ &\leq \max[-L, \limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(F \cap K_L)] \\ &\leq \max[-L, -\inf_{F \cap K_L} \Lambda^*] \\ &\leq \max[-L, -\inf_F \Lambda^*]. \end{aligned}$$

Finally, letting L increases to infinity, we obtain the upper bound (3.9). \square

3.2. Lower bound. In this section, we prove the lower bound inequality for the family of measures $\{\mu_\epsilon, \epsilon > 0\}$, i.e. for all open subsets G of X' ,

$$\liminf_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(G) \geq -\inf_{x \in G} \Lambda^*(x).$$

Let $G \subset X'$ be an open set, $y \in G$, $\xi \in X$ and $\delta > 0$. Consider the following subset of G defined by

$$P_{\delta, y, \xi} := \{z \in G, \langle z, \xi \rangle < \delta + \langle y, \xi \rangle\}.$$

Next, for all $\epsilon > 0$, we define as follows the measure $\tilde{\mu}_\epsilon$

$$\frac{d\tilde{\mu}_\epsilon}{d\mu_\epsilon}(z) = \exp \left[\left\langle z, \frac{\xi}{b(\epsilon)} \right\rangle - \Lambda_{\mu_\epsilon} \left(\frac{\xi}{b(\epsilon)} \right) \right]. \quad (3.10)$$

Then, the Logarithmic moment generating function of the measure $\tilde{\mu}_\epsilon$ is given by

$$b(\epsilon)\Lambda_{\tilde{\mu}_\epsilon} \left(\frac{\lambda}{b(\epsilon)} \right) = b(\epsilon)\Lambda_{\mu_\epsilon} \left(\frac{\lambda + \xi}{b(\epsilon)} \right) - b(\epsilon)\Lambda_{\mu_\epsilon} \left(\frac{\xi}{b(\epsilon)} \right), \quad \lambda \in X.$$

Hence, we have

$$\tilde{\Lambda}(\lambda) := \limsup_{\epsilon \rightarrow 0} b(\epsilon)\Lambda_{\tilde{\mu}_\epsilon} \left(\frac{\lambda}{b(\epsilon)} \right) = \Lambda(\lambda + \xi) - \Lambda(\xi).$$

Let $\tilde{\Lambda}^*$ denotes the Fenchel-Legendre transform of $\tilde{\Lambda}$. Then, for all $z \in X'$, we have

$$\tilde{\Lambda}^*(z) = \Lambda^*(z) + \Lambda(\xi) - \langle z, \xi \rangle, \quad \xi \in X.$$

Lemma 3.6. *The family of probability measures $\{\tilde{\mu}_\epsilon, \epsilon > 0\}$ on X' is exponentially tight with normalization $b(\epsilon)$. Then, for all $\delta > 0$ and all L large enough, $\xi \in X$,*

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \tilde{\mu}_\epsilon(K_L^c) < 0. \quad (3.11)$$

where K_L is defined by (3.7),

and

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \tilde{\mu}_\epsilon(P_{\delta, y, \xi}^c \cap K_L) < 0. \quad (3.12)$$

As a consequence, we have

$$\lim_{\delta \rightarrow 0} \liminf_{\epsilon \rightarrow 0} b(\epsilon) \log \tilde{\mu}_\epsilon(P_{\delta, y, \xi}) = 0. \quad (3.13)$$

Proof. For all $\epsilon > 0$,

$$\begin{aligned} \tilde{\mu}_\epsilon(K_L^c) &= \tilde{\mu}_\epsilon \left(\left\{ z \in X'; \left| \left\langle z, \frac{\xi}{b(\epsilon)} \right\rangle \right| > \frac{L}{b(\epsilon)} + \frac{2l(\xi)}{b(\epsilon)} \right\} \right) \\ &\leq e^{-\frac{L}{b(\epsilon)} - \frac{2l(\xi)}{b(\epsilon)}} \int_{X'} e^{\langle z, \frac{\xi}{b(\epsilon)} \rangle} \tilde{\mu}_\epsilon(dz) + e^{-\frac{L}{b(\epsilon)} - \frac{2l(\xi)}{b(\epsilon)}} \int_{X'} e^{\langle z, \frac{-\xi}{b(\epsilon)} \rangle} \tilde{\mu}_\epsilon(dy). \end{aligned}$$

