

ON A NONSYMMETRIC ORNSTEIN-UHLENBECK SEMIGROUP AND ITS GENERATOR*

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ABSTRACT. If we add a simple rotation term to both the Ornstein-Uhlenbeck semigroup and the H-derivative, then analogue to the classical Malliavin calculus on the real Wiener space [I. Shigekawa, Stochastic analysis, 2004], we get a normal but nonsymmetric Ornstein-Uhlenbeck operator L on the complex Wiener space. The eigenfunctions of the operator L are given. In addition, the hypercontractivity for the nonsymmetric Ornstein-Uhlenbeck semigroup is shown.

1. Introduction

In [1], the following stochastic differential equation is considered:

$$\begin{cases} dZ_t = -\alpha Z_t dt + \sqrt{2\sigma^2} d\zeta_t, & t \geq 0, \\ Z_0 = z_0 \in \mathbb{C}^1, \end{cases} \quad (1.1)$$

where $Z_t = X_1(t) + iX_2(t)$, $\alpha = ae^{i\theta} = r + i\Omega$ with $a > 0$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and $\zeta_t = B_1(t) + iB_2(t)$ is a complex Brownian motion. Clearly, when $\Omega \neq 0$, the generator of the process is a 2-dimensional not symmetric but normal Ornstein-Uhlenbeck (OU) operator

$$A = \sigma^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (-rx + \Omega y) \frac{\partial}{\partial x} - (\Omega x + ry) \frac{\partial}{\partial y} \quad (1.2)$$

$$= 4\sigma^2 \frac{\partial^2}{\partial z \partial \bar{z}} - \alpha z \frac{\partial}{\partial z} - \bar{\alpha} \bar{z} \frac{\partial}{\partial \bar{z}}, \quad (1.3)$$

where we denote by $\frac{\partial f}{\partial z} = \frac{1}{2}(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y})$, $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y})$ the formal derivative of f at point $z = x + iy$. Note that $\Im(\alpha) \neq 0$ in Eq.(1.1) is the key point for the nonsymmetric property. The eigenfunctions of A are the so called complex Hermite polynomials [2]¹ which can be generated iteratively by the complex creation operator acting on the constant 1. Let $B = \begin{bmatrix} -r & \Omega \\ -\Omega & -r \end{bmatrix}$ and $B_0 = \begin{bmatrix} \cos \Omega t & \sin \Omega t \\ -\sin \Omega t & \cos \Omega t \end{bmatrix}$.

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¹It is called the Hermite-Laguerre-Itô polynomials in [1].

Then $e^{tB} = e^{-rt}B_0(t)$ and the associated OU semigroup of A is

$$\begin{aligned} P_t\varphi(z_0) &= \int_{\mathbb{R}^2} \varphi(e^{-rt}B_0(t)z_0 + \sqrt{1 - e^{-2rt}}z) \mu(dz) \\ &= \int_{\mathbb{C}} \varphi(e^{-\alpha t}z_0 + \sqrt{1 - e^{-2rt}}z) \mu(dz), \end{aligned} \quad (1.4)$$

where the stationary distribution is $d\mu = \frac{r}{2\pi\sigma^2} \exp\left\{-\frac{r(x^2+y^2)}{2\sigma^2}\right\} dx dy$ and we write a function $\varphi(x, y)$ of the two real variables x and y as the function $\varphi(z)$ of the complex argument $x + iy$ (i.e., we use the complex representation of \mathbb{R}^2 in (1.4)). For simplicity, we can choose that $a = 1$ and $r = \sigma^2 = \cos \theta$ then (1.4) becomes

$$\int_{\mathbb{C}} \varphi(e^{-(\cos \theta + i \sin \theta)t}z_0 + \sqrt{1 - e^{-2t \cos \theta}}z) \mu(dz). \quad (1.5)$$

If we let z_0, z be in the infinite dimensional space ($C_0([0, T] \rightarrow \mathbb{C}^1)$), we can define the nonsymmetric OU semigroup on ($C_0([0, T] \rightarrow \mathbb{C}^1)$) (see Definition 2.1). This idea is similar to the symmetric case [9]. This is the topic of Section 2.

