

## ANALYSING SYSTEMIC RISK CONTRIBUTION USING A CLOSED FORMULA FOR CONDITIONAL VALUE AT RISK THROUGH COPULA

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**ABSTRACT.** The main challenge by the analysis and the regulation of **systemic risk** is the measurement of the adverse financial effect that the bankruptcy of one single financial institution can cause to the **financial system**. One of the main tools that has been proposed for this purpose is the risk measure  $\Delta CoVaR$  of Adrian and Brunnermeier in [2]. The main contribution of this paper is to propose a general and flexible framework for the computation of  $\Delta CoVaR$  in a more general stochastic setting compared to those provided so far. The formula that we propose here is based on **Copula's** theory. It allows us to stay not only in the Gaussian but also in the non-Gaussian setting. We also discuss the properties of our formula and analyse many examples, involving in particular **elliptical** and **Archimedean** copula, as well as **convex combination of copulas**. We also propose alternative models to those in [2].

### 1. Introduction

With the last crisis it became clear that the failure of certain financial institutions (the so called system relevant financial institutions) can produce an adverse impact on the whole financial system (systemic risk). The inability of standard risk-measurement tools like Value-at-Risk ( $VaR$ ) to capture this kind of risk (since their focus is on an institution in isolation: micro risk management) poses a new risk-management challenge to the financial regulators and academics. This problem can be summarized into two questions:

- (1) How to identify System-relevant Financial Institutions ?
- (2) How to quantify the marginal risk contribution of one single financial institute to the system ?

As an academic response to the second question Adrian and Brunnermeier [2] proposed  $CoVaR$  method as a tool to analyse the adverse financial effect of the failure of a single financial institution on the financial system. They defined the term  $CoVaR$  as the Value-at-Risk ( $VaR$ ) of the financial system conditional on the state of one given financial institution and quantified the risk contribution (i.e. how much an institution adds to the risk of the system) of a given financial

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institution by the measure  $\Delta CoVaR$ . This is defined as the difference between  $CoVaR$  conditional on the given financial institution being in distress and the  $CoVaR$  when it is not.

The implementation of  $CoVaR$  involves variables characterizing a single financial institution  $i$  (e.g. the loss incurred by the financial institution  $i$  denoted by  $L^i$ ) and the financial system  $s$  (e.g. the loss incurred by the financial system  $s$  denoted by  $L^s$ ) respectively and variables characterizing the interdependency structure between single financial institutions  $i$  and the financial system  $s$ . This macro-dimension of  $CoVaR$  allows the integration of the dependence structure of  $i$  and  $s$  in risk-measurement contrary to the standard risk measures ("micro-risk measure" e.g. VaR) where only variables characterizing the financial institution alone are considered. The  $CoVaR$  concept can be thus used by regulatory institutions as a macro-prudential tool (or as a basis for the development of other tools) for the regulation of systemic risk. Its computation represents an open problem, although some approaches have been proposed: Adrian and Brunnermeier [2] proposed an estimation method based on "linear quantile regression", Jäger-Ambrożewicz, M. [12] developed a closed formula for the special case where the random vector  $(L^i, L^s)$  is modelled by a bivariate normal distribution. In all these approaches there are some difficulties to flexibly model the stochastic behaviours of financial institution's specific variables and their dependence structure (interconnection), since only bivariate normal distribution is considered.

Our aim is thus to provide a more flexible framework, including also the non-Gaussian case, for the implementation of the  $CoVaR$  concept which allows the integration of stylized features of marginal losses as skewness, fat tails and interdependence properties like linear, non-linear and positive or negative tail dependence. To do this, we first propose an improved definition of  $CoVaR$  which makes it mathematically tractable (see Definition 1.5), and based on copula theory we propose a general formula for  $CoVaR$  and hence also for  $\Delta CoVaR$  (see Theorem 3.2).

We first recall here the definition of the Value-at-Risk ( $VaR$ ) in order to define the  $CoVaR$  as a conditional  $VaR$  following [2].