Thus,

$$\begin{aligned} b(\epsilon) \log \tilde{\mu}_\epsilon(K_L^c) &\leq -L - 2l(\xi) + b(\epsilon)\Lambda_{\tilde{\mu}_\epsilon} \left(\frac{\xi}{b(\epsilon)} \right) + b(\epsilon)\Lambda_{\tilde{\mu}_\epsilon} \left(\frac{-\xi}{b(\epsilon)} \right) \\ &= -L - 2l(\xi) + b(\epsilon)\Lambda_{\mu_\epsilon} \left(\frac{\lambda + \xi}{b(\epsilon)} \right) - b(\epsilon)\Lambda_{\mu_\epsilon} \left(\frac{\xi}{b(\epsilon)} \right) \\ &\quad + b(\epsilon)\Lambda_{\mu_\epsilon} \left(\frac{-\lambda - \xi}{b(\epsilon)} \right) - b(\epsilon)\Lambda_{\mu_\epsilon} \left(\frac{-\xi}{b(\epsilon)} \right). \end{aligned}$$

Then,

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \tilde{\mu}_\epsilon(K_L^c) \leq -L$$

which proves that the family of probability measures $\{\tilde{\mu}_\epsilon, \epsilon > 0\}$ on X' is exponentially tight with normalization $b(\epsilon)$. Hence, for each $L > 0$ we obtain (3.11).

For any $L > 0$,

$$P_{\delta, y, \xi}^c \cap K_L = \{z \in X', \sup(-L - 2l(\xi), \delta + \langle y, \xi \rangle) \leq \langle z, \xi \rangle \leq L + 2l(\xi)\},$$

so we have

$$\begin{aligned}
 b(\epsilon) \log \tilde{\mu}_\epsilon(P_{\delta,y,\xi}^c \cap K_L) &= b(\epsilon) \log \left(\int_{P_{\delta,y,\xi}^c \cap K_L} \tilde{\mu}_\epsilon(dz) \right) \\
 &\leq b(\epsilon) \log \left[\int_{X'} e^{L\langle z, \frac{\xi}{b(\epsilon)} \rangle} \tilde{\mu}_\epsilon(dz) \right] - L\delta - L\langle y, \xi \rangle \\
 &= b(\epsilon) \Lambda_{\tilde{\mu}_\epsilon} \left(\frac{L\xi}{b(\epsilon)} \right) - L\delta - L\langle y, \xi \rangle.
 \end{aligned}$$

Hence,

$$\limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \tilde{\mu}_\epsilon(P_{\delta,y,\xi}^c \cap K_L) \leq \tilde{\Lambda}(L\xi) - L\delta - L\langle y, \xi \rangle.$$

Then, for each L large enough, we obtain (3.12).

Next, we prove the inequality (3.13).

$$\tilde{\mu}_\epsilon(P_{\delta,y,\xi}^c) \leq \tilde{\mu}_\epsilon(P_{\delta,y,\xi}^c \cap K_L) + \tilde{\mu}_\epsilon(K_L^c).$$

Thus,

$$\begin{aligned}
 \limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \tilde{\mu}_\epsilon(P_{\delta,y,\xi}^c) &\leq \limsup_{\epsilon \rightarrow 0} b(\epsilon) \log \left[\tilde{\mu}_\epsilon(P_{\delta,y,\xi}^c \cap K_L) + \tilde{\mu}_\epsilon(K_L^c) \right] \\
 &< 0.
 \end{aligned}$$

This inequality implies that

$$\tilde{\mu}_\epsilon(P_{\delta,y,\xi}) \xrightarrow{\epsilon \rightarrow 0} 1 \text{ for all } \delta > 0. \text{ As a consequence,}$$

$$\lim_{\delta \rightarrow 0} \liminf_{\epsilon \rightarrow 0} b(\epsilon) \log \tilde{\mu}_\epsilon(P_{\delta,y,\xi}) = 0.$$

□

Thanks to Lemma 3.6 we conclude the next theorem.

