

CONVERGENCE OF SCHEME FOR DECOUPLED FORWARD BACKWARD STOCHASTIC DIFFERENTIAL EQUATION

HANI ABIDI AND HABIB OUERDIANE

ABSTRACT. We study the convergence rate of Bouchard-Touzi-Zhang scheme (in short B-T-Z) for Decoupled Forward Backward Stochastic Differential Equation, this convergence is controlled by the stability of truncation error and the Markovian property of its processes. Then, we present the algorithm used and provide some numerical results. Finally, we give a fundamental stability property for the reflected Backward Stochastic Differential Equation in a Markovian framework.

1. Introduction

We are interested by the discrete time approximation of a decoupled forward backward stochastic differential equation (FBSDE), which has a solution given by the triplet (X, Y, Z) satisfying:

$$\begin{aligned} X_t &= X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s \\ Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s \end{aligned} \quad (1.1)$$

where $b : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are Lipschitz-continuous functions, $g : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable with continuous and bounded first derivative, T is a given positive constant and W is a Brownian motion.

The BSDE's study was performed by Pardoux and Peng [10] who proved the existence and the uniqueness of BSDE solution under the Lipschitz continuity assumption. Many BSDE don't have an explicit solution, however it's approximated by many schemes, such as the four steps scheme resolved by Ma Protter and al. [6]. More recently, other authors developed some schemes to discretize the BSDE like [1] [7].

Then, El Karoui and al. [5] introduced the reflected BSDE (RBSDE in short) with one continuous lower barrier $L = (L_t)$. More precisely, a solution of such equation, associated with a terminal value ξ , is a triplet $(Y_t, Z_t, A_t)_{0 \leq t \leq T}$ of

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adapted processes valued in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$, satisfying:

$$Y_t = \xi + \int_t^T f(Y_s, Z_s) ds + A_T - A_t - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T \text{ a.s.} \quad (1.2)$$

and $Y_t \geq L_t$ a.s. For any $0 \leq t \leq T$, A_t is non-decreasing continuous process. The role of A_t is to push upward the process Y in a minimal way, in order to keep it above $L = (L_t)$. In this way it satisfies $\int_0^T (Y_s - L_s) dA_s = 0$. The authors in [5] have proved that the equation (1.2) has a unique solution when ξ is square integrable, f is uniformly Lipschitz with respect to (y, z) and $L = (L_t)$ is a continuous process.

The aim of our study is to reformulate the convergence rate of B-T-Z scheme using another method namely the tree method combining weak Taylor scheme. Chassagneux and D. Crisan [3] performed this kind of method for autonomous functions used in (1.1), i.e. $f(t, x, y, z) = f(y, z)$, $b(s, x) = b(x)$ and $\sigma(s, x) = \sigma(x)$.

The advantage of this method is: firstly, it does not allow Monte Carlo error because the approximation of the conditional expectation is based on the forward approximation of X by the tree method. Secondly, not like Malliavin method [1] and the regression method [7], this require to estimate only a small number of conditional expectations at each step. Finally, it is more simple to implement.

The outline of our paper is presented below. Section 2 is devoted to define the *Itô* Taylor expansion announced by Kloeden and Platen [8]. In section 3, we study the convergence of B-T-Z scheme for FBSDE which is based on a fundamental stability property, an *Itô* Taylor expansion and the approximation of Brownian motion with backward random walk in a discrete time-grid $0 = t_0 < t_1 < \dots < t_n = T, n \in \mathbb{N}$. In Section 4, we give some estimates to prove the stability property of reflected backward stochastic differential equation by using the Penalization approximation.

2. Preliminaries and Notations

In the sequel, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $(W_t)_{t \geq 0}$ is a 1-dimensional Brownian motion defined on a fixed interval $[0, T]$, with a fixed $T > 0$.

2.0.1. Notations. For using later, We denote by:

- $\{\mathcal{F}\}_{0 \leq t \leq T}$ the natural filtration generated by the Brownian motion W .
- C_b : set of bounded process.
- $L^2(\mathcal{F}_t)$ the space of \mathcal{F}_t -meas. random variable ξ such that $\mathbf{E}[|\xi|^2] < \infty$.
- \mathcal{S}^2 the space of prog. meas process Y such that $E[\sup_{0 \leq t \leq T} |Y_t|^2] < \infty$.
- \mathcal{H}^2 the space of prog. meas process Z such that $\left(E \int_0^T |Z_s|^2 ds\right)^{\frac{1}{2}} < \infty$.

Now, our effort concentrates on the following FBSDE:

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s. \end{cases} \quad (2.1)$$

We relate our FBSDE (2.1) to the parabolic differential equation

$$\begin{cases} L^0 u(t, x) + f(t, x, u(t, x), L^1(u)(t, x)) = 0 \\ u(T, x) = g(x) \end{cases} \quad (2.2)$$

where $u : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ and

$$L^0 = \frac{\partial}{\partial t} + b \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} ; L^1 = \sigma \frac{\partial}{\partial x}. \quad (2.3)$$

Theorem 2.1. *If $u \in C^{1,2}([0, T] \times \mathbb{R})$ solves (2.2), then $u(t, X_t) = Y_t$ and $Z_t = \sigma(t, X_t) \frac{\partial}{\partial x}(t, X_t)$ where $\{(Y_t, Z_t)_{0 \leq t \leq T}\}$ is the unique solution of the BSDE (2.1).*

Proof. By applying Itô formula to $u(t, X_t)$, we have:

$$du(t, X_t) = L^0 u(s, X_s) ds + L^1 u(s, X_s) dW_s \quad (2.4)$$

Since u solves (2.2), it follows that

$$\begin{aligned} -du(t, X_t) &= f(t, X_t, u(t, X_t), \sigma(t, X_t) \partial_x u(t, X_t)) dt \\ &\quad - \sigma(t, X_t) \partial_x u(t, X_t) dW_s. \end{aligned}$$

with $u(T, X_T) = g(X_T)$. Thus $\{u(t, X_t), \sigma(t, X_t) \partial_x u(t, X_t), t \in [0, T]\}$ is equal to the unique solution of BSDE (2.1), and the result is obtained. \square

2.0.2. Approximation of backward stochastic differential equation. [1] Let X be the solution on $[0, T]$ of the stochastic differential equation:

$$X_t = X_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s \quad (2.5)$$

where $b : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are assumed to be C-Lipschitz i.e

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| < C|x - y|.$$

In order to approximate the above FBSDE (2.1), we introduce first the approximation of X_t . Let $\pi : \{0 = t_0 < t_1 < \dots < t_n = T\}$ be a partition of the interval $[0, T]$. Throughout this paper, we shall use the grid notations:

$$h_i = |t_i - t_{i-1}|, |\pi| = \max_{0 \leq i \leq n} h_i, \text{ and } \Delta W_i = W_{t_i} - W_{t_{i-1}}.$$

The forward component X will be approximated by the classical Euler scheme :

$$\begin{aligned} \widehat{X}_{t_0} &= X_{t_0} \\ \widehat{X}_{t_i} &= \widehat{X}_{t_{i-1}} + b(t_{i-1}, \widehat{X}_{t_{i-1}}) h_i + \sigma(t_{i-1}, \widehat{X}_{t_{i-1}}) \Delta W_i. \end{aligned} \quad (2.6)$$

for $i=1, \dots, n$. Under the Lipschitz conditions on b and σ , the following L^2 estimate for the error due to the Euler scheme (2.6):

$$\lim_{|\pi| \rightarrow 0} \sup |\pi|^{-\frac{1}{2}} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X_t - \widehat{X}_t|^2 + \max_{1 \leq i \leq n} \sup_{t_{i-1} \leq t \leq t_i} |X_t - X_{t_{i-1}}|^2 \right]^{\frac{1}{2}} < \infty, \quad (2.7)$$

see [8].

We shall denote by $\{\mathcal{F}_{t_i}\}_{0 \leq i \leq n}$ is the associated discrete time filtration:

$$\mathcal{F}_{t_i} = \sigma(W_{t_j}, j \leq i)$$

and $E[\cdot|\mathcal{F}_{t_i}] = \mathbf{E}_{t_i}[\cdot]$ the related conditional expectation. Now we define the discrete time approximation of FBSDE (2.1). We need to introduce a continuous-time approximation of (Y, Z) .

$$\begin{aligned} Y_{t_i} &= Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(s, X_s, Y_s, Z_s) ds + \int_{t_i}^{t_{i+1}} Z_s dW_s \\ &= \mathbf{E}_{t_i}[Y_{t_{i+1}} + \int_{t_i}^{t_{i+1}} f(s, X_s, Y_s, Z_s) ds] \end{aligned}$$

Then we define the explicit scheme

$$\bar{Y}_{t_i} = \mathbf{E}_{t_i}[Y_{t_{i+1}} + h_i f(t_{i+1}, X_{t_{i+1}}, Y_{t_{i+1}}, Z_{t_{i+1}})].$$

To obtain an approximation to Z , we observe that

$$\int_{t_i}^{t_{i+1}} Z_s dW_s = Y_{t_{i+1}} - Y_{t_i} + \int_{t_i}^{t_{i+1}} f(s, X_s, Y_s, Z_s) ds$$

So we define

$$\bar{Z}_{t_i} = \mathbf{E}_{t_i}[H^i Y_{t_{i+1}}]$$

where $H^i = \frac{1}{h_i}(W_{t_{i+1}} - W_{t_i})$.