**Definition 1.1** (Value-at-Risk). Given some confidence level  $\alpha \in (0, 1)$  the  $VaR$  of a portfolio at the confidence level  $\alpha$  is given by the smallest number  $l$  such that the probability that the loss  $L$  exceeds  $l$  is no larger than  $(1 - \alpha)$ . Formally

$$\begin{aligned} VaR_\alpha &:= \inf \{l \in \mathbb{R} : Pr(L > l) \leq 1 - \alpha\} \\ &= \inf \{l \in \mathbb{R} : Pr(L \leq l) \geq \alpha\}. \end{aligned}$$

We will employ the notation of quantile as provided in the following definition (cf. [15] Definition 2.12).

**Definition 1.2** (Generalized inverse and quantile function).

- a) Given an increasing function  $T : \mathbb{R} \rightarrow \mathbb{R}$ , the *generalized inverse* of  $T$  is defined by

$$T(y) := \inf \{x \in \mathbb{R} : T(x) \geq y\}.$$

- b) Given a distribution function  $F$ , the generalized inverse  $F^{\leftarrow}$  is called the *quantile function* of  $F$ . For  $\alpha \in (0, 1)$  we have

$$q_\alpha(F) = F^{\leftarrow}(\alpha) := \inf \{x \in \mathbb{R} : F(x) \geq \alpha\}.$$

Note that, if  $F$  is continuous and strictly increasing, we simply have

$$q_\alpha(F) = F^{-1}(\alpha), \quad (1.1)$$

where  $F^{-1}$  is the (ordinary) inverse of  $F$ . Thus suppose that the distribution  $F$  of the loss  $L$  is continuous and strictly increasing. It follows

$$VaR_\alpha = F^{-1}(\alpha). \quad (1.2)$$

We note that typical values taken for  $\alpha$  are 0.99 or 0.995.

*Assumption 1.3.* Henceforth we consider only random variables which have strictly positive density function. Also in case we consider a bivariate joint distribution  $H(x, y)$  we assume that it has strictly positive density and its marginal distributions have strictly positive densities.

Due to this assumption all considered distribution functions  $F$  are continuous and strictly increasing.

Let us describe the loss incurred by the financial institution  $i$  and that incurred by the financial system  $s$  by two random variables  $L^i$  and  $L^s$  with univariate distribution functions  $F_i$  and  $F_s$  respectively. At least since the financial crisis it is clear that the dependency between the financial system and the financial institution  $i$  must be analysed more seriously. A step towards such an analysis is done by assuming that the random variables  $L^i$  and  $L^s$  are stochastically dependent and that their joint behaviour is determined by a bivariate joint distribution function.

Following this, Adrian and Brunnermeier [2] define  $CoVaR_\alpha^{s|C(L^i)}$  as the Value-at-Risk at the level  $\alpha$  of an financial system  $s$  conditional on some event  $C(L^i)$  depending on the loss  $L^i$  incurred by financial institution  $i$ . Thus  $CoVaR_\alpha^{s|C(L^i)}$  can be implicitly defined as the  $\alpha$ -quantile of the conditional probability of the financial system's loss.

$$Pr\left(L^s \leq CoVaR_\alpha^{s|C(L^i)} | C(L^i)\right) = \alpha. \quad (1.3)$$

In their work Adrian and Brunnermeier considered the case where the condition  $C(L^i)$  refers to the loss  $L^i$  incurred by the financial institution  $i$  being exactly at its Value-at-Risk and at its mean. We generalize this approach by allowing  $L^i$  to assumes any value  $l \in \mathbb{R}$ . We have in this case in the context of (1.3) the following expression

$$Pr\left(L^s \leq CoVaR_\alpha^{s|L^i=l} | L^i = l\right) = \alpha. \quad (1.4)$$

*Remark 1.4.* The so defined  $CoVaR_\alpha^{s|L^i=l}$  is consistent with respect to the (stochastic) independence, in the sense  $CoVaR_\alpha^{s|L^i=l} = VaR_\alpha^s$  when  $L^i$  and  $L^s$  are (stochastically) independent.

Following [5] (Definition 4.7) we can define in the context of Assumption 1.3, a conditional probability of the form  $Pr(L^s \leq h | L^i = l)$  for fixed  $h$  as a function of  $l$  as follows

$$Pr(L^s \leq h | L^i \leq y) = \int_{-\infty}^y Pr(L^s \leq h | L^i = l) f_i(l) dl \quad \forall y \in \mathbb{R}. \quad (1.5)$$

Consider the function  $R_l(h) := Pr(L^s \leq h | L^i = l)$ . As  $R_l(h)$  is strictly increasing, it follows that it is invertible. Based on this we provide an alternative definition, for  $CoVaR_\alpha^{s|L^i=l}$  which is more tractable from a mathematical point of view than that proposed in [2].