The topic of Section 3 is how to obtain a concrete expression of the generator L of the above OU semigroup with rotation. We extend the definition of the Gateaux derivative and the H-derivative to the function $F : B \rightarrow \mathbb{C}$ and consider the derivative of the function $F(x + e^{i\theta}ty)$ with $t \in \mathbb{R}$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ at $t = 0$ (i.e., here the rotation term in the derivative corresponds to the one in the above OU semigroup). Furthermore, since we consider complex-value functions, we need the conjugate-linear functional. This idea also comes from the symmetric case[9].

In Section 4, we recall the Itô-Wiener chaos decomposition and give all the eigenfunctions of the generator L . In addition, we show the hypercontractivity for the above OU semigroup along almost the same lines as the symmetric case.

2. The Nonsymmetric OU Semigroup

By the complex representation of \mathbb{R}^2 , the planar Brownian motion (B^1, B^2) will be written $B = B^1 + iB^2$. Let H_1 be the 1-dimensional Cameron-Martin space [9], and denote H the complex Hilbert space $H = H_1 + iH_1$ with the natural inner product

$$\langle h, k \rangle = \int_0^T \dot{h}(s) \overline{\dot{k}(s)} ds. \quad (2.1)$$

Clearly, one can choose a c.o.n.s of H to be $\left\{ \frac{\varphi_m}{\sqrt{2}}, \frac{\bar{\varphi}_m}{\sqrt{2}} : m = 1, 2, \dots \right\}$.

We look the 2-dimensional Wiener space as a complex Wiener space ($C_0([0, T] \rightarrow \mathbb{C}^1), \mu$). The characteristic function of μ is

$$\int_B \exp\left\{\sqrt{-1}\Re(\langle \omega, \varphi \rangle)\right\} d\mu(\omega) = \exp\left\{-\frac{1}{2}|\varphi|_{H^*}^2\right\}, \quad \forall \varphi \in B^*. \quad (2.2)$$

Definition 2.1. Let the above notation prevail. We define *transition probability* on B as follows. For $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $t \geq 0, \Omega \in \mathbb{R}, x \in B, A \in \mathcal{B}(B)$ (the Borel σ -field generated by all open sets),

$$P_t(x, A) = \int_B 1_A(e^{-(\cos \theta + i \sin \theta)t}x + \sqrt{1 - e^{-2t \cos \theta}}y) \mu(dy). \quad (2.3)$$

The following property about the measure μ is well known.

Proposition 2.2. *For any $a \in \mathbb{R}$, the induced measure of μ under the mapping $x \mapsto e^{ia}x$ is identical to μ , that is to say, μ is rotation invariant. In addition, for any $t \geq 0$, denote the induced measure of μ under the mapping $x \mapsto \sqrt{t}x$ by μ_t , then $\mu_t \star \mu_s = \mu_{t+s}$ (\star is the convolution operator).*

An argument similar to the one used in the real case [9, p21] shows that $P_t(x, A)$ satisfies the Chapman-Kolmogorov equation.

$$\begin{aligned}
 & \int_B P_t(x, dy)P_s(y, A) \\
 &= \int_B P_s(e^{-te^{i\theta}}x + \sqrt{1 - e^{-2t \cos \theta}}y, A)\mu(dy) \\
 &= \int_B \int_B 1_A(e^{-se^{i\theta}}(e^{-te^{i\theta}}x + \sqrt{1 - e^{-2t \cos \theta}}y) + \sqrt{1 - e^{-2s \cos \theta}}z)\mu(dz)\mu(dy) \\
 &= \int_B \int_B 1_A(e^{-(s+t)e^{i\theta}}x + e^{-s \cos \theta}\sqrt{1 - e^{-2t \cos \theta}}y + \sqrt{1 - e^{-2s \cos \theta}}z)\mu(dz)\mu(dy) \\
 & \quad (\text{by the rotation invariant of the measure } \mu(dy)) \\
 &= \int_B \int_B 1_A(e^{-(s+t)e^{i\theta}}x + y)\mu_{e^{-2s \cos \theta}(1 - e^{-2t \cos \theta})} \star \mu_{1 - e^{-2s \cos \theta}}(dy) \\
 &= \int_B 1_A(e^{-(s+t)e^{i\theta}}x + \sqrt{1 - e^{2(s+t) \cos \theta}}y)\mu(dy) \\
 &= P_{s+t}(x, A).
 \end{aligned}$$