Theorem 3.7. *For every open subset set $G \subset X'$,*

$$\liminf_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(G) \geq - \inf_{x \in G} \Lambda^*(x). \quad (3.14)$$

Proof. By using equality (3.10), we obtain

$$\begin{aligned}
 b(\epsilon) \log \mu_\epsilon(P_{\delta,y,\xi}) &= b(\epsilon) \Lambda_{\mu_\epsilon} \left(\frac{\xi}{b(\epsilon)} \right) - \langle y, \xi \rangle \\
 &\quad + b(\epsilon) \log \int_{P_{\delta,y,\xi}} \exp \left[\left\langle y - z, \frac{\xi}{b(\epsilon)} \right\rangle \right] d\tilde{\mu}_\epsilon(z) \\
 &\geq b(\epsilon) \Lambda_{\mu_\epsilon} \left(\frac{\xi}{b(\epsilon)} \right) - \langle y, \xi \rangle - \delta + b(\epsilon) \log \tilde{\mu}_\epsilon(P_{\delta,y,\xi}).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \liminf_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(G) &\geq \lim_{\delta \rightarrow 0} \liminf_{\epsilon \rightarrow 0} b(\epsilon) \log \mu_\epsilon(P_{\delta,y,\xi}) \\
 &\geq \Lambda(\xi) - \langle y, \xi \rangle + \lim_{\delta \rightarrow 0} \liminf_{\epsilon \rightarrow 0} b(\epsilon) \log \tilde{\mu}_\epsilon(P_{\delta,y,\xi}) \\
 &\geq -\Lambda^*(y) + \lim_{\delta \rightarrow 0} \liminf_{\epsilon \rightarrow 0} b(\epsilon) \log \tilde{\mu}_\epsilon(P_{\delta,y,\xi}). \quad (3.15)
 \end{aligned}$$

Hence, by equation (3.13) Theorem 3.7 follows. □

4. Examples

4.1. Generalized Gross heat equation perturbed by the space-time white noise. It is well known that in infinite dimensional complex analysis, the convolution operator on a general function space is defined as a continuous operator which commutes with the translation operator.

Let us define the convolution $\Phi * \varphi$ of a distribution $\Phi \in \mathcal{F}_\theta^*(N')$ and a test function $\varphi \in \mathcal{F}_\theta(N')$ to be the function

$$(\Phi * \varphi)(x) = \ll \Phi, t_{-x}\varphi \gg, \quad x \in N', \quad (4.1)$$

where $t_x\varphi$ denotes the translation operator, i.e.,

$$t_x\varphi(y) = \varphi(y - x), \quad y \in N'.$$

Note that $\Phi * \varphi \in \mathcal{F}_\theta(N')$ for any $\varphi \in \mathcal{F}_\theta(N')$ and the convolution product is given in terms of the dual pairing as $(\Phi * \varphi)(0) = \ll \Phi, \varphi \gg$ for any $\Phi \in \mathcal{F}_\theta^*(N')$ and $\varphi \in \mathcal{F}_\theta(N')$. We can generalize the above convolution product for generalized functions as follows.

Let $\Phi, \Psi \in \mathcal{F}_\theta^*(N')$ be given, then the convolution product $\Phi * \varphi$ is defined by

$$\ll \Phi * \Psi, \varphi \gg = \ll \Phi, \Psi * \varphi \gg, \quad \forall \varphi \in \mathcal{F}_\theta(N'). \quad (4.2)$$

The Gross Laplacian Δ_G is given by

$$\Delta_G \varphi(x) = \sum_{n \geq 0} (n+2)(n+1) \langle x^{\otimes n}, \langle \tau, \varphi^{(n+2)} \rangle \rangle, \quad x \in N' \quad (4.3)$$

for $\varphi \in \mathcal{F}_\theta(N')$ such that $\varphi(x) = \sum_{n \geq 0} \langle x^{\otimes n}, \varphi^{(n)} \rangle$ and τ is the trace operator defined by $\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle$, for $\xi, \eta \in N$. In fact, the Gross Laplacian Δ_G is a convolution operator given by

$$\Delta_G(\Psi) = \mathcal{T} * \Psi, \quad \Psi \in \mathcal{F}_\theta^*(N'), \quad (4.4)$$