**Definition 1.5.** Assume that  $L^i$  and  $L^s$  satisfy Assumption 1.3. then for a given  $\alpha \in (0, 1)$  and for a fixed  $l$ ,  $CoVaR_\alpha^{s|L^i=l}$  is defined as:

$$\begin{aligned} CoVaR_\alpha^{s|L^i=l} &:= \inf \{h \in \mathbb{R} : Pr(L^s > h | L^i = l) \leq 1 - \alpha\} \\ &= \inf \{h \in \mathbb{R} : Pr(L^s \leq h | L^i = l) \geq \alpha\} \\ &= R_l^{-1}(\alpha). \end{aligned}$$

For a fixed  $\alpha$  we define the function

$$CoVaR_\alpha^{s|i}(l) := CoVaR_\alpha^{s|L^i=l}, \quad \forall l \in \mathbb{R}. \quad (1.6)$$

$\Delta CoVaR_\alpha^{s|i}$  is defined by Adrian and Brunnermeier [2] as the difference between  $CoVaR_\alpha^{s|C(L^i)}$  condition on the institution  $i$  being under distress (i.e.  $C(L^i) = \{L^i = VaR_\alpha^i\}$ ) and the  $CoVaR_\alpha^{s|C(L^i)}$  condition on the financial institution having mean loss (i.e.  $C(L^i) = \{L^i = E[L^i]\}$ ).

**Definition 1.6.**

$$\Delta CoVaR_\alpha^{s|i} := CoVaR_\alpha^{s|L^i=VaR_\alpha^i} - CoVaR_\alpha^{s|L^i=E(L^i)}. \quad (1.7)$$

Using (1.6) we can rewrite (1.7) as follows.

**Definition 1.7.** For  $l_1, l_2 \in \mathbb{R}$  such that  $l_1 \geq l_2$ , we define

$$\Delta CoVaR_\alpha^{s|i}(l_1, l_2) := CoVaR_\alpha^{s|i}(l_1) - CoVaR_\alpha^{s|i}(l_2) \quad (1.8)$$

## 2. A Brief Introduction to Copulas

In this section we introduce the notion of copula and give some basic definitions and important properties needed later. For detailed analysis of copulas, we refer the reader to e.g. [13], [15], [16] or [17] and the references therein.

In order to introduce the concept of copula, we recall some important remarks upon which it is built.

*Remark 2.1* (cf. [15] Proposition 5.2). Assume  $F$  is a distribution function such that its inverse function  $F^{-1}$  is well defined.

- (1) **Quantile transformation.** Let  $U$  be a standard uniform distributed random variable (i.e.  $U \sim U(0, 1)$ ), then  $Pr(F^{-1}(U) \leq x) = F(x)$ .

- (2) **Probability transformation.** Let  $X$  be a random variable with distribution function  $F$ , then  $F(X)$  has a uniform standard distribution i.e.  $F(X) \sim U(0, 1)$ .

**Definition 2.2** (2-dimensional copula (cf. [16] Definition 2.2.2)). A *2-dimensional copula* is a (distribution) function  $C : [0, 1]^2 \rightarrow [0, 1]$  with the following satisfying:

- Boundary conditions:
  - 1) For every  $u \in [0, 1] : C(0, u) = C(u, 0) = 0$ .
  - 2) For every  $u \in [0, 1] : C(1, u) = u$  and  $C(u, 1) = u$ .
- Monotonicity condition:
  - 3) For every  $(u_1, u_2), (v_1, v_2) \in [0, 1] \times [0, 1]$  with  $u_1 \leq u_2$  and  $v_1 \leq v_2$  we have

$$C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0.$$

Conditions 1) and 3) imply that the so defined 2-copula  $C$  is a bivariate joint distribution function (cf. [16] Definition 2.3.2) and Condition 2) implies that the copula  $C$  has standard uniform margins. From this it follows that a bivariate copulas is increasing with respect to each of its arguments i.e. for every  $u$  and  $v \in [0, 1]$  the maps  $v \mapsto C(u, v)$  and  $u \mapsto C(u, v)$  are each increasing.