Consider now the following natural explicit discrete-time approximation (Y_i, Z_i) of (Y_{t_i}, Z_{t_i}) , for $i:1\dots n$, [1]:

$$\begin{aligned} (Y_n, Z_n) &= (u(t_n, X_{t_n}), L^1 u(t_n, X_{t_n})) \\ Y_i &= \mathbf{E}_{t_i}[Y_{i+1} + h_i f(t_i, X_{i+1}, Y_{i+1}, Z_{i+1})] \\ Z_i &= \mathbf{E}_{t_i}[H^i Y_{i+1}]. \end{aligned} \tag{2.8}$$

2.0.3. Truncation error. The global errors that we need to control here are

$$\varepsilon(Y, \pi) = \sup_{0 \leq i \leq n} \mathbf{E}[|Y_{t_i} - Y_i|^2] \text{ and } \varepsilon(Z, \pi) = \sum_{i=0}^n h_i \mathbf{E}[|Z_{t_i} - Z_i|^2]$$

To control these errors we will use the local truncation error for the pair (Y, Z) defined as

$$\eta_i = \eta_i^Y + \eta_i^Z$$

where

$$\eta_i^Y = \frac{1}{h_i^2} |Y_{t_i} - \bar{Y}_{t_i}|^2 \text{ and } \eta_i^Z = |Z_{t_i} - \bar{Z}_{t_i}|^2$$

and

$$\begin{aligned} \bar{Y}_{t_i} &= \mathbf{E}_{t_i}[Y_{t_{i+1}} + h_i f(t_i, X_{t_{i+1}}, Y_{t_{i+1}}, Z_{t_{i+1}})] \\ \bar{Z}_{t_i} &= \mathbf{E}_{t_i}[H^i Y_{t_{i+1}}]. \end{aligned} \tag{2.9}$$

The global truncation error for a given grid π is given by:

$$\begin{aligned} \Gamma(\pi) &= \Gamma_Y(\pi) + \Gamma_Z(\pi) \\ \Gamma_Y(\pi) &= \sum_{i=0}^{n-1} h_i \eta_i^Y \text{ and } \Gamma_Z(\pi) = \sum_{i=0}^{n-1} h_i \eta_i^Z. \end{aligned}$$

3. Forward-Backward Stochastic Differential Equation

3.1. Itô Taylor expansion. This section discusses the stochastic version of Taylor expansion to understand how the stochastic integration methods are obtained. In the next part, we will use some assumptions like:

(\mathbf{H}_r) : the coefficient b, σ in (1.1) belongs to $C_b^{3,6}$ and f belongs to $C_b^{3,6,6,6}$ and the value function u belongs to $C_b^{3,6}$.

First, we recall how we can obtain the stochastic version of the Taylor expansion: Let X_t be the solution of this EDS:

$$\begin{cases} dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \\ X(t_0) = X_0 \end{cases} \quad (3.1)$$

which has the following integral form:

$$X_t = X_0 + \int_{t_0}^t b(s, X_s)ds + \int_{t_0}^t \sigma(s, X_s)dW_s \quad \forall t \in [t_0, T] \quad (3.2)$$

Lemma 3.1. For $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \in C^{3,6}(\mathbb{R})$, we have:

$$\begin{aligned} u(t, X_t) &= u(t_0, X_{t_0}) + L^0 u(t_0, X_{t_0})(t - t_0) + L^1 u(t_0, X_{t_0})(W_t - W_{t_0}) \\ &\quad + L^0 L^1 u(t_0, X_{t_0}) \int_{t_0}^t \int_{t_0}^s dz dW_s + L^1 L^1 u(t_0, X_{t_0}) \int_{t_0}^t \int_{t_0}^s dW_z dW_s \\ &\quad + L^0 L^0 u(t_0, X_{t_0}) \int_{t_0}^t \int_{t_0}^s dz ds + L^1 L^0 u(t_0, X_{t_0}) \int_{t_0}^t \int_{t_0}^s dW_z ds + R \end{aligned}$$

where

$$\begin{aligned} R &= \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^0 L^0 L^0 u(z, X_z) dr dz ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^0 L^0 L^0 u(z, X_z) dW_r dz ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^0 L^1 L^0 u(z, X_z) dr dW_z ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^1 L^1 L^0 u(z, X_z) dW_r dW_z ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^0 L^0 L^1 u(z, X_z) dr dz dW_s \\ &\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^1 L^0 L^1 u(z, X_z) dW_r dz dW_s \\ &\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^0 L^1 L^1 u(z, X_z) dr dW_z dW_s \\ &\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^1 L^1 L^1 u(z, X_z) dW_r dW_z dW_s. \end{aligned}$$

Proof. Using the Itô formula, we have for $u : [t_0, T] \times \mathbb{R} \rightarrow \mathbb{R} \in C^{3,6}(\mathbb{R})$ and $t_0 \leq t \leq T$:

$$\begin{aligned}
u(t, X_t) &= u(t_0, X_{t_0}) + \int_{t_0}^t \left(\frac{\partial}{\partial t} u(s, X_s) + b(s, X_s) \frac{\partial}{\partial x} u(s, X_s) \right. \\
&\quad \left. + \frac{1}{2} \sigma^2(s, X_s) \frac{\partial^2}{\partial x^2} u(s, X_s) \right) ds \\
&\quad + \int_{t_0}^t \sigma(s, X_s) \frac{\partial}{\partial x} u(s, X_s) dW_s \\
&= u(t_0, X_{t_0}) + \int_{t_0}^t L^0(u(s, X_s)) ds + \int_{t_0}^t L^1(u(s, X_s)) dW_s.
\end{aligned}$$

Applying again the Itô formula to $L^0 u$ and $L^1 u$, we obtain:

$$\begin{aligned}
u(t, X_t) &= u(t_0, X_{t_0}) + \int_{t_0}^t \left(L^0 u(t_0, X_{t_0}) \right. \\
&\quad \left. + \int_{t_0}^s L^0 L^0(u(z, X_z)) dz + \int_{t_0}^s L^1 L^0(u(z, X_z)) dW_z \right) ds \\
&\quad + \int_{t_0}^t \left(L^1 u(t_0, X_{t_0}) + \int_{t_0}^s L^0 L^1(u(z, X_z)) dz \right. \\
&\quad \left. + \int_{t_0}^s L^1 L^1(u(z, X_z)) dW_z \right) dW_s \\
&= u(t_0, X_{t_0}) + L^0 u(t_0, X_{t_0})(t - t_0) + L^1 u(t_0, X_{t_0})(W_t - W_{t_0}) \\
&\quad + L^0 L^1 u(t_0, X_{t_0}) \int_{t_0}^t \int_{t_0}^s dz dW_s + L^1 L^1 u(t_0, X_{t_0}) \int_{t_0}^t \int_{t_0}^s dW_z dW_s \\
&\quad + L^0 L^0 u(t_0, X_{t_0}) \int_{t_0}^t \int_{t_0}^s dz ds + L^1 L^0 u(t_0, X_{t_0}) \int_{t_0}^t \int_{t_0}^s dW_z ds + R
\end{aligned}$$

where

$$\begin{aligned}
R &= \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^0 L^0 L^0 u(z, X_z) dr dz ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^1 L^0 L^0 u(z, X_z) dW_r dz ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^0 L^1 L^0 u(z, X_z) dr dW_z ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^1 L^1 L^0 u(z, X_z) dW_r dW_z ds \\
&\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^0 L^0 L^1 u(z, X_z) dr dz dW_s \\
&\quad + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^1 L^0 L^1 u(z, X_z) dW_r dz dW_s
\end{aligned}$$

$$\begin{aligned}
& + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^0 L^1 L^1 u(z, X_z) dr dW_z dW_s \\
& + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^1 L^1 L^1 u(z, X_z) dW_r dW_z dW_s.
\end{aligned}$$

□

Remark 3.2. The notation $R = O(h^p)$, for $p \geq 1$ means that $|R| \leq \lambda_t h^p$ where λ_t is a positive random variable satisfying

$$\mathbf{E}|\lambda_t^p| \leq C_p,$$

for all $p > 0$.

3.2. General convergence results. The aim of this paper is to compute the global numerical error which is the sum of the discrete time approximation error controlled by the rest integral of Itô Taylor expansion given by Theorem 3.7 and the numerical error induced by the approximation of the conditional expectations given in Theorem 3.9. The numerical scheme presented above are still theoretical. We first establish a stability property the scheme.

3.2.1. Stability of the scheme. For this rate of convergence, we are going to introduce the truncation error between the B-T-Z scheme and the perturbed scheme given by:

$$\tilde{Y}_i = \mathbf{E}_{t_i}[\tilde{Y}_{i+1} + h_i f(t_i, \tilde{X}_i, \tilde{Y}_{i+1}, \tilde{Z}_{i+1})] + \xi_i^Y \quad (3.3)$$

$$\tilde{Z}_i = \mathbf{E}_{t_i}[H^i \tilde{Y}_{i+1}] + \xi_i^Z. \quad (3.4)$$

where ξ_i^Y and ξ_i^Z are the two perturbed errors belongs in $L^2(\mathcal{F}_{t_i})$ for all $i \leq n$.

Definition 3.3. (L^2 -stability) The scheme given in (3.3) is said to be L^2 -stable if

$$\begin{aligned}
\max_i \mathbf{E}[|\delta Y_i|^2] + \sum_{i=0}^{n-1} h_i \mathbf{E}[|\delta Z_i|^2] & \leq C \left(\mathbf{E}[|\pi|^2 + |\delta Y_n|^2 + |\pi| |\delta Z_n|^2] \right. \\
& \left. + \sum_{i=0}^{n-1} h_i \mathbf{E}\left[\frac{1}{h_i^2} |\xi_i^Y|^2 + |\xi_i^Z|^2 \right] \right),
\end{aligned}$$

where $\delta X_i = X_i - \hat{X}_i$, $\delta Y_i = Y_i - \tilde{Y}_i$ and $\delta Z_i = Z_i - \tilde{Z}_i$, for any sequences ξ_i^Y , ξ_i^Z of $L^2(\mathcal{F}_{t_i})$ -random variable and terminal values (Y_n, Z_n) , $(\tilde{Y}_n, \tilde{Z}_n)$ belonging to $\mathcal{S}^2 \times \mathcal{H}^2$.