where \mathcal{T} is a distribution in $\mathcal{F}_\theta^*(N')$ whose Laplace transform is given by:

$$\widehat{\mathcal{T}}(z) = \langle \tau, z^{\otimes 2} \rangle.$$

For $t > 0$, define $\tilde{\gamma}_t(\cdot) = \gamma(\cdot/\sqrt{t})$, where γ is the standard Gaussian measure on the space X' . Then the probability γ_t induces a positive distribution $\tilde{\gamma}_t$ in $\mathcal{F}_\theta^*(N')$ given by

$$\ll \tilde{\gamma}_t, \varphi \gg = \int_{X'} \varphi(x) d\tilde{\gamma}_t(x) = \int_{X'} \varphi(\sqrt{t}x) d\gamma(x), \quad \forall \varphi \in \mathcal{F}_\theta(N'). \quad (4.5)$$

For details, see the book [21].

The space-time white noise defined by

$$Z(t, x) = \langle \cdot, \delta_t \otimes \tilde{\delta}_x \rangle, \quad t \in \mathbb{R}, x \in X',$$

where δ_t is the Dirac delta function at t and $\tilde{\delta}_x$ is the Kubo-Yokoi delta function at x (see the book [21].) Equivalently, the space-time white noise $Z(t, x)$ is given by

$$Z(t, x)(\varphi) = \varphi(t, x), \quad \varphi \in \mathcal{F}_\theta(\mathbb{C} \times N'),$$

where $\mathcal{F}_\theta(\mathbb{C} \times N')$ is defined in Remark (2.1).

In view of the canonical topological isomorphism $\mathcal{F}_\theta(\mathbb{C} \times N') \simeq \mathcal{F}_\theta(\mathbb{C}) \widehat{\otimes} \mathcal{F}_\theta(N')$, for each fixed $t \in \mathbb{R}$ and $x \in X'$, with $\widehat{\otimes}$ is the symmetric tensor product. We have

$$\ll Z(t, x), g \otimes \psi \gg = g(t)\psi(x), \quad g \in \mathcal{F}_\theta(\mathbb{C}), \psi \in \mathcal{F}_\theta(N'). \tag{4.6}$$

Note that for a fixed $g \in \mathcal{F}_\theta(\mathbb{C})$, the action of $Z(t, x)$ on ψ in Equation (4.6) is a generalized function in $\mathcal{F}_\theta^*(N')$. For the generalized Gross heat equation perturbed by the space-time white noise $Z(t, x)$, we have the following theorem.

Theorem 4.1. [5] *Let θ be a Young function such that $\lim_{r \rightarrow +\infty} \frac{\theta(r)}{r^2} < \infty$. Let $F \in \mathcal{F}_\theta^*(N')_+$. Then the generalized Gross heat equation perturbed by the space-time white noise $Z(t, x)$*

$$\frac{\partial U_t}{\partial t} = \frac{1}{2} \Delta_G U_t + \alpha Z(t, \cdot), \quad U_0 = F, \alpha \in \mathbb{R}_+, \tag{4.7}$$

has a unique solution in $\mathcal{F}_\theta^*(N')_+$ given by

$$U_t = F * \tilde{\gamma}_t + \alpha \int_0^t \tilde{\gamma}_{t-s} * Z(s, \cdot) ds. \tag{4.8}$$

Moreover, let μ_{U_t} be the associated measure with the solution of the equation (4.7), and its laplace transform is given by the following equation,

$$\widehat{\mu}_{U_t}(\xi) = \widehat{F}(\xi) e^{\frac{1}{2}t|\xi|_0^2} + \frac{2\alpha}{|\xi|_0^2} e^{\frac{1}{2}t|\xi|_0^2} e^{|\xi|_0^2} \left(1 - e^{-\frac{1}{2}t|\xi|_0^2} \right). \tag{4.9}$$