The associated Markov process to $P_t(x, A)$ is called a complex-valued Ornstein-Uhlenbeck process. Similar to the real case [9, Proposition 2.2], it follows Kolmogorov's criterion and the rotation invariance of μ that the Ornstein-Uhlenbeck process is realized as a measure on $C([0, \infty) \rightarrow B)$.

The associated semigroup $\{T_t, t \geq 0\}$ is defined as follows: for a bounded Borel measurable function F ,

$$T_t F(x) = \int_B F(e^{-(\cos \theta + i \sin \theta)t}x + \sqrt{1 - e^{-2t \cos \theta}}y) \mu(dy). \quad (2.4)$$

An argument similar to the one used in [9, Proposition 2.3, 2.4] shows that

Proposition 2.3. μ is a unique invariant measure, i.e.,

$$\int_B P(t, A)\mu(dx) = \mu(A), \quad \forall A \in \mathcal{B}(B).$$

And $\{T_t, t \geq 0\}$ is a strongly continuous contraction semigroup in $L^p(B, \mu)$ ($p \geq 1$).

3. The Ornstein-Uhlenbeck Operator and the Complex H-derivative

The generator of $\{T_t\}$ is called the Ornstein-Uhlenbeck operator, denoted by L . We will obtain a concrete expression of L in this section. Since there is a rotation term in the transition probabilities $P_t(x, A)$, to obtain a concrete expression of L , we need the complex H-derivative along a direction $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Definition 3.1. A function $F : B \rightarrow \mathbb{C}$ is *complex Gateaux differentiable* at $x \in B$ along the direction $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if there exist $\varphi_1, \varphi_2 \in B^*$ such that

$$\frac{d}{dt}F(x + e^{i\theta}ty)\Big|_{t=0} = \langle y, \varphi_1 \rangle + \langle \bar{y}, \varphi_2 \rangle, \quad \forall y \in B. \quad (3.1)$$

(φ_1, φ_2) is called a Gateaux derivative of F at x along the direction θ , denoted by $G_\theta F(x)$.

Remark 3.2. Here we look φ_1 as a linear functional on B , and φ_2 a conjugate-linear functional. And we inherit the notation in [9] that

$${}_B\langle x, G_\theta F(x) \rangle_{B^*} = \langle x, \varphi_1 \rangle + \langle \bar{x}, \varphi_2 \rangle.$$

Definition 3.3. A function $F : B \rightarrow \mathbb{C}$ is *complex H -differentiable* at $x \in B$ along the direction $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if there exist $h_1, h_2 \in H$ such that

$$\frac{d}{dt}F(x + e^{i\theta}th)\Big|_{t=0} = \langle h, h_1 \rangle + \langle h_2, h \rangle, \quad \forall h \in H. \quad (3.2)$$

(h_1, h_2) is called a *complex H -derivative* of F at x along the direction θ , denoted by $D_\theta F(x)$. When $\theta = 0$, we denote $D_\theta F(x)$ by $DF(x)$ instead.

We can define higher order differentiability. For simplicity, we only present the 2-th case here.

Definition 3.4. F is said to be 2-th *H -differentiable along the direction θ* if there exists a mapping $(\Phi_1, \Phi_2, \Phi_3, \Phi_4) : H \times H \rightarrow \mathbb{C}^4$ such that $\forall h_1, h_2 \in H$,

$$\frac{\partial^2}{\partial t_1 \partial t_2} F(x + e^{i\theta}t_1h_1 + t_2h_2)\Big|_{t_1=t_2=0} = \sum_{j=1}^4 \Phi_j(h_1, h_2) := \Phi(h_1, h_2), \quad (3.3)$$

where Φ_1 and $\bar{\Phi}_2$ are the bilinear forms², and Φ_3 and $\bar{\Phi}_4$ are the sesquilinear forms³. Φ is called the 2-th H -derivative of F at x along θ , denoted by $DD_\theta F(x)$.