Theorem 3.4. *If f is a Lipschitz-continuous, then the scheme is L^2 -stable in the meaning of the definition 3.3, for $|\pi|$ small enough.*

Proof. In the following, $C > 0$ will denote a generic constant may take different values from line to line. Recalling (2.8), by the Cauchy-Schwartz inequality, we

compute, for $1 \geq \eta \geq 0$ to be fixed later on that:

$$\begin{aligned} |\delta Y_i|^2 &\leq \left(1 + \frac{h_i}{\eta}\right) (\mathbf{E}_{t_i} |\delta Y_{i+1}|)^2 + Ch_i^2 \left(1 + \frac{\eta}{h_i}\right) (\mathbf{E}_{t_i} (|\delta X_i|^2 + |\delta Y_{i+1}|^2 \\ &\quad + |\delta Z_{i+1}|^2) + C \frac{\eta}{h_i} |\xi_i^Y|^2). \end{aligned} \quad (3.5)$$

$$|\delta Z_i|^2 \leq C \left[\frac{1}{h_i} (\mathbf{E}_{t_i} (|\delta Y_{i+1}|^2) - (\mathbf{E}_{t_i} |\delta Y_{i+1}|)^2) + |\xi_i^Z|^2 \right]. \quad (3.6)$$

For $\epsilon > 0$, small enough, we get

$$\begin{aligned} |\delta Y_i|^2 + \epsilon h_i |\delta Z_i|^2 &\leq \left(1 + \frac{h_i}{\eta} - C\epsilon\right) (\mathbf{E}_{t_i} |\delta Y_{i+1}|)^2 \\ &\quad + \left[Ch_i^2 \left(1 + \frac{\eta}{h_i}\right) + C\epsilon \right] \mathbf{E}_{t_i} (|\delta Y_{i+1}|^2) \\ &\quad + Ch_i^2 \left(1 + \frac{\eta}{h_i}\right) \mathbf{E}_{t_i} (|\delta X_i|^2) + C \frac{\eta}{h_i} |\xi_i^Y|^2 + Ch_i |\xi_i^Z|^2 \\ &\quad + Ch_i^2 \left(1 + \frac{\eta}{h_i}\right) \mathbf{E}_{t_i} (|\delta Z_{i+1}|^2). \end{aligned}$$

we may use Jensen's inequality to $(\mathbf{E}_{t_i} |\delta Y_{i+1}|)^2$, we obtain:

$$\begin{aligned} |\delta Y_i|^2 + \epsilon h_i |\delta Z_i|^2 &\leq \left(1 + \frac{h_i}{\eta} + Ch_i^2 \left(1 + \frac{\eta}{h_i}\right)\right) (\mathbf{E}_{t_i} |\delta Y_{i+1}|^2) \\ &\quad + Ch_i^2 \left(1 + \frac{\eta}{h_i}\right) \mathbf{E}_{t_i} (|\delta Z_{i+1}|^2) \\ &\quad + Ch_i^2 \left(1 + \frac{\eta}{h_i}\right) \mathbf{E}_{t_i} (|\delta X_i|^2) + C \frac{\eta}{h_i} |\xi_i^Y|^2 + Ch_i |\xi_i^Z|^2. \end{aligned} \quad (3.7)$$

By (2.7) and applying the expectation,

$$\begin{aligned} \mathbf{E} |\delta Y_i|^2 + \epsilon h_i \mathbf{E} |\delta Z_i|^2 &\leq \left(1 + \frac{h_i}{\eta} + Ch_i^2 \left(1 + \frac{\eta}{h_i}\right)\right) (\mathbf{E} |\delta Y_{i+1}|^2) \\ &\quad + Ch_i^2 \left(1 + \frac{\eta}{h_i}\right) \mathbf{E} (|\delta Z_{i+1}|^2) \\ &\quad + Ch_i^3 \left(1 + \frac{\eta}{h_i}\right) + Ch_i \mathbf{E} \left[\frac{\eta}{h_i^2} |\xi_i^Y|^2 + |\xi_i^Z|^2 \right]. \end{aligned}$$

For h_i small enough, we can compute that:

$$\begin{aligned} \mathbf{E} |\delta Y_i|^2 + \epsilon h_i \mathbf{E} |\delta Z_i|^2 &\leq (1 + Ch_i) \left[\mathbf{E} |\delta Y_{i+1}|^2 + \epsilon h_i \mathbf{E} (|\delta Z_{i+1}|^2) \right] \\ &\quad + Ch_i^3 \left(1 + \frac{\eta}{h_i}\right) + Ch_i \mathbf{E} \left[\frac{\eta}{h_i^2} |\xi_i^Y|^2 + |\xi_i^Z|^2 \right]. \end{aligned}$$

Using the discrete version of Gronwall's Lemma, we obtain

$$\sup_{1 \leq i \leq n} \mathbf{E} |\delta Y_i|^2 \leq C \left(|\pi|^2 + \mathbf{E} [|\delta Y_n|^2 + |\pi| |\delta Z_n|^2] + \sum_{i=1}^n h_i \mathbf{E} \left(\frac{1}{h_i^2} |\xi_i^Y|^2 + |\xi_i^Z|^2 \right) \right)$$

for small π . This concludes the proof for the Y-part. For the Z-part, the proof is concluded by summing over i in (3.7). \square

Lemma 3.5. *We assume that H_r holds and $L^0L^0u, L^1L^0u, L^0L^1u$ have polynomial growth with respect to X , we get:*

$$\begin{aligned}
i) \mathbf{E}_{t_i} [H^i (\int_{t_i}^{t_{i+1}} \int_{t_i}^s L^0L^0u(z, X_z) dz ds)] &= O(|\pi|^{\frac{3}{2}}). \\
ii) \mathbf{E}_{t_i} [H^i (\int_{t_i}^{t_{i+1}} \int_{t_i}^s L^0L^1u(z, X_z) dz dW_s)] &= O(|\pi|). \\
iii) \mathbf{E}_{t_i} [H^i (\int_{t_i}^{t_{i+1}} \int_{t_i}^s L^1L^0u(z, X_z) dW_z ds)] &= O(|\pi|). \\
iv) \mathbf{E}_{t_i} [H^i (\int_{t_i}^{t_{i+1}} \int_{t_i}^s L^1L^1u(z, X_z) dW_z dW_s)] &= 0.
\end{aligned}$$

Proof. Observing that $H^i = \frac{1}{h_i} \int_{t_i}^{t_{i+1}} dW_s$. By using the Cauchy Schwartz inequality and lemma 5.7.5 in [8], we get

$$\begin{aligned}
&\mathbf{E}_{t_i} [H_i \int_{t_i}^{t_{i+1}} \int_{t_i}^s L^0L^0u(z, X_z) dz ds] \\
&\leq h_i^{-\frac{1}{2}} \left(\mathbf{E}_{t_i} [\int_{t_i}^{t_{i+1}} \int_{t_i}^s (L^0L^0u(z, X_z))^2 dz ds] \right)^{\frac{1}{2}} = O(|\pi|^{\frac{3}{2}}).
\end{aligned}$$

Let we turn to ii) and iii), we recall that L^1L^0u, L^0L^1u have polynomial growth, by lemma 5.7.2 [8], we have:

$$\begin{aligned}
&\mathbf{E}_{t_i} [H^i (\int_{t_i}^{t_{i+1}} \int_{t_i}^s L^0L^1u(z, X_z) dz dW_s + \int_{t_i}^{t_{i+1}} \int_{t_i}^s L^1L^0u(z, X_z) dW_z ds)] \\
&= \frac{1}{h_i} \mathbf{E}_{t_i} [\int_{t_i}^{t_{i+1}} \int_{t_i}^s L^0L^1u(z, X_z) dz ds + \int_{t_i}^{t_{i+1}} \int_{t_i}^s L^1L^0u(z, X_z) dz ds] \\
&= O(|\pi|).
\end{aligned}$$

Finally, by lemma 5.7.1 in [8], we obtain:

$$\begin{aligned}
&\mathbf{E}_{t_i} [H^i (\int_{t_i}^{t_{i+1}} \int_{t_i}^s L^1L^1u(z, X_z) dW_z dW_s)] \\
&= \frac{1}{h_i} \mathbf{E}_{t_i} [(\int_{t_i}^{t_{i+1}} \int_{t_i}^s L^1L^1u(z, X_z) dW_z ds)] = 0.
\end{aligned}$$

□

We are now ready to state our first result, which provides an error estimate of the approximation scheme

Theorem 3.6. *Under (H_r) , we get:*

$$\Gamma(\pi) = O(|\pi|^2). \quad (3.8)$$

Proof. We recall that:

$$\begin{aligned}\bar{Y}_{t_i} &= \mathbf{E}_{t_i}[Y_{t_{i+1}} + h_i f(t_i, X_{i+1}, Y_{t_{i+1}}, Z_{t_{i+1}})] \\ \bar{Z}_{t_i} &= \mathbf{E}_{t_i}[H^i Y_{t_{i+1}}].\end{aligned}$$

For the part of Z , we have

$$\begin{aligned}\bar{Z}_{t_i} &= \mathbf{E}_{t_i}[H^i Y_{t_{i+1}}] \\ &= \mathbf{E}_{t_i}[H^i u(t_{i+1}, X_{t_{i+1}})] \\ &= \mathbf{E}_{t_i}\left[H^i \left(u(t_i, X_{t_i}) + h_i L^0 u(t_i, X_{t_i}) + L^1 u(t_i, X_{t_i})(W_{t_{i+1}} - W_{t_i}) + R_i\right)\right] \\ &= L^{(1)}u(t_i, X_{t_i}) + \mathbf{E}_{t_i}\left[H^i \left(\int_{t_i}^{t_{i+1}} \int_{t_i}^s L^0 L^0 u(z, X_z) dz ds \right. \right. \\ &\quad \left. \left. + \int_{t_i}^{t_{i+1}} \int_{t_i}^s L^1 L^0 u(z, X_z) dW_z ds + \int_{t_i}^{t_{i+1}} \int_{t_i}^s L^0 L^1 u(z, X_z) dz dW_s \right. \right. \\ &\quad \left. \left. + \int_{t_i}^{t_{i+1}} \int_{t_i}^s L^1 L^1 u(z, X_z) dW_z dW_s\right)\right].\end{aligned}$$

By lemma 3.5, we get

$$\bar{Z}_{t_i} = Z_{t_i} + O_{t_i}(|\pi|).$$

Indeed, by the truncation error definition for the Z component, we have:

$$\Gamma_Z(\pi) = O(|\pi|^2).$$

Now the error for the Y -part is:

$$\begin{aligned}\bar{Y}_{t_i} &= \mathbf{E}_{t_i}\left[Y_{t_{i+1}} + h_i (f(t_{i+1}, X_{t_{i+1}}, Y_{t_{i+1}}, Z_{t_{i+1}}))\right] \\ &= \mathbf{E}_{t_i}\left[u(t_{i+1}, X_{t_{i+1}}) - h_i L^0 u(t_{i+1}, X_{t_{i+1}})\right] \\ &= Y_{t_i} + O(|\pi|^2).\end{aligned}$$

So

$$\Gamma_Y(\pi) = O(|\pi|^2).$$

□

Theorem 3.7. *Under hypothesis H_r :*

$$\varepsilon(Y, \pi) + \varepsilon(Z, \pi) \leq C|\pi|^2$$

where C is a positive random variable.