**Theorem 2.3** (Sklar's theorem, cf. [16] Theorem 2.3.3). *Let  $H$  be a joint distribution function with marginal distribution functions  $F$  and  $G$ , then there exists a copula  $C$  such that for all  $x, y \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$*

$$H(x, y) = C[F(x), G(y)]. \quad (2.1)$$

*If  $F$  and  $G$  have density, then  $C$  is unique. Conversely, if  $C$  is a copula and  $F$  and  $G$  are distribution functions, then the function  $H$  defined by (2.1) is a joint distribution function with margins  $F$  and  $G$ .*

This theorem is very important because it asserts that, using copula function, it is possible to represent each bivariate distribution function as a composition of two univariate distribution functions and a given copula function. Thus, we can use the copula to extract the dependence structure among the components  $X$  and  $Y$  of the vector  $(X, Y)$ , independently of the marginal distribution  $F$  and  $G$ . This allows us to model the dependence structure and marginals separately.

*Remark 2.4.* Assume  $(X, Y)$  is a bivariate random variables with copula  $C$  and joint distribution  $H$  satisfying Assumption 1.3, with marginals distribution function  $F$  and  $G$ , then the transformed random variables  $U = F(X)$  and  $V = F(Y)$  have standard uniform distribution and  $C(U, V)$  is the joint distribution of  $(U, V)$ . In fact

$$C(u, v) = C(\Pr(U \leq u), \Pr(V \leq v)).$$

**Corollary 2.5** (cf. e.g. [16] Corollary 2.3.7). *Let  $H$  denote a bivariate distribution function with margins  $F$  and  $G$  satisfying Assumption 1.3, then there exist an unique copula  $C$  such that for all  $(u, v) \in [0, 1]^2$  it holds:*

$$C(u, v) = H(F^{-1}(u), G^{-1}(v)).$$

*Remark 2.6.* As long as Assumption 1.3 is satisfied any copula  $C$  assumed here is associated to a joint distribution functions that have strictly positive density, i.e. there exist a unique strictly positive function  $c : [0, 1]^2 \rightarrow [0, \infty)$ , called "copula density", such that

$$C(u, v) = \int_0^v \int_0^u c(s, t) ds dt \quad \forall u, v \in [0, 1]. \quad (2.2)$$

### 3. Analysing Systemic Risk Contribution Using $\Delta CoVaR$ : A Copula Approach

The aim of this section is to improve the quality of systemic risk analysis by providing a general and flexible framework for the calculation and the theoretical analysis of the term  $CoVaR_\alpha^{s|L^i=l}$  for a large class of stochastic setting. To do this, we will use the relation between conditional probability and copula to rewrite the implicit definition of  $CoVaR_\alpha^{s|L^i=l}$  in terms of copula and obtain so a general formula. Then using our formula we will highlight some important properties of  $CoVaR_\alpha^{s|L^i=l}$  and  $\Delta CoVaR^{s|i}$ .

We provide in the following theorem a general formula for the computing of  $CoVaR_\alpha^{s|L^i=l}$ . We do this by assuming that the joint distribution  $H$  of  $L^i$  and  $L^s$  satisfy Assumption 1.3 and we denote by  $f_i$  and  $f_s$  the density function of  $L^i$  and  $L^s$  respectively.

Let  $C$  be the copula associated to  $H$ , i.e.

$$H(x, y) = C(F_i(x), F_s(y)). \quad (3.1)$$

Due to Assumption 1.3 we have that the Copulas  $C$  have strictly positive density function  $c$ . We define the function

$$g(v, u) := \frac{\partial C(u, v)}{\partial u}.$$

*Remark 3.1.* Under Assumption 1.3 the function  $g(v, u)$  is well defined and for each fixed  $u \in [0, 1]$  invertible with respect to the parameter  $v$ . In fact, by differentiating (2.2) with respect to  $u$  and applying the Fubini's theorem we obtain

$$g(v, u) = \frac{\partial C(u, v)}{\partial u} = \int_0^v c(u, t) dt. \quad (3.2)$$

Since the copula density  $c$  is strictly positive, it follows that for a fixed  $u \in [0, 1]$  the function  $g(v, u)$  is strictly increasing and thus invertible with respect to  $v$ .