Definition 3.5. Let Φ be as in Definition 3.4. Φ is said to be of *trace class* if the supremum

$$\sup \sum_{n=1}^{\infty} \sum_{i=1}^4 |\Phi_i(h_n, k_n)|$$

is finite, where k_n and h_n run over all c.o.n.s of H . Furthermore, the trace of Φ is defined by

$$\text{tr } \Phi = \sum_{n=1}^{\infty} \Phi_1(h_n, \bar{h}_n) + \Phi_2(h_n, \bar{h}_n) + \Phi_3(h_n, h_n) + \Phi_4(h_n, h_n). \quad (3.4)$$

Here $\{h_n\}$ is a c.o.n.s of H , and this does not depend on a choice of c.o.n.s.

Remark 3.6. An argument similar to the one used in [7, p44] shows that there exist bounded conjugate-linear operators A_1, A_2 such that $\Phi_1(h_1, h_2) = (h_1, A_1 h_2)$ and $\Phi_2(h_1, h_2) = (A_2 h_2, h_1)$.

²Here the bar is used for the conjugate instead of for the closure operator.

³The definition of sesquilinear is that the first argument is linear and the second one is conjugate-linear.

\mathcal{S} stands for all functions $F : B \rightarrow \mathbb{C}$ such that there exist $n \in \mathbb{N}, \varphi_1, \dots, \varphi_n \in B^*$, $f \in C^\infty(\mathbb{C}^n)$ so that

$$F(x) = f(\langle x, \varphi_1 \rangle, \langle x, \varphi_2 \rangle, \dots, \langle x, \varphi_n \rangle). \quad (3.5)$$

Here we assume that f with its derivatives has polynomial growth. If $F \in \mathcal{S}$, then the two derivative are given in the following forms. Let $z_j = \langle x, \varphi_j \rangle$, $j = 1, \dots, n$ and denote

$$\partial_j f = \frac{\partial}{\partial z_j} f(z_1, \dots, z_n), \quad \bar{\partial}_j f = \frac{\partial}{\partial \bar{z}_j} f(z_1, \dots, z_n).$$

If $\varphi \in B^*$, $c\varphi$ means that $(c\varphi)(x) = c\varphi(x)$. Then the Gâteaux derivative is

$$G_\theta F(x) = \left(e^{i\theta} \sum_{j=1}^n \varphi_j \partial_j f, \quad e^{-i\theta} \sum_{j=1}^n \varphi_j \bar{\partial}_j f \right), \quad (3.6)$$

$${}_B \langle x, G_\theta F(x) \rangle_{B^*} = \sum_{j=1}^n [e^{i\theta} z_j \partial_j f + e^{-i\theta} \bar{z}_j \bar{\partial}_j f]. \quad (3.7)$$

The H-derivative is given by

$$D_\theta F(x) = \left(e^{i\theta} \sum_{j=1}^n \varphi_j \partial_j f, \quad e^{-i\theta} \sum_{j=1}^n \varphi_j \bar{\partial}_j f \right), \quad (3.8)$$

$$D_\theta F(x)(h) = \sum_{j=1}^n [e^{i\theta} \langle h, \varphi_j \rangle \partial_j f + e^{-i\theta} \langle \varphi_j, h \rangle \bar{\partial}_j f], \quad (3.9)$$

where we adopt the convention that B^* is the subspace of H^* . (3.9) implies that the 2-th H-derivative is given by

$$\begin{aligned} DD_\theta^2 F(x)(h_1, h_2) &= \sum_{j, k=1}^n [e^{i\theta} \langle h_1, \varphi_j \rangle (\langle h_2, \varphi_k \rangle \partial_k \partial_j f + \langle \varphi_k, h_2 \rangle \bar{\partial}_k \bar{\partial}_j f) \\ &\quad + e^{-i\theta} \langle \varphi_j, h_1 \rangle (\langle h_2, \varphi_k \rangle \partial_k \bar{\partial}_j f + \langle \varphi_k, h_2 \rangle \bar{\partial}_j \bar{\partial}_k f)]. \end{aligned}$$