Proof. Now, we consider that (Y, Z) is a solution of perturbed scheme, setting

$$\xi_i^Y = Y_{t_i} - \bar{Y}_{t_i} \text{ and } \xi_i^Z = Z_{t_i} - \bar{Z}_{t_i}.$$

By Theorem 3.6, we get thus

$$\sum_{i=0}^{n-1} h_i \mathbf{E}\left[\frac{1}{h_i^2} |\xi_i^Y|^2 + |\xi_i^Z|^2\right] \leq C|\pi|^2.$$

So we can conclude by stability error given by theorem 3.4, that:

$$\varepsilon(Y, \pi) + \varepsilon(Z, \pi) \leq C|\pi|^2.$$

□

3.3. Error of implementable scheme for FBSDE. In this section, we are interested to the error due to the numerical illustration of the result presented above. Firstly, by the approximation of multiple $It\hat{o}$ integral [8] p 225, let we consider a weak Taylor approximation of X denoted by \widehat{X} and defined by:

$$\begin{aligned}\widehat{X}_{i+1} &= \widehat{X}_i + L^0(\widehat{X}_i)h_i + L^1(\widehat{X}_i)\Delta\widehat{W}_{t_i} \\ &+ \frac{1}{2}L^1L^1(\widehat{X}_i)\left((\Delta\widehat{W}_{t_i})^2 - h_i\right) + \frac{1}{2}L^0L^0(\widehat{X}_i)h_i^2 \\ &+ \frac{1}{2}[L^1L^0 + L^0L^1](\widehat{X}_i)h_i\Delta\widehat{W}_{t_i}.\end{aligned}\quad (3.9)$$

We denote by $(\widehat{\mathcal{F}}_t)$ the filtration generated by (\widehat{X}_t) and $\widehat{\mathbf{E}}_t[\cdot]$ the related conditional expectation. The empirical scheme we use in practice is the following:

$$\begin{aligned}(\widehat{Y}_n, \widehat{Z}_n) &= (u(t_n, \widehat{X}_{t_n}), L^1u(t_n, \widehat{X}_{t_n})) \\ \widehat{Y}_i &= \widehat{\mathbf{E}}_{t_i}[\widehat{Y}_{i+1} + h_i f(t_i, \widehat{X}_{i+1}, \widehat{Y}_{i+1}, \widehat{Z}_{i+1})] \\ \widehat{Z}_i &= \widehat{\mathbf{E}}_{t_i}[\widehat{H}^i \widehat{Y}_{i+1}].\end{aligned}$$

where $\widehat{H}^i = \frac{\Delta\widehat{W}_{t_i}}{h_i}$.

Let now give a lemma to illustrate our convergence result.

Lemma 3.8. *For u smooth enough*

$$i) \mathbf{E}_{t_i}[u(t_{i+1}, \widehat{X}_{t_{i+1}})] = u(t_i, \widehat{X}_{t_i}) + h_i L^0 u(t_i, \widehat{X}_{t_i}) + O(h_i^2). \quad (3.10)$$

$$ii) \mathbf{E}_{t_i}[H^i u(t_{i+1}, \widehat{X}_{t_{i+1}})] = L^1 u(t_i, \widehat{X}_{t_i}) + O(h_i). \quad (3.11)$$

Proof. For ii) it is the same proof like Theorem 3.6 in a discrete time. Now we pass to prove i), For v smooth enough and by lemma 5.7.1 in [8], we get:

$$\mathbf{E}_{t_i}[u(t_{i+1}, X_{t_{i+1}})] = u(t_i, X_{t_i}) + L^0 v(t_i, X_{t_i})h_i + L^0 L^0 u(t_i, X_{t_i})\frac{h_i^2}{2} + O(h_i^3).$$

□

Theorem 3.9. *If we assume (H_r) is verified and \widehat{X} is given by 3.9, then*

$$Y_0 - \widehat{Y}_0 = O(h_i^2).$$

Proof. Let us define $\widetilde{Y}_i = u(t_i, \widehat{X}_{t_i})$, $\widetilde{Z}_i = L^1 u(t_i, \widehat{X}_{t_i})$ and

$$\begin{aligned}\check{Y}_i &= \widehat{\mathbf{E}}_{t_i}[\widetilde{Y}_{i+1} + h_i f(t_{i+1}, \widehat{X}_{i+1}, \widetilde{Y}_{i+1}, \widetilde{Z}_{i+1})] \\ \check{Z}_i &= \widehat{\mathbf{E}}_{t_i}[\widehat{H}^i \widetilde{Y}_{i+1}].\end{aligned}$$

Analog to the proof of theorem 3.6, by using lemma 3.8, we compute that:

For the Y Part,

$$\begin{aligned}\check{Y}_{t_i} &= \widehat{\mathbf{E}}_{t_i}\left[u(t_{i+1}, \widehat{X}_{t_{i+1}}) - h_i L^0 u(t_{i+1}, \widehat{X}_{t_{i+1}})\right] \\ &= \widetilde{Y}_i + O(|\pi|^2).\end{aligned}$$

For the Z part,

$$\begin{aligned}\check{Z}_{t_i} &= \widehat{\mathbf{E}}_{t_i}[H^i Y_{t_{i+1}}] \\ &= \widehat{\mathbf{E}}_{t_i}[H^i u(t_{i+1}, X_{t_{i+1}})] \\ &= \check{Z}_i + O(|\pi|).\end{aligned}$$

If $(\widehat{Y}, \widehat{Z})$ is the solution of perturbed scheme (\check{Y}, \check{Z}) it follows that $\xi^Y = \widehat{Y} - \check{Y}$ and $\xi^Z = \widehat{Z} - \check{Z}$, and by proposition 3.4

$$Y_0 - \widehat{Y}_0 = O(|\pi|^2).$$

□

Now, we explicit the description of tree method combining weak Taylor scheme for the forward process with binomial approximation of Brownian motion.

Firstly, we describe how to obtain the tree associated to \widehat{X} , then we introduce a binomial approximation of brownian motion, given by, for $i \leq n-1$,

$$P(\widehat{W}_{t_{i+1}} - \widehat{W}_{t_i} = \sqrt{h_i}) = P(\widehat{W}_{t_{i+1}} - \widehat{W}_{t_i} = -\sqrt{h_i}) = \frac{1}{2}.$$

From the definition of $\Delta\widehat{W}_{t_i}$, we see that each node of \widehat{X}_i will give two new nodes denoted $\widehat{X}_{i+1}^{\widehat{X}_i, \sqrt{h}}$, $\widehat{X}_{i+1}^{\widehat{X}_i, -\sqrt{h}}$ depending on the value of the value taken by $\Delta\widehat{W}_{t_i}$. Secondly, we turn to the approximation of backward component, which is computed along the tree built for \widehat{X}_i . We observe that the scheme can be written

$$\begin{aligned}(\widehat{Y}_n, \widehat{Z}_n) &= (u(t_n, \widehat{X}_{t_n}), L^1 u(t_n, \widehat{X}_{t_n})) \\ \widehat{Y}_i &= \widehat{\mathbf{E}}_{t_i}[\widehat{Y}_{i+1} + h_i f(t_i, \widehat{X}_{i+1}, \widehat{Y}_{i+1}, \widehat{Z}_{i+1})](\widehat{X}_i) \\ \widehat{Z}_i &= \widehat{\mathbf{E}}_{t_i}[\widehat{H}^i \widehat{Y}_{i+1}](\widehat{X}_i).\end{aligned}$$

The above quantitie can be computed for each node \widehat{X}_i using the quadrature rule:

$$\widehat{\mathbf{E}}_{t_i}[\varphi(\widehat{X}_{i+1})](\widehat{X}_i) = \frac{1}{2}[\varphi(\widehat{X}_{i+1}^{\widehat{X}_i, \sqrt{h}})] + \frac{1}{2}[\varphi(\widehat{X}_{i+1}^{\widehat{X}_i, -\sqrt{h}})].$$

Finally, we need to compute $\widehat{\mathbf{E}}_{t_i}[\widehat{Y}_{i+1}]$, $\widehat{\mathbf{E}}_{t_i}[f(t_i, \widehat{X}_{i+1}, \widehat{Y}_{i+1}, \widehat{Z}_{i+1})]$ and $\widehat{\mathbf{E}}_{t_i}[\widehat{H}^i \widehat{Y}_{i+1}]$ using the same quadrature rule.

4. Implementable Scheme for FBSDE

To illustrate our previous results, we will focus on the simple case where $X = W$. So this step is based on the following BSDE

$$Y_t = g(W_T) + \int_t^T f(s, W_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$

For $i < n$, the BTZ scheme can be approximated numerically by:

$$\begin{cases} Y_i(x) = \mathbb{E}_{t_i}[Y_{i+1}(W_{t_{i+1}}^{t_i, x})] + h_i f(t_i, x, Y_i(x), Z_i(x)) \\ Z_i(x) = \mathbb{E}_{t_i}[Y_{i+1}(W_{t_{i+1}}^{t_i, x}) \frac{W_{t_{i+1}}^{t_i, x} - x}{h_i}] \end{cases} \quad (4.1)$$

where $W_{t_{i+1}}^{t_i, x}$ is a Brownian motion started by x in t_i .