**Theorem 3.2.** *Under Assumption 1.3 for all  $l \in \mathbb{R}$  and a given  $\alpha \in (0, 1)$ ,  $CoVaR_\alpha^{s|L^i=l}$  is given by*

$$CoVaR_\alpha^{s|L^i=l} = F_s^{-1}(g^{-1}(\alpha, F_i(l))). \quad (3.3)$$

*Proof.* Recall that the implicit definition of  $CoVaR_\alpha^{s|L^i=l}$  is given by:

$$\begin{aligned} Pr(L^s \leq CoVaR_\alpha^{s|L^i=l} | L^i = l) &= \alpha \\ \Leftrightarrow Pr(F_s(L^s) \leq F_s(CoVaR_\alpha^{s|L^i=l}) | F_i(L^i) = F_i(l)) &= \alpha. \end{aligned}$$

We define  $V := F_s(L^s)$ ,  $U := F_i(L^i)$ ,  $v := F_s(CoVaR_\alpha^{s|L^i=l})$  and  $u := F_i(l)$  such that

$$Pr\left(F_s(L^s) \leq F_s(CoVaR_\alpha^{s|L^i=l}) \mid F_i(L^i) = F_i(l)\right) = Pr(V \leq v \mid U = u).$$

Due to Assumption 1.3 it follows from Remark 2.1 that  $V$  and  $U$  are standard uniform distributed and hence continuous.

The conditional probability  $Pr(V \leq v \mid U = u)$  can be thus computed as follows (cf. e.g. ([5] Equation. (4.4)) and ([17] Page 263))

$$\begin{aligned} Pr(V \leq v \mid U = u) &= \lim_{\Delta u \rightarrow 0^+} \frac{Pr(V \leq v, u \leq U \leq u + \Delta u)}{Pr(u \leq U \leq u + \Delta u)} \\ &= \lim_{\Delta u \rightarrow 0^+} \frac{C(v, u + \Delta u) - C(v, u)}{\Delta u} \\ &= \frac{\partial C(v, u)}{\partial u} = g(v, u). \end{aligned} \quad (3.4)$$

We have thus the following equivalence

$$\begin{aligned} Pr\left(F_s(L^s) \leq F_s(CoVaR_\alpha^{s|L^i=l}) \mid F_i(L^i) = F_i(l)\right) \\ = g\left(F_s(CoVaR_\alpha^{s|L^i=l}), F_i(l)\right). \end{aligned}$$

Based on this equivalence and due to the fact that the function  $g(v, u)$  is invertible with respect to  $v$  for any fixed  $u \in [0, 1]$  (see Remark 3.1), we are able to derive the explicit expressions of  $CoVaR_\alpha^{s|L^i=l}$ . We do this by expressing  $v$  as a function of  $\alpha$  and  $u$  as follow

$$v = g^{-1}(\alpha, u).$$

By replacing  $v$  by  $F_s(CoVaR_\alpha^{s|L^i=l})$  and  $u$  by  $F_i(l)$  we obtain

$$F_s(CoVaR_\alpha^{s|L^i=l}) = g^{-1}(\alpha, F_i(l)).$$

Thus

$$CoVaR_\alpha^{s|L^i=l} = F_s(CoVaR_\alpha^{s|L^i=l}) = F_s^{-1}(g^{-1}(\alpha, F_i(l))).$$

□

One important advantage of our formula is that, the expression of  $CoVaR_\alpha^{s|L^i=l}$  (see Equation (3.3)) can be separated into two distinct components.

- (1) On the one hand the marginal distributions  $F_i$  and  $F_s$ , which represent the purely univariate features of the single financial institution  $i$  and the financial system  $s$  respectively.
- (2) On the other hand the function  $g^{-1}$ , which represents the dependency structure between the single financial institution  $i$  and the system  $s$ .

This separation in the spirit of Sklar's theorem is very important for the analysis of systemic risk because it allows to investigate the effect of the marginal distributions  $F_i$  and  $F_s$  and the assumed copula  $C$  to the systemic risk contribution.