As a consequence the Laplace transform of μ_ϵ is the image measure of μ_{U_t} by the map:

$$\begin{aligned} g_\epsilon : X' &\longrightarrow X' \\ \lambda &\longmapsto a(\epsilon)\lambda, \end{aligned}$$

is given by,

$$\widehat{\mu}_\epsilon(\xi) = \widehat{F}(a(\epsilon)\xi) e^{\frac{1}{2}ta^2(\epsilon)|\xi|_0^2} + \frac{2\alpha}{a^2(\epsilon)|\xi|_0^2} e^{\frac{1}{2}ta^2(\epsilon)|\xi|_0^2} e^{a^2(\epsilon)|\xi|_0^2} \left(1 - e^{-\frac{1}{2}ta^2(\epsilon)|\xi|_0^2} \right). \tag{4.10}$$

Furthermore, the The logarithmic moment generating function for the measure μ_ϵ is given by

$$\Lambda_{\mu_\epsilon}(\xi) = \log(\widehat{\mu}_\epsilon(\xi)) = \frac{1}{2}ta^2(\epsilon)|\xi|_0^2 + \log \left(\widehat{F}(a(\epsilon)\xi) + \frac{2\alpha}{a^2(\epsilon)|\xi|_0^2} (1 - e^{-\frac{1}{2}ta^2(\epsilon)|\xi|_0^2}) \right).$$

In the following, we give some examples of α and F , and we prove large deviation principle for a specified family of measures $\{\mu_\epsilon, \epsilon > 0\}$.

- (1) Let, μ_{U_t} be the associated measure with the solution of the equation (4.7) for the particular case where $\alpha = 0$, the Young function θ is given by: $\theta(x) = \frac{x^2}{2}$ and F is a the standard Gaussian distribution on $S'(\mathbb{R})$. Then the family of measures $\{\mu_\epsilon, \epsilon > 0\}$ of image measure of the measure μ_{U_t} under the map $\xi \mapsto a(\epsilon)\xi$, satisfies the full large deviation principle with normalization $b(\epsilon) = la^2(\epsilon)$, $l \neq 0$ and with rate function Λ^* given by :

$$\Lambda^*(y) = \begin{cases} \frac{1}{2l(1+l)}|\xi|_0^2, & \text{if } \xi \in \mathcal{L}^2(\mathbb{R}, dx), \\ +\infty, & \text{if } \xi \in S'(\mathbb{R}) \setminus \mathcal{L}^2(\mathbb{R}, dx). \end{cases} \tag{4.11}$$

Here is the proof. If $\alpha = 0$, the Young function θ is given by : $\theta(x) = \frac{x^2}{2}$ and F is the standard Gaussian distribution on $S'(\mathbb{R})$, then the solution of (4.7) is a positive generalized function and given by the explicit formulae:

$$U_t = \tilde{\gamma}_{t+1} \in \mathcal{F}_\theta^*(S'(\mathbb{R}))_+.$$

So Theorem 2.3 guarantees the existence and uniqueness of a Radon measure μ_{U_t} on $S'(\mathbb{R})$ associated to U_t such that :

$$\widehat{\mu}_{U_t}(\xi) = e^{\frac{1}{2}(1+t)|\xi|_0^2}.$$

The logarithmic moment generating function for the measure μ_ϵ is given by

$$\Lambda_{\mu_\epsilon}(\xi) = \log(\widehat{\mu}_\epsilon(\xi)) = \theta^*(a(\epsilon)|\xi|_0) = \frac{1}{2}(1+t)a(\epsilon)^2|\xi|_0^2, \forall \xi \in S(\mathbb{R}).$$

For $l \neq 0$, we take $b(\epsilon) = la^2(\epsilon)$.

Then

$$\Lambda(\xi) = \lim_{\epsilon \rightarrow 0} b(\epsilon) \Lambda_{\mu_\epsilon} \left(\frac{\xi}{b(\epsilon)} \right) = \frac{1}{2}(1+t)l|\xi|_0^2 < \infty, \quad \forall \xi \in S(\mathbb{R}).$$

And its Legendre transform Λ^* is given by

$$\Lambda^*(y) = \begin{cases} \frac{1}{2l(1+t)}|\xi|_0^2, & \text{if } \xi \in \mathcal{L}^2(\mathbb{R}, dx), \\ +\infty, & \text{if } \xi \in S'(\mathbb{R}) \setminus \mathcal{L}^2(\mathbb{R}, dx). \end{cases} \quad (4.12)$$

Therefore, our desired result.

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