For this convergence method, we approximate the Brownian motion by a random walk. The Donsker Theorem gives an approximation for Brownian motion as follows, choosing a large integer $k > 1$,

$$W_k = \frac{1}{\sqrt{k}} \sum_{i=1}^{tk} \xi^i$$

and ξ is a random walk.

In our paper, we are interested by the ξ^i defined by:

$$\mathbb{P}(\xi^1 = 1) = \mathbb{P}(\xi^2 = -1) = \frac{1}{2}.$$

We denote by \widehat{W} the approximation of W , and we define recursively :

$$\widehat{W}_{t_{i+1}} - \widehat{W}_{t_i} = \sqrt{h} \xi^i.$$

with $\widehat{W}_0 = 0$.

In the following, we describe how to obtain a reasonable scheme for (Y, Z) which is given by Chassagneux and Dan Crisan [3] for the Runge Kutta scheme. The convergence analysis is based on the stability of the scheme and some kinds of truncation error. We are particularly interested by the initial value of Y , which is deterministic but we will need to measure the error at each date to obtain the error at the beginning. So for the Y , a sensible choice is

$$\varepsilon(Y, \pi) = \sup_{0 \leq i \leq n} \mathbf{E}[|Y_{t_i} - \widehat{Y}_i|^2].$$

The error for the process Z here is given by this following structure

$$\varepsilon(Z, \pi) = \mathbf{E}\left[\int_0^T |Z_{t_i} - \widehat{Z}_i|^2 ds\right].$$

Notations :

First, we need to consider 'functional' version of the schemes above. Let us introduce the following operator, related to the theoretical schemes

$$R_i^Z[\varphi^Y, \varphi^Z](x) = \mathbf{E}[H^i \varphi^Y(W_{t_{i+1}}^{t_i, x})]$$

$$R_i^Y[\varphi^Y, \varphi^Z](x) = \mathbf{E}[\varphi(W_{t_{i+1}}^{t_i, x}) + h_i f(t_{i+1}, W_{t_{i+1}}^{t_i, x}, \varphi^Y(W_{t_{i+1}}^{t_i, x}), \varphi^Z(W_{t_{i+1}}^{t_i, x}))].$$

Similarly, let us define for the fully discrete scheme-operators

$$\widehat{R}_i^Z[\varphi^Y, \varphi^Z](x) = \widehat{\mathbf{E}}[\widehat{H}^i \varphi^Y(\widehat{W}_{t_{i+1}}^{t_i, x})]$$

$$\widehat{R}_i^Y[\varphi^Y, \varphi^Z](x) = \widehat{\mathbf{E}}[\varphi(\widehat{W}_{t_{i+1}}^{t_i, x}) + h_i f(t_{i+1}, \widehat{W}_{t_{i+1}}^{t_i, x}, \varphi^Y(\widehat{W}_{t_{i+1}}^{t_i, x}), \varphi^Z(\widehat{W}_{t_{i+1}}^{t_i, x}))].$$

Using the B-T-Z scheme, we give the functional version as follows:

$$\begin{cases} \widehat{y}_i(x) = R_i^Y[\widehat{y}_{i+1}, \widehat{z}_{i+1}](x) \\ \widehat{z}_i(x) = R_i^Z[\widehat{y}_{i+1}, \widehat{z}_{i+1}](x) \end{cases}$$

given initial data $(\widehat{y}_n, \widehat{z}_n) = (u(t_n, \cdot), \partial_x u(t_n, \cdot))$. Due to the Markov property of the discrete process $(\widehat{W}_t^{0, x})$, it is easily checked that

$$\widehat{Y}_i = \widehat{y}_i(\widehat{W}_t^{0, x}) \text{ and } \widehat{Z}_i = \widehat{z}_i(\widehat{W}_t^{0, x}).$$

Finally, we define:

$$\tilde{Y}_i = u(t_i, \widehat{W}_{t_i}^{0,x}) \text{ and } \tilde{Z}_i = \partial_x u(t_i, \widehat{W}_{t_i}^{0,x}).$$

Observe that $\tilde{Y}_0 = u(0, x)$ and that, $(\widehat{Y}_n, \widehat{Z}_n) = (\tilde{Y}_n, \tilde{Z}_n)$.

Stability :

The key observation is that $(\tilde{Y}_i, \tilde{Z}_i)$ can be seen as a perturbed version

$$\begin{cases} \tilde{Y}_i = \mathbf{E}_{t_i}[\tilde{Y}_{i+1} + h_i f(t_{i+1}, X_{i+1}, \tilde{Y}_{i+1}, \tilde{Z})] + {}^s \xi_i^Y \\ \tilde{Z}_i = \mathbf{E}_{t_i}[H^i \tilde{Y}_{i+1}] + {}^s \xi_i^Z \end{cases}$$

where the local error due to the 'space-discretization' is:

$$\begin{cases} {}^s \xi_i^Y = (R_i^Y - \widehat{R}_i^Y)[u(t_{i+1}, \cdot), \partial_x u(t_{i+1}, \cdot)](\widehat{W}_{t_i}^{0,x}) \\ {}^s \xi_i^Z = (R_i^Z - \widehat{R}_i^Z)[u(t_{i+1}, \partial_x u(t_{i+1}, \cdot)](\widehat{W}_{t_i}^{0,x}) \end{cases}$$

Now, we can conclude the error with computing the next Theorem.

Theorem 4.1. *Under (\mathbf{H}_r) , If f is a Lipshitz function, then*

$$E[|\delta Y_0|^2] + E[|\delta Z_0|^2] \leq c|\pi|^2. \quad (4.2)$$

where $\delta Y_0 = Y_0 - \tilde{Y}_0$ and $\delta Z_0 = Z_0 - \tilde{Z}_0$.

Proof. In a first time, we are interested by the convergence of Y . This proof use the Markovian property of Brownian motion for changing the conditional expectation considered difficult to approximate . Then, we approximate the truncation error using the *Itô*-Taylor expansion. So we can define the local error ${}^s \xi_i^Y$:

$$\begin{aligned} {}^s \xi_i^Y &= R_i^Y - \widehat{R}_i^Y[\varphi^Y, \varphi^Z](x) = \mathbf{E}[\varphi^Y(W_{t_{i+1}}^{t_i,x})] - \mathbf{E}[\varphi^Y(\widehat{W}_{t_{i+1}}^{t_i,x})] \\ &\quad + h_i(\mathbf{E}[f(t_{i+1}, W_{t_{i+1}}^{t_i,x}, \varphi^Y(W_{t_{i+1}}^{t_i,x}), \varphi^Z(W_{t_{i+1}}^{t_i,x}))]) \\ &\quad - \mathbf{E}[f(t_{i+1}, \widehat{W}_{t_{i+1}}^{t_i,x}, \varphi^Y(\widehat{W}_{t_{i+1}}^{t_i,x}), \varphi^Z(\widehat{W}_{t_{i+1}}^{t_i,x}))]). \end{aligned}$$

Let $\Delta \widehat{W}_{t_i} = \widehat{W}_{t_{i+1}} - \widehat{W}_{t_i}$, from the Taylor expansion, we have:

$$\begin{aligned} \varphi^Y(\widehat{W}_{t_{i+1}}^{t_i,x}) &= \varphi^Y(x + \sqrt{h_i} \xi^i) \\ &= \varphi^Y(x) + (\varphi^Y)'(x) \sqrt{h_i} \xi^i + \frac{1}{2}(\varphi^Y)^{(2)}(x) h_i (\xi^i)^2 \\ &\quad + \frac{1}{3!}(\varphi^Y)^{(3)}(x) h_i^{\frac{3}{2}} (\xi^i)^3 + \frac{1}{4!}(\varphi^Y)^{(4)}(x) h_i^2 (\xi^i)^4 + O(h_i^4). \end{aligned}$$

So, we get

$$\mathbf{E}[\varphi^Y(\widehat{W}_{t_{i+1}}^{t_i,x})] = \varphi^Y(x) + \frac{1}{2}(\varphi^Y)^{(2)}(x) h_i + \frac{1}{4!}(\varphi^Y)^{(4)}(x) h_i^2 + O(h_i^3).$$

Let $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$, using the *Itô*-Taylor expansion, we have:

$$\begin{aligned} \mathbf{E}[\varphi^Y(W_{t_{i+1}}^{t_i,x})] &= \mathbf{E}[\varphi^Y(W_{t_i}^{t_i,x} + \Delta W_{t_i})] \\ &= \mathbf{E}\left[\varphi^Y(x) + L^{(0)}\varphi^Y(x)h_i + L^{(1)}\varphi^Z(x)\Delta W_{t_i} + L^{(0)}L^{(0)}\varphi^Y(x)\frac{h_i^2}{2} \right. \\ &\quad \left. + L^{(1)}L^{(1)}\varphi^Y(x)\int_{t_i}^{t_{i+1}}\int_{t_i}^s dW_z dW_s + L^{(1)}L^{(0)}\varphi^Y(x)\int_{t_i}^{t_{i+1}}\int_{t_i}^s dz dW_s \right. \\ &\quad \left. + L^{(0)}L^{(1)}\varphi^Y(x)\int_{t_i}^{t_{i+1}}\int_{t_i}^s dW_z ds\right]. \end{aligned}$$

Then, we obtain

$$\mathbf{E}[\varphi^Y(W_{t_{i+1}}^{t_i,x}) - \varphi^Y(\widehat{W}_{t_{i+1}}^{t_i,x})] = O(h_i^2).$$