*Remark 3.3.* We can see from Equation (3.3) that  $CoVaR_\alpha^{s|L^i=l}$  is nothing other than a quantile of the loss distribution  $F_s$  of the financial system  $s$ . In fact we have

$$CoVaR_\alpha^{s|L^i=l} = F_s^{-1}(\tilde{\alpha}) \quad \text{with } \tilde{\alpha} := g^{-1}(\alpha, F_i(l)). \quad (3.5)$$

So, as  $CoVaR_\alpha^{s|L^i=l}$  can be expressed as the quantile of  $F_s$  with respect to the adjusted level  $\tilde{\alpha}$ , it follows that  $CoVaR_\alpha^{s|L^i=l}$  as a function of  $\tilde{\alpha}$  has the same properties like a simply Value-at-Risk. For example, the following properties hold.

*Property 3.4.*

- a)  $CoVaR_\alpha^{s|L^i=l}$  increases when the marginal distribution of the system ( $F_s$ ) has leptokurtosis (heavy-tailed) and positive skewness. (cf. [3] IV.2.8.1).
- b) If the loss  $L^s$  of the financial system  $s$  is assumed to be normal distributed with mean  $\mu_s$  and standard deviation  $\sigma_s$ , then

$$CoVaR_\alpha^{s|L^i=l} = \sigma_s \Phi^{-1}(\tilde{\alpha}) + \mu_s, \quad (3.6)$$

$\Phi$  denotes the standard normal distribution function and  $\tilde{\alpha}$  is defined according to Equation (3.5). Moreover

$$\Delta CoVaR_\alpha^{s|i} = \sigma_s (\Phi^{-1}(\tilde{\alpha}_d) - \Phi^{-1}(\tilde{\alpha}_m)), \quad (3.7)$$

where  $\tilde{\alpha}_d$  and  $\tilde{\alpha}_m$  are the adjusted levels when the financial institution  $i$  is under distress and has its mean loss respectively, i.e.

$$\tilde{\alpha}_d = g^{-1}(\alpha, F_i(VaR_\alpha^i)) \quad \text{and} \quad \tilde{\alpha}_m = g^{-1}(\alpha, F_i(E[L^i])).$$

In general the following Corollary of Theorem 3.2 holds:

**Corollary 3.5.** *Under Assumption 1.3, the risk measure  $\Delta CoVaR_\alpha^{s|i}$  is computed using Definition 1.6 as follows:*

$$\begin{aligned} \Delta CoVaR_\alpha^{s|i} &= CoVaR_\alpha^{s|L^i=VaR_\alpha^i} - CoVaR_\alpha^{s|L^i=E[L^i]} \\ &= F_s^{-1}(g^{-1}(\alpha, F_i(VaR_\alpha^i))) - F_s^{-1}(g^{-1}(\alpha, F_i(E[L^i]))) \\ &= F_s^{-1}(g^{-1}(\alpha, \alpha)) - F_s^{-1}(g^{-1}(\alpha, F_i(\mu_i))), \end{aligned}$$

where  $\mu_i = E[L^i]$ .

*Remark 3.6.* If we assume a symmetric distribution for  $L^i$ , then it holds

$$\Delta CoVaR_\alpha^{s|i} = F_s^{-1}(g^{-1}(\alpha, \alpha)) - F_s^{-1}(g^{-1}(\alpha, 0.5)). \quad (3.8)$$

*Remark 3.7.* In practice the conditional level  $l$  for the financial institution  $i$  is implicitly defined through a given confidence level  $\beta \in (0, 1)$  by

$$l = F_i^{-1}(\beta), \quad (3.9)$$

The confidence level  $\beta$  is specified by the regulatory institution and represents the probability with which the financial institution  $i$  remains solvent over a given period of time horizon.



Based on this information we can express  $CoVaR_\alpha^{s|L^i=l}$  as follow:

$$CoVaR_\alpha^{s|L^i=l} = F_s^{-1} \left( g^{-1}(\alpha, \beta) \right). \quad (3.10)$$

We observe that for a given marginal distribution function  $F_s$ ,  $CoVaR_\alpha^{s|L^i=l}$  can be expressed as a function of  $\alpha$  and  $\beta$ . This motivates the following definitions.