If, in addition, $\left\{ \frac{\varphi_m}{\sqrt{2}}, \frac{\bar{\varphi}_m}{\sqrt{2}} \right\}$ is an orthonormal system of H^* ,

$$\text{tr } DD_\theta F(x) = 4 \cos \theta \sum_{j=1}^n \partial_j \bar{\partial}_j f. \quad (3.10)$$

Proposition 3.7. For $F \in \mathcal{S}$,

$$LF(x) = \text{tr } DD_\theta F(x) - {}_B \langle x, G_\theta F(x) \rangle_{B^*}. \quad (3.11)$$

Proof. Suppose that $F \in \mathcal{S}$ is given by (3.5). We may assume that $\left\{ \frac{\varphi_j}{\sqrt{2}}, \frac{\bar{\varphi}_j}{\sqrt{2}} \right\}$ is an orthonormal system of H^* . Thus $\xi = (\langle x, \varphi_1 \rangle, \dots, \langle x, \varphi_n \rangle) \in \mathbb{C}^n$ has a $2n$ -dimensional standard normal distribution and we have

$$T_t F(x) = \int_{\mathbb{C}^n} f(e^{-(\cos \theta + i \sin \theta)t} \xi + \sqrt{1 - e^{-2t \cos \theta}} \eta) (2\pi)^{-n} e^{-|\eta|^2/2} d\eta.$$

When $t > 0$,

$$\begin{aligned}
& \frac{d}{dt} T_t F(x) \\
&= \frac{d}{dt} \int_{\mathbb{C}^n} f(e^{-(\cos \theta + i \sin \theta)t} \xi + \sqrt{1 - e^{-2t \cos \theta}} \eta) (2\pi)^{-n} e^{-|\eta|^2/2} d\eta \\
&= \sum_{j=1}^n \int_{\mathbb{C}^n} \left(-\xi_j e^{i\theta} e^{-e^{i\theta}t} + \frac{\eta_j \cos \theta e^{-2t \cos \theta}}{\sqrt{1 - e^{-2t \cos \theta}}} \right) \partial_j f(e^{-e^{i\theta}t} \xi + \sqrt{1 - e^{-2t \cos \theta}} \eta) u(d\eta) \\
&\quad + \sum_{j=1}^n \int_{\mathbb{C}^n} \left(-\bar{\xi}_j e^{-i\theta} e^{-e^{-i\theta}t} + \frac{\bar{\eta}_j \cos \theta e^{-2t \cos \theta}}{\sqrt{1 - e^{-2t \cos \theta}}} \right) \bar{\partial}_j f(e^{-e^{i\theta}t} \xi + \sqrt{1 - e^{-2t \cos \theta}} \eta) u(d\eta) \\
&= -e^{i\theta} e^{-e^{i\theta}t} \sum_{j=1}^n \xi_j \int_{\mathbb{C}^n} \partial_j f u(d\eta) - e^{-i\theta} e^{-e^{-i\theta}t} \sum_{j=1}^n \bar{\xi}_j \int_{\mathbb{C}^n} \bar{\partial}_j f u(d\eta) \\
&\quad + 4 \cos \theta e^{-2t \cos \theta} \sum_{j=1}^n \int_{\mathbb{C}^n} \partial_j \bar{\partial}_j f u(d\eta).
\end{aligned}$$

The last equation follows from the formula for integration by parts of the complex creation operator (see Lemma 2.3 of [1]). An argument similar to the one used in Proposition 2.7 of [9] shows that the convergence takes place in the topology of $L^p(B)$. Let $t \rightarrow 0$, we have

$$LF(x) = -e^{i\theta} \sum_{j=1}^n \xi_j \partial_j f - e^{-i\theta} \sum_{j=1}^n \bar{\xi}_j \bar{\partial}_j f + 4 \cos \theta \sum_{j=1}^n \partial_j \bar{\partial}_j f, \quad (3.12)$$

which is exact (3.11). \square

4. Itô-Wiener Chaos Decomposition, Eigenfunctions and the Hypercontractivity

Definition 4.1 (Definition of the Hermite-Laguerre-Itô polynomials). Let $m, n \in \mathbb{N}$ and $z = x + iy$ with $x, y \in \mathbb{R}$. We define the sequence on \mathbb{C}

$$\begin{aligned}
J_{0,0}(z) &= 1, \\
J_{m,n}(z) &= 2^{m+n} (\partial^*)^m (\bar{\partial}^*)^n 1.
\end{aligned} \quad (4.1)$$