Under the lipschitz assumption of f , we remark that:

$$(R_i^Y - \widehat{R}_i^Y)[\varphi^Y, \varphi^Z] = O(h_i^2).$$

Using the matching moment property of $W^{t_i,x}$, we easily obtain that

$$\mathbf{E}\left[|R_i^Y[\varphi^Y, \varphi^Z] - \widehat{R}_i^Y[\varphi^Y, \varphi^Z]|^2(W_{t_i}^{0,x})\right] \leq C|\pi|^4.$$

Now, by using the binary random walk to approximate the Brownian motion, we can see that:

$$\mathbf{E}\left[\varphi^Y(\widehat{W}_{t_{i+1}}^{t_i,x})\frac{\widehat{W}_{t_{i+1}}^{t_i,x} - x}{h_i}\right] = (\varphi^Y)'(x) + \frac{1}{3!}(\varphi^Y)^{(3)}(x)h_i + O(h_i^2).$$

On the other hand:

$$\begin{aligned} \mathbf{E}\left[\varphi^Y(W_{t_{i+1}}^{t_i,x})\frac{\Delta W_{t_i}}{h_i}\right] &= \mathbf{E}\left[\varphi^Y(W_{t_i}^{t_i,x} + \Delta W_{t_i})\frac{\Delta W_{t_i}}{h_i}\right] \\ &= \mathbf{E}\left[\left(\varphi^Y(x) + L^{(0)}\varphi^Y(x)h_i + L^{(1)}\varphi^Y(x)\Delta W_{t_i} \right. \right. \\ &\quad \left. \left. + L^{(0)}L^{(0)}\varphi^Y(x)\frac{h_i^2}{2} + L^{(1)}L^{(1)}\varphi^Y(x)\int_{t_0}^t\int_{t_0}^s dW_z dW_s \right. \right. \\ &\quad \left. \left. + L^{(1)}L^{(0)}\varphi^Y(x)\int_{t_0}^t\int_{t_0}^s dW_z ds + L^{(0)}L^{(1)}\varphi^Y(x)\int_{t_0}^t\int_{t_0}^s dz dW_s\right)\frac{\Delta W_{t_i}}{h_i}\right]. \end{aligned}$$

We get the following result:

$$|{}^s\xi_i^Z| = O(h_i).$$

Since f is a Lipschitz-function, then the B-T-Z scheme (3.3) is L^2 -stable. So, we obtain:

$$\begin{aligned} \mathbf{E}[|\delta Y_0|^2] &\leq \left(\sum_{i=0}^{n-1} h_i \mathbf{E}\left[\frac{1}{h_i^2}(|{}^s\xi_i^Y|^2) + |{}^s\xi_i^Z|^2\right]\right) \\ &= O(h_i^2). \end{aligned}$$

□

Remark 4.2. We remark that if

$$X_t = \int_0^t a(s, X_s) ds + \int_0^t b(s, X_s) dW_s \quad (4.3)$$

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s.$$

Under some assumptions, we can use the Girsanov Theorem [8] and the martingal representation to change X_t given in (4.3) to

$$dX_t = c(X_t) d\widetilde{W}_t$$

where \widetilde{W}_t is a Brownian motion respect to $\widetilde{\mathbb{P}}$ and $c \in L^2(\mathcal{F}_t)$. After that, we can use the same idea to get the convergence of scheme.

4.1. Example of numerical simulation: In this subsection, we focus on the problem of the simulation of the B-T-Z scheme approximation. This algorithm is based on the approximation of Brownian motion by a binary random walk. Then we approximate the conditional expectation of Euler scheme in a backward direction. We consider the process

$$(X_t, Y_t, Z_t) = (W_t, \frac{1}{1 + \exp(-W_t - \frac{t}{4})}, \frac{\exp(-W_t - \frac{t}{4})}{(1 + \exp(-W_t - \frac{t}{4}))^2}).$$

This process is a solution of the (decoupled) FBSDE

$$X_t = W_t \quad (4.4)$$

$$Y_t = g(X_T) + \int_t^T f(Y_s, Z_s) ds - \int_t^T Z_s dW_s \quad (4.5)$$

where the driver f is given by

$$f(y) = -z(\frac{3}{4} - y)$$

and

$$g_T(x) = \frac{1}{1 + \exp(-x - \frac{T}{4})}.$$

Let $h = \frac{1}{n}$, this process Y and Z defined in (4.5) are approximated as follows:

$$\begin{cases} y_i(x) = \mathbb{E}_{t_i} [y_{i+1}(W_{t_{i+1}}^{t_i, x})] + h_i f(y_i(x), z_i(x)) \\ z_i(x) = \mathbb{E}_{t_i} [y_{i+1}(W_{t_{i+1}}^{t_i, x}) \frac{W_{t_{i+1}}^{t_i, x} - x}{h_i}] \end{cases} \quad (4.6)$$

where $W_{t_{i+1}}^{t_i, x}$ is a Brownian motion started by x in t_i . Using the C++, we test the error between the Euler scheme and the exact solution for the initial value Y_0 and Z_0 . Table 1 and 2 shows the decrease of Y and Z errors, if we reduce the step discretization h . We also note that the slope of the two curves are equal to -1.

$\log(n)$	$\log(\text{error of } Y)$
1.3862	-6.189
1.791	-6.561
2.3025	-7.045
2.564	-7.298
2.708	-7.438

tab.1.error for the Y-part

$\log(n)$	$\log(\text{error of } Z)$
1.386	-4.575
1.791	-4.938
2.079	-5.205
2.302	-5.415
2.708	-5.804

tab.2.error for the Z-part

5. Reflected BSDE's With Continuous Barrier

In this section, we gives some estimates for the extension of B-T-Z scheme for reflected FBSDE

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

$$Y_t = g(X_T) + \int_t^T f(Y_s, Z_s)ds + A_T - A_t - \int_t^T Z_s dW_s$$

where A_t is an increased continuous process, whose roles are to keep the process Y up to the continuous lower barrier L :

$$\int_0^T (Y_s - L_s)dA_s = 0.$$

5.1. Penalization approximation. We consider here the following sequence of BSDE's

$$Y_t^m = g(\widehat{X}_T) + \int_t^T f(Y_s^m, Z_s^m)ds + m \int_t^T (l(\widehat{X}_s) - Y_s^m)_+ ds - \int_t^T Z_s^m dW_s$$

where $m \rightarrow \infty$.

If we use an explicit B-T-Z-scheme, this would lead to the following method:

$$Y_i^{\pi, m} = \mathbb{E}_{t_i}[Y_{i+1}^{\pi, m} + h_i f(Y_{i+1}^{\pi, m}, Z_i^{\pi, m}) + mh_i(l(\widehat{X}_i) - Y_{i+1}^{\pi, m})_+].$$

$$Z_i^{\pi, m} = \mathbb{E}_{t_i}[H^i Y_{i+1}^{\pi, m}].$$

In the next part, we are interested by the optimal case, where $m = \frac{n}{T}$.

5.2. General a priori estimates. The two results which will be showed, establish the stability estimations for a class of RBSDE. To estimate the discrete RBSDE, let's introduce an extension of B-T-Z scheme:

$$Y_{j, N} = g(X_{T, j}), Y_{j, i} = \mathbf{E}_i[Y_{j, i+1} + f_{j, i}(Y_{j, i+1}, Z_{j, i})h_i + (l_j(X_i) - Y_{j, i+1})^+],$$

$$h_i Z_{j, i} = E_i[Y_{j, i+1} \Delta W_i],$$

where $i \in \{0, \dots, N-1\}$, $j \in \{1, 2\}$. To establish the stability result, the lemma below, is an intermediate result to give the global estimations.

Denote by

$$\Delta \xi = g(X_{T, 1}) - g(X_{T, 2}), \Delta Y_i = Y_{1, i} - Y_{2, i},$$

$$\Delta f_i = f_1(Y_{1, i+1}, Z_{1, i}) - f_2(Y_{2, i+1}, Z_{2, i})$$

$$\Delta \tilde{l}(X_i)^+ = (l_1(X_i) - Y_{1, i+1})^+ - (l_2(X_i) - Y_{2, i+1})^+.$$

Lemma 5.1. (*Local estimates*). For $j \in \{1, 2\}$, assume that $g(X_{T,j})$ is in $L_2(F_T)$. for each $i \in \{0, \dots, N\}$, assume that $f_{1,i}(Y_{1,i+1}, Z_{1,i})$ is in $L_2(F_T)$ and $f_{2,i}(y, z)$ is Lipschitz continuous w.r.t y and z , with a finite lipschitz constant $L_{f_{2,i}} \geq 0$. Then, for any $h_i \leq T$ and $\gamma_i \geq 0$ satisfying $6(h_i + \frac{1}{\gamma_i}) \leq 1$, it follows that

$$|\Delta Y_i|^2 \leq (1 + (\gamma_i + \frac{1}{2})h_i)\mathbf{E}_i(|\Delta Y_{i+1}|^2) + 3(h_i + \frac{1}{\gamma_i})h_i\mathbf{E}_i[|\Delta f_i|^2] \quad (5.1)$$

$$+(1 + \gamma_i h_i)\mathbf{E}_i[\Delta l(X_i)]^2. \quad (5.2)$$

Proof. Preliminary estimates for ΔZ_i : Since the Brownian motion is conditionally centered, it follows that:

$$h_i \Delta Z_i = \mathbf{E}_i[(\Delta Y_{i+1} - \mathbf{E}_i[\Delta Y_{i+1}])\Delta W_i].$$

By the cauchy-Schwartz inequality, we get:

$$|\Delta Z_i|^2 \leq \frac{1}{h_i} \left(\mathbf{E}_i[(\Delta Y_{i+1})^2] - (\mathbf{E}_i[\Delta Y_{i+1}])^2 \right)$$

Let:

$$Y_{j,i} = \mathbf{E}_i[Y_{j,i+1} + f_{j,i}(Y_{j,i+1}, Z_{j,i})h_i + (l_j(X_i) - Y_{j,i+1})^+].$$