**Definition 3.8.**

$$\begin{aligned} CoVaR_\alpha^\beta &:= CoVaR_\alpha^{s|L^i=F_i^{-1}(\beta)} \\ \Delta CoVaR_\alpha^\beta &:= CoVaR_\alpha^{s|L^i=F_i^{-1}(\beta)} - CoVaR_\alpha^{s|L^i=E(L^i)} \\ &:= CoVaR_\alpha^\beta - CoVaR_\alpha^{s|L^i=E(L^i)} \end{aligned}$$

It follows

$$\Delta CoVaR_\alpha^\beta = F_s^{-1} \left( g^{-1}(\alpha, \beta) \right) - F_s^{-1} \left( g^{-1}(\alpha, F_i(\mu_i)) \right)$$

The bivariate Gaussian copula is defined as follows (cf. [16] Equation 2.3.6):

$$C_\rho^{Gau}(u, v) = \Phi_2 \left( \Phi(u)^{-1}, \Phi(v)^{-1} \right),$$

where  $\Phi_2$  denotes the bivariate standard normal distribution function with linear correlation coefficient  $\rho$ , and  $\Phi$  the univariate standard normal distribution function. Hence,

$$C_\rho^{Gau}(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left( \frac{2\rho st - s^2 - t^2}{2(1-\rho^2)} \right) ds dt.$$

According to Theorem 3.2 we have the following formula for  $CoVaR_\alpha^{s|L^i=l}$  when the dependence between  $L^i$  and  $L^s$  is modelled by a Gaussian copula.

**Proposition 3.9.** *Assume that the copula of  $L^i$  and  $L^s$  is the Gaussian copula, then*

$$CoVaR_\alpha^{s|L^i=l} = F_s^{-1} \left( \Phi \left( \rho \Phi^{-1}(F_i(l)) + \sqrt{1-\rho^2} \Phi^{-1}(\alpha) \right) \right). \quad (3.11)$$

where  $F_i$  and  $F_s$  represent the univariate distribution function of  $L^i$  and  $L^s$  respectively.

*Proof.* We first note that  $C_\rho(u, v)$  can be expressed as

$$C_\rho^{Gau}(u, v) = \int_0^u \Phi \left( \frac{\Phi^{-1}(v) - \rho \Phi^{-1}(t)}{\sqrt{1-\rho^2}} \right) dt. \quad (3.12)$$

In fact let  $X = (U, V)$  be a standard Gaussian random vector with correlation  $\rho$ , then we have:

$$\Phi_2(u, v) = Pr(U \leq u, V \leq v) = \int_{-\infty}^u \int_{-\infty}^v \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left( \frac{2\rho st - s^2 - t^2}{2(1-\rho^2)} \right) ds dt.$$

This implies

$$\frac{\partial \Phi_2(u, v)}{\partial u} = \phi(u) \cdot \Phi \left( \frac{v - u\rho}{\sqrt{1-\rho^2}} \right),$$

where  $\phi$  denotes the density function of the standard univariate normal distribution. Therefore, we have

$$\Phi_2(u, v) = \int_{-\infty}^u \phi(x) \cdot \Phi\left(\frac{v - x\rho}{\sqrt{1 - \rho^2}}\right) dx.$$

The expression of the bivariate Gaussian copula is thus

$$C_\rho(u, v) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v), \rho) = \int_{-\infty}^{\Phi^{-1}(u)} \phi(x) \cdot \Phi\left(\frac{\Phi^{-1}(v) - x\rho}{\sqrt{1 - \rho^2}}\right) dx.$$

By making the substitution  $t = \Phi(x)$ , we obtain

$$C_\rho(u, v) = \int_0^u \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(t)}{\sqrt{1 - \rho^2}}\right) dt.$$

According to Theorem 3.2 we have:

$$g(v, u) = \frac{\partial C_\rho(u, v)}{\partial u} = \Phi\left(\frac{\Phi^{-1}(v) - \rho\Phi^{-1}(u)}{\sqrt{1 - \rho^2}}\right). \quad (3.13)$$

The function  $g(v, u)$  is strictly monotone with respect to  $v$  and its inverse is given by

$$g^{-1}(\alpha, u) = \Phi\left(\rho\Phi^{-1}(u) + \sqrt{1 - \rho^2}\Phi^{-1}(\alpha)\right). \quad (3.14)$$

Thus by Equation 3.3 it follows that

$$CoVaR_\alpha^{s|L^i=l} = F_s^{-1}\left(\Phi\left(\rho\Phi^{-1}(F_i(l)) + \sqrt{