We call it the *Hermite-Laguerre-Itô polynomial* in the present paper.

One can show [1] that $\left\{ (m!n!2^{m+n})^{-\frac{1}{2}} J_{m,n}(z) : m, n \in \mathbb{N} \right\}$ is an orthonormal basis of $L^2_{\mathbb{C}}(\nu)$ with $d\nu = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx dy$ and

$$\left[e^{i\theta} z \frac{\partial}{\partial z} + e^{-i\theta} \bar{z} \frac{\partial}{\partial \bar{z}} - 4 \cos \theta \frac{\partial^2}{\partial z \partial \bar{z}} \right] J_{m,n}(z) = [(m+n) \cos \theta + i(m-n) \sin \theta] J_{m,n}(z). \quad (4.2)$$

For a sequence $\mathbf{m} = \{m_k\}_{k=1}^{\infty}$, write $|\mathbf{m}| = \sum_k m_k$.

Definition 4.2. Take a complete orthonormal system $\left\{ \frac{\varphi_k}{\sqrt{2}}, \frac{\bar{\varphi}_k}{\sqrt{2}} \right\} \subseteq B^*$ in H^* and fix it throughout the section. For two sequences $\mathbf{m} = \{m_k\}_{k=1}^{\infty}$, $\mathbf{n} = \{n_k\}_{k=1}^{\infty}$ of

nonnegative integrals with finite sum, define

$$\mathbf{J}_{\mathbf{m},\mathbf{n}}(x) := \prod_k \frac{1}{\sqrt{2^{m_k+n_k} m_k! n_k!}} J_{m_k, n_k}(\langle x, \varphi_k \rangle). \quad (4.3)$$

We name it the *Fourier-Hermite-Itô polynomial*. For two $m, n \in Z_+$, the closed subspace spanned by $\{\mathbf{J}_{\mathbf{m},\mathbf{n}}(x); |\mathbf{m}| = m, |\mathbf{n}| = n\}$ in $L^2_{\mathbb{C}}(B, \mu)$ is called the Itô-Wiener chaos of degree of (m, n) and is denoted by $\mathcal{H}_{m,n}$.

Theorem 4.3. *For any fixed integer $m, n \geq 0$, the collection of functions*

$$\{\mathbf{J}_{\mathbf{m},\mathbf{n}}; |\mathbf{m}| = m, |\mathbf{n}| = n\} \quad (4.4)$$

is an orthogonal basis for the space $\mathcal{H}_{m,n}$. And if (m, n) varies then the collection of functions

$$\{\mathbf{J}_{\mathbf{m},\mathbf{n}}; |\mathbf{m}| = m, |\mathbf{n}| = n, m, n \geq 0\} \quad (4.5)$$

is an orthogonal basis for the space $L^2_{\mathbb{C}}(B, \mu)$. And $L^2_{\mathbb{C}}(B, \mu)$ has the Itô-Wiener expansion in the following way:

$$L^2_{\mathbb{C}}(B, \mu) = \bigoplus_{m=0}^{\infty} \bigoplus_{n=0}^{\infty} \mathcal{H}_{m,n}. \quad (4.6)$$

The project from $L^2_{\mathbb{C}}(B, \mu)$ to $\mathcal{H}_{m,n}$ is denoted by $\mathbf{J}_{m,n}$.

The above theorem is well known, which is exact Example 3.32 of [3, p31] which can be shown from view of the Gaussian Hilbert spaces. The reader can also give an elementary proof using an argument similar to Theorem 9.5.4 and 9.5.7 of [4].