We get:

$$\begin{aligned} \Delta Y_i &= \mathbf{E}_i \left[\Delta Y_{i+1} + h_i [f_1(Y_{1,i+1}, Z_{1,i}) - f_2(Y_{2,i+1}, Z_{2,i})] + \Delta \tilde{l}(X_i) \right] \\ &= \mathbf{E}_i \left[\Delta Y_{i+1} + h_i [f_1(Y_{1,i+1}, Z_{1,i}) - f_2(Y_{1,i+1}, Z_{1,i}) \right. \\ &\quad \left. + f_2(Y_{1,i+1}, Z_{1,i}) - f_2(Y_{2,i+1}, Z_{2,i})] + \Delta \tilde{l}(X_i) \right] \end{aligned}$$

$$\Delta Y_i = \mathbf{E}_i \left[\Delta Y_{i+1} + h_i [\Delta f_i + f_2(Y_{1,i+1}, Z_{1,i}) - f_2(Y_{2,i+1}, Z_{2,i})] + \Delta \tilde{l}(X_i) \right].$$

We denote now by:

$$\Delta G_i = \Delta f_i + f_2(Y_{1,i+1}, Z_{1,i}) - f_2(Y_{2,i+1}, Z_{2,i}).$$

Therefore, we can distinguish four cases:

I- If $l_1(X_i) \leq Y_{1,i+1}$ and $l_2(X_i) \leq Y_{2,i+1}$

$$\Delta Y_i = \mathbf{E}_i \left[\Delta Y_{i+1} + h_i \Delta G_i \right].$$

II-If $l_1(X_i) \leq Y_{1,i+1}$ and $l_2(X_i) \geq Y_{2,i+1}$

$$\begin{aligned} \Delta Y_i &= \mathbf{E}_i \left[\Delta Y_{i+1} + h_i \Delta G_i + Y_{2,i+1} - l_2(X_i) \right] \\ &\leq \mathbf{E}_i \left[Y_{1,i+1} - l_2(X_i) + h_i \Delta G_i \right] \\ &\leq \mathbf{E}_i \left[\Delta Y_{i+1} + h_i \Delta G_i \right]. \end{aligned}$$

II- If $l_1(X_i) \geq Y_{1,i+1}$ and $l_2(X_i) \leq Y_{2,i+1}$

$$\begin{aligned}\Delta Y_i &= \mathbf{E}_i \left[\Delta Y_{i+1} + h_i \Delta G_i - Y_{1,i+1} + l_1(X_i) \right] \\ &= \mathbf{E}_i \left[-Y_{2,i+1} + l_1(X_i) + h_i \Delta G_i \right] \\ &\leq \mathbf{E}_i \left[\Delta l(X_i) + h_i \Delta G_i \right].\end{aligned}$$

II-If $l_1(X_i) \geq Y_{1,i+1}$ and $l_2(X_i) \geq Y_{2,i+1}$

$$\begin{aligned}\Delta Y_i &= \mathbf{E}_i \left[\Delta Y_{i+1} + h_i \Delta G_i - Y_{1,i+1} + l_1(X_i) + Y_{2,i+1} - l_2(X_i) \right] \\ &= \mathbf{E}_i \left[\Delta l(X_i) + h_i \Delta G_i \right].\end{aligned}$$

Estimates for ΔY_i . We distinguish two cases.

First case: let $l_1(X_i) \leq Y_{1,i+1}$, We have

$$\Delta Y_i = \mathbf{E}_i [\Delta Y_{i+1} + h_i \Delta f_i + h_i (f_{2,i}(Y_{1,i+1}, Z_{1,i}) - f_{2,i}(Y_{2,i+1}, Z_{2,i}))].$$

Combining the Young inequality and the lipschitz property of $f_{2,i}$, we deduce that:

$$\begin{aligned}(\Delta Y_i)^2 &\leq (1 + \gamma h_i) (\mathbf{E}_i [\Delta Y_{i+1}])^2 \\ &\quad + 3(h_i + \frac{1}{\gamma}) h_i \left[\mathbf{E}_i [|\Delta f_i|^2] + L_{f_{2,i}}^2 \mathbf{E}_i [(\Delta Y_{i+1})^2] + L_{f_{2,i}}^2 |\Delta Z_i|^2 \right] \\ &\leq (1 + \gamma h_i - 3L_{f_{2,i}}^2 (h_i + \frac{1}{\gamma})) (\mathbf{E}_i [\Delta Y_{i+1}])^2 + 3(h_i + \frac{1}{\gamma}) h_i \mathbf{E}_i [|\Delta f_i|^2] \\ &\quad + \left[3(h_i + \frac{1}{\gamma}) h_i L_{f_{2,i}}^2 + 3L_{f_{2,i}}^2 (h_i + \frac{1}{\gamma}) \right] \mathbf{E}_i [(\Delta Y_{i+1})^2].\end{aligned}$$

The assumption on γ_i and h_i ensure that $1 + \gamma_i h_i - 3L_{f_{2,i}}^2 (h_i + \frac{1}{\gamma_i}) \geq 0$ for any h_i , whence, applying Jensen's inequality to the term in $(\mathbf{E}_i \Delta Y_{i+1})^2$, it follows that:

$$\begin{aligned}(\Delta Y_i)^2 &\leq (1 + \gamma_i h_i - 3L_{f_{2,i}}^2 (h_i + \frac{1}{\gamma_i})) (\mathbf{E}_i [(\Delta Y_{i+1})^2]) + 3(h_i + \frac{1}{\gamma_i}) h_i \left[\mathbf{E}_i [|\Delta f_i|^2] \right] \\ &\quad + \left[3(h_i + \frac{1}{\gamma_i}) h_i L_{f_{2,i}}^2 + 3L_{f_{2,i}}^2 (h_i + \frac{1}{\gamma_i}) \right] \mathbf{E}_i [\Delta Y_{i+1}^2] \\ &= (1 + \gamma_i h_i + 3L_{f_{2,i}}^2 h_i (h_i + \frac{1}{\gamma_i})) (\mathbf{E}_i [\Delta Y_{i+1}^2]) + 3(h_i + \frac{1}{\gamma_i}) h_i |\Delta f_i|^2.\end{aligned}$$

Finally, we use $3(h_i + \frac{1}{\gamma_i}) h_i L_{f_{2,i}}^2 \leq \frac{h_i}{2}$ to complete the first case.

Second case: let $l_1(X_i) \geq Y_{1,i+1}$. We have

$$\Delta Y_i = \mathbf{E}_i [\Delta l(X_i) + h_i \Delta f_i + h_i (f_2(Y_{1,i+1}, Z_{1,i}) - f_2(Y_{2,i+1}, Z_{2,i}))],$$

so we get:

$$\begin{aligned}(\Delta Y_i)^2 &\leq (1 + \gamma_i h_i) (\mathbf{E}_i [\Delta l(X_i)])^2 \\ &\quad + 3(h_i + \frac{1}{\gamma_i}) h_i \left[\mathbf{E}_i [|\Delta f_i|^2] + L_{f_{2,i}}^2 \mathbf{E}_i [(\Delta Y_{i+1})^2] + L_{f_{2,i}}^2 |\Delta Z_i|^2 \right]\end{aligned}$$

$$\begin{aligned}
&\leq 3L_{f_2,i}^2(h_i + \frac{1}{\gamma_i}) \left[\mathbf{E}[(\Delta Y_{i+1})^2] - (\mathbf{E}_i[\Delta Y_{i+1}])^2 \right] \\
&\quad + 3(h_i + \frac{1}{\gamma_i}) h_i L_{f_2,i}^2 \mathbf{E}_i[(\Delta Y_{i+1})^2] \\
&\quad + 3(h_i + \frac{1}{\gamma_i}) \mathbf{E}_i[|\Delta f_i|^2] + K_l(1 + \gamma_i h_i) (\mathbf{E}_i[\Delta l(X_i)])^2.
\end{aligned}$$

The application of $6(h_i + \frac{1}{\gamma_i}) \leq 1$, gives:

$$\begin{aligned}
(\Delta Y_i)^2 &\leq 3L_{f_2,i}^2(h_i + \frac{1}{\gamma_i})(h_i + 2) \mathbf{E}_i[(\Delta Y_{i+1})^2] \\
&\quad + 3(h_i + \frac{1}{\gamma_i}) h_i [\mathbf{E}_i[|\Delta f_i|^2] + (1 + \gamma_i h_i) (\mathbf{E}_i[\Delta l(X_i)])^2] \\
&\leq (1 + \frac{h_i}{2}) \mathbf{E}_i[(\Delta Y_{i+1})^2] + 3(h_i + \frac{1}{\gamma_i}) h_i [\mathbf{E}_i[|\Delta f_i|^2] \\
&\quad + (1 + \gamma_i h_i) (\mathbf{E}_i[\Delta l(X_i)])^2].
\end{aligned}$$

□

Proposition 5.2. (Global pointwise estimates). For $j \in \{1, 2\}$, assume that ξ_j is in $L_2(\mathcal{F}_T)$. Moreover, for each $i \in \{0, \dots, N-1\}$, assume that $f_{1,i}(Y_{1,i+1}, Z_{1,i})$ is in $L_2(\mathcal{F}_T)$ and $f_{2,i}(y, z)$ is Lipschitz continuous w.r.t y and z , with a finite Lipschitz constant $L_{f_2,i} \geq 0$. Then, for any time grid π and $\gamma \in (0, \infty)^N$ satisfying $6(h_i + \frac{1}{\gamma_k}) L_{f_2,k}^2 \leq 1$ for all $k \leq N-1$, we have for $0 \leq i \leq N$:

$$\begin{aligned}
|\Delta Y_i|^2 \Gamma_i &+ \sum_{k=i}^{N-1} h_k \mathbf{E}_i(|\Delta Z_k|^2) \Gamma_k \\
&\leq C \left(\Gamma_N \mathbf{E}_i(|\Delta \xi|^2) + 3 \sum_{k=i}^{N-1} \left(\frac{1}{\gamma_k} + h_i \right) h_i \mathbf{E}_i[|\Delta f_k|^2] \Gamma_k \right) \\
&\quad + 6 \sum_{k=i}^{N-1} (1 + \gamma_i h_i) (\mathbf{E}_i[\Delta l(X_i)])^2,
\end{aligned}$$

where $\Gamma_i = \prod_{k=0}^{i-1} (1 + \gamma_k h_k)$ and $C = (1 + T)e^{T/2}$.