Theorem 4.4. *Let $\mathbf{J}_{\mathbf{m},\mathbf{n}}(x)$ be a Fourier-Hermite-Itô polynomial defined by (4.3). Denote $m = |\mathbf{m}|$, $n = |\mathbf{n}|$. Then*

$$L\mathbf{J}_{\mathbf{m},\mathbf{n}}(x) = -[(m+n)\cos\theta + i(m-n)\sin\theta]\mathbf{J}_{\mathbf{m},\mathbf{n}}(x), \quad (4.7)$$

$$T_t\mathbf{J}_{\mathbf{m},\mathbf{n}}(x) = e^{-[(m+n)\cos\theta + i(m-n)\sin\theta]t}\mathbf{J}_{\mathbf{m},\mathbf{n}}(x). \quad (4.8)$$

Proof. Proposition 3.7 and (4.2) imply (4.7) directly. (4.8) follows from (4.7) and the semigroup equation (or say: Kolmogorov's equation). \square

In fact, (4.8) is an alternative procedure for introducing the OU semigroup. Similar to the symmetric OU semigroup(see [6, p54]), we define a nonsymmetric OU semigroup:

Definition 4.5. The *nonsymmetric OU semigroup* is the one-parameter semigroup $\{T_t, t \geq 0\}$ of contraction operators on $L^2_{\mathbb{C}}(B)$ defined by

$$T_t F(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-[(m+n)\cos\theta + i(m-n)\sin\theta]t} \mathbf{J}_{m,n} F \quad (4.9)$$

for any $F \in L^2_{\mathbb{C}}(B)$.

Finally, along almost the same lines as the proof of [9, Theorem 2.11], we have the hypercontractivity of the OU semigroup.

Proposition 4.6. *For the fixed $t \geq 0$ and $p > 1$, set $q(t) = e^{2t \cos \theta} (p - 1) + 1$. Then*

$$\|T_t F\|_{q(t)} \leq \|F\|_p, \quad \forall F \in L^p(B, \mu). \quad (4.10)$$

Proof. Since \mathcal{S} is dense in $L^p(B, \mu)$, it is enough to show this when $B = \mathbb{C}^n$. It is enough to show that for any $0 < a \leq f, g \leq b$ where f, g are Borel functions on \mathbb{C}^n , the following inequality holds [9, Theorem 2.11].

$$\int_{\mathbb{C}^n} T_t f(\xi) g(\xi) (2\pi)^{-n} e^{-|\xi|^2/2} d\xi \leq \|f\|_p \|g\|_{q(t)}. \quad (4.11)$$

Let $\zeta_t = (\zeta_t^{(1)}, \zeta_t^{(2)}, \dots, \zeta_t^{(n)})'$, $0 \leq t \leq 1$ be an n -dimensional standard complex Brownian motion. $\bar{\zeta}_t$ is the complex conjugate. Let $\tilde{\zeta}_t$ be an independent copy of ζ_t . For a given $0 < \lambda < 1$ and $a \in \mathbb{R}$, define

$$\hat{\zeta}_t = \lambda e^{ia} \zeta_t + \sqrt{1 - \lambda^2} \tilde{\zeta}_t. \quad (4.12)$$

Clearly, $\hat{\zeta}_t$ is still a standard complex Brownian motion. Set $\mathcal{F}_t^\zeta = \sigma(\zeta_s; 0 \leq s \leq t)$, $\mathcal{F}_t^{\hat{\zeta}} = \sigma(\hat{\zeta}_s; 0 \leq s \leq t)$. Define martingales

$$\begin{aligned} M_t &= E[f^p(\hat{\zeta}_1) | \mathcal{F}_t^{\hat{\zeta}}], \quad 0 \leq t \leq 1, \\ N_t &= E[g^{q'}(\zeta_1) | \mathcal{F}_t^\zeta], \quad 0 \leq t \leq 1, \end{aligned}$$

where q' is the conjugate number of

$$q = 1 + (p - 1)/\lambda^2.$$

It follows from the martingale (on filtrations induced by the complex Brownian motion) representation theorem that

$$M_t = M_0 + \int_0^t \theta_s d\hat{\zeta}_s + \int_0^t \vartheta_s d\bar{\zeta}_s, \quad N_t = N_0 + \int_0^t \phi_s d\zeta_s + \int_0^t \varphi_s d\bar{\zeta}_s.$$