Proof. Let $\lambda_i = (1 + (\gamma_{i-1} + \frac{1}{2})\lambda_{i-1})$, where $\lambda_0 = 1$. Multiplying the both sides of the equation by λ_i , we obtain

$$|\Delta Y_i|^2 \lambda_i \leq \mathbf{E}_i[|\Delta Y_{i+1}|^2] \lambda_{i+1} + 3(h_i + \frac{1}{\gamma_i}) h_i \mathbf{E}_i[|\Delta f_i|^2] \lambda_i + (1 + \gamma_i h_i) \mathbf{E}_i[\Delta l(X_i)]^2 \lambda_i,$$

summing both sides of the inequality from i to $N-1$ and taking the conditional expectation $\mathbf{E}_i[\cdot]$, we deduce that:

$$\begin{aligned}
(\Delta Y_i)^2 \lambda_i &\leq \lambda_N \mathbf{E}_i(|\Delta \xi|^2) + 3 \sum_{k=i}^{N-1} \left(\frac{1}{\gamma_k} + h_k \right) \\
&\quad \times h_k \mathbf{E}_i[|\Delta f_k|^2] \lambda_k + \sum_{k=i}^{N-1} (1 + \gamma_k h_k) \mathbf{E}_i[\Delta l(X_k)]^2 \lambda_k.
\end{aligned}$$

From the simple inequality $\Gamma_i \leq \lambda_i = \exp(\sum_{k=0}^i \ln(1 + (\gamma_k + \frac{1}{2})h_k)) \leq e^{T/2}\Gamma_i$, it follows that, for all $i \in \{0, \dots, N\}$,

$$\begin{aligned} (\Delta Y_i)^2 \Gamma_i &\leq e^{T/2} \Gamma_N \mathbf{E}_i(|\Delta \xi|^2) + 3e^{T/2} \sum_{k=i}^{N-1} \left(\frac{1}{\gamma_k} + h_k\right) h_k \mathbf{E}_i[|\Delta f_k|^2] \Gamma_k \\ &\quad + \sum_{k=i}^{N-1} (1 + \gamma_k h_k) \mathbf{E}_i[\Delta l(X_k)]^2 \Gamma_k. \end{aligned}$$

For the estimation of ΔZ_i , we remark that $\sum_{k=i}^{N-1} h_k \mathbf{E}_i[|\Delta Z_k|^2] \Gamma_k$ is bounded above by:

$$\begin{aligned} &\sum_{k=i}^{N-1} \Gamma_{k+1} (\mathbf{E}_i[(\Delta Y_{k+1})^2] - \mathbf{E}_i[\mathbf{E}_k(\Delta Y_{k+1})^2]) \\ &\leq \Gamma_N \mathbf{E}_i[|\Delta \xi|^2] + \sum_{k=i+1}^{N-1} \Gamma_k (\mathbf{E}_i[(\Delta Y_k)^2] - (1 + \gamma_k h_k) \mathbf{E}_i[\mathbf{E}_k(\Delta Y_{k+1})^2]). \end{aligned}$$

Therefore, we can distinguish two cases: If $\mathbf{I}_1(\mathbf{X}_i) \leq \mathbf{Y}_{1,i+1}$, we can see that $\mathbf{E}_i[(\Delta Y_k)^2] - (1 + \gamma_k h_k) \mathbf{E}_i[\mathbf{E}_k(\Delta Y_{k+1})^2]$ is bounded above by:

$$3\left(\frac{1}{\gamma_k} + h_k\right) h_k \left[\mathbf{E}_i[|\Delta f_k|^2] + L_{f_2,k}^2 \mathbf{E}_i[(\Delta Y_{k+1})^2] + L_{f_2,k}^2 \mathbf{E}_i[|\Delta Z_k|^2] \right].$$

By using the Jensen Inequality and the statement $6(h_i + \frac{1}{\gamma_k})L_{f_2,k}^2 \leq 1$, it follows that $\sum_{k=i}^{N-1} h_k \mathbf{E}_i[|\Delta Z_k|^2] \Gamma_k$ is bounded above by:

$$\begin{aligned} &\Gamma_N \mathbf{E}_i(\Delta \xi^2) + 3 \sum_{k=i+1}^{N-1} \left(\frac{1}{\gamma_k} + h_k\right) h_k L_{f_2,k}^2 \mathbf{E}_i[|\Delta Z_k|^2] \Gamma_k \\ &\quad + 3 \sum_{k=i+1}^{N-1} \left(\frac{1}{\gamma_k} + h_k\right) h_k \mathbf{E}_i(|\Delta f_k|^2) \Gamma_k \\ &\quad + 3 \sum_{k=i+1}^{N-1} \left(\frac{1}{\gamma_k} + h_k\right) h_k L_{f_2,k}^2 \mathbf{E}_i[(\Delta Y_{k+1})^2] \Gamma_k. \end{aligned}$$

Using the assumptions on γ_k and Δ_k of the proposition statement, it follows that, for the first case, $\sum_{k=i}^{N-1} h_k \mathbf{E}_i[|\Delta Z_k|^2] \Gamma_k$ is bounded by:

$$\begin{aligned} &2\Gamma_N \mathbf{E}_i[|\Delta \xi^2|] + 6 \sum_{k=i+1}^{N-1} \left(\frac{1}{\gamma_k} + h_k\right) h_k \mathbf{E}_i(|\Delta f_k|^2) \Gamma_k \\ &\quad + \sum_{k=i+1}^{N-1} h_k \mathbf{E}_i[(\Delta Y_{k+1})^2] \Gamma_k \\ &\leq (2 + Te^{T/2}) \Gamma_N \mathbf{E}_i[|\Delta \xi^2|] + (6 + 3Te^{T/2}) \sum_{k=i+1}^{N-1} \left(\frac{1}{\gamma_k} + h_k\right) h_k \mathbf{E}_i(|\Delta f_k|^2) \Gamma_k. \end{aligned}$$

where the estimate on ΔY is used in the last inequality. I- If $\mathbf{I}_1(\mathbf{X}_i) \geq \mathbf{Y}_{1,i+1}$, we can see that $(\mathbf{E}_i[(\Delta Y_k)^2] - (1 + \gamma_k h_k) \mathbf{E}_i[\mathbf{E}_k(\Delta Y_{k+1})^2])$ is bounded above by:

$$(1 + \gamma_i h_i) \mathbf{E}_i[\Delta l(X_i)]^2 + 3\left(\frac{1}{\gamma_k} + h_k\right) h_k \left[\mathbf{E}_i[|\Delta f_k|^2] + L_{f_2,k}^2 \mathbf{E}_i[(\Delta Y_{k+1})^2] + L_{f_2,k}^2 \mathbf{E}_i[|\Delta Z_k|^2] \right] - (1 + \gamma_k h_k) \mathbf{E}_i[\mathbf{E}_k(\Delta Y_{k+1})^2].$$

Using the Jensen Inequality and the statement $6(h_i + \frac{1}{\gamma_k}) L_{f_2,k}^2 \leq 1$, it follows that $\sum_{k=i}^{N-1} h_k \mathbf{E}_i[|\Delta Z_k|^2] \Gamma_k$ is bounded above by:

$$\begin{aligned} & \Gamma_N \mathbf{E}_i(\Delta \xi^2) + 3 \sum_{k=i+1}^{N-1} \left(\frac{1}{\gamma_k} + h_k\right) h_k L_{f_2,k}^2 \mathbf{E}_i[|\Delta Z_k|^2] \Gamma_k \\ & + 3 \sum_{k=i+1}^{N-1} \left(\frac{1}{\gamma_k} + h_k\right) h_k \mathbf{E}_i(|\Delta f_k|^2) \Gamma_k \\ & + \frac{1}{2} \sum_{k=i+1}^{N-1} \mathbf{E}_i[(\Delta Y_{k+1})^2] \Gamma_k + 3 \sum_{k=i+1}^{N-1} (1 + \gamma_i h_i) \mathbf{E}_i[\Delta l(X_i)]^2 \Gamma_k. \end{aligned}$$

Therefore:

$$\begin{aligned} & 2\Gamma_N \mathbf{E}_i[|\Delta \xi^2|] + 6 \sum_{k=i+1}^{N-1} \left(\frac{1}{\gamma_k} + h_k\right) h_k \mathbf{E}_i(|\Delta f_k|^2) \Gamma_k \\ & + \sum_{k=i+1}^{N-1} h_k \mathbf{E}_i[(\Delta Y_{k+1})^2] \Gamma_k + 6 \sum_{k=i+1}^{N-1} (1 + \gamma_i h_i) \mathbf{E}_i[\Delta l(X_i)]^2 \Gamma_k \\ & \leq (2 + Te^{T/2}) \Gamma_N \mathbf{E}_i[|\Delta \xi^2|] + (6 + 3Te^{T/2}) \sum_{k=i+1}^{N-1} \left(\frac{1}{\gamma_k} + h_k\right) \\ & \quad \times h_k \mathbf{E}_i(|\Delta f_k|^2) \Gamma_k + 6 \sum_{k=i+1}^{N-1} (1 + \gamma_i h_i) \mathbf{E}_i[\Delta l(X_i)]^2 \Gamma_k. \end{aligned}$$

□

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HANI ABIDI: DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF TUNIS UNIVERSITY OF TUNIS EL-MANAR, 1060 TUNIS, TUNISIA
E-mail address: `abidiheni@gmail.com`

HABIB OUERDIANE: DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES OF TUNIS UNIVERSITY OF TUNIS EL-MANAR, 1060 TUNIS, TUNISIA
E-mail address: `habib.ouerdiane@fst.rnu.tn`