Since $M_t, N_t \in \mathbb{R}$, $\vartheta_s = \bar{\theta}_s$ and $\varphi_s = \bar{\phi}_s$. It follows from the Itô's table that $dM_t dM_t = 4|\theta_t|^2 dt$, $dN_t dN_t = 4|\phi_t|^2 dt$ and $dM_t dN_t = 2\lambda(e^{ia}\theta_t\varphi_t + e^{-ia}\vartheta_t\phi_t)dt$. By Itô's formula, we have

$$\begin{aligned} d(M_t^{1/p} N_t^{1/q'}) &= \frac{1}{p} M_t^{1/p-1} N_t^{1/q'} dM_t + \frac{1}{q'} M_t^{1/p} N_t^{1/q'-1} dN_t \\ &\quad + \frac{1}{2} \frac{1}{p} \left(\frac{1}{p} - 1\right) M_t^{1/p-2} N_t^{1/q'} dM_t dM_t \\ &\quad + \frac{1}{p} \frac{1}{q'} M_t^{1/p-1} N_t^{1/q'-1} dM_t dN_t \\ &\quad + \frac{1}{2} \frac{1}{q'} \left(\frac{1}{q'} - 1\right) M_t^{1/p} N_t^{1/q'-2} dN_t dN_t. \end{aligned}$$

Note that $\sqrt{(p-1)(q'-1)} = \lambda$, therefore,

$$\begin{aligned} & E(M_t^{1/p} N_t^{1/q'}) - E(M_0^{1/p} N_0^{1/q'}) \\ &= -2E\left[\int_0^t M_t^{1/p-2} N_t^{1/q'-2} \left[\frac{1}{p}\left(1 - \frac{1}{p}\right) N_t^2 |\theta_t|^2 - 2\frac{1}{p} \frac{1}{q'} \lambda M_t N_t \Re(e^{ia} \theta_t \varphi_t)\right.\right. \\ &\quad \left.\left. + \frac{1}{q'}\left(1 - \frac{1}{q'}\right) M_t^2 |\phi_t|^2\right] dt\right] \\ &= -2E\left[\int_0^t M_t^{1/p-2} N_t^{1/q'-2} \left|\frac{\sqrt{p-1}}{p} N_t \theta_t - \frac{\sqrt{q'-1}}{q'} e^{ia} M_t \phi_t\right|^2 dt\right] \\ &\leq 0. \end{aligned}$$

Let $t = 1$ in the above inequality displayed, we have

$$E(f(\hat{\zeta}_1)g(\zeta_1)) \leq E[f^p(\hat{\zeta}_1)]^{1/p} E[g^{q'}(\zeta_1)]^{1/q'}.$$

From the definition of $\hat{\zeta}$ and letting $\lambda = e^{-t \cos \theta}$, $a = -t \sin \theta$, the above inequality displayed is exact (4.11). This ends the proof. \square

An argument similar to the one used in [9, Proposition 2.14, 2.15] shows the following boundedness of operator in $L^p(B, \mu)$ ($p > 1$).

Corollary 4.7. *$\mathcal{H}_{m,n}$, the Itô-Wiener chaos of degree of (m, n) , is a closed subspace in $L^p(B, \mu)$ ($p > 1$) and its norms $\|\cdot\|_p$ in $L^p(B, \mu)$ ($p > 1$) are equivalent to each other. In addition, the project operator $J_{m,n}$ is bounded in $L^p(B, \mu)$ ($p > 1$) and satisfies that*

$$J_{m,n} J_{i,j} = J_{i,j} J_{m,n} = \delta_{m,i} \delta_{n,j} J_{m,n} \tag{4.13}$$

$$T_t J_{m,n} = J_{m,n} T_t = e^{-(m+n)t \cos \theta - i(m-n)t \sin \theta} J_{m,n}. \tag{4.14}$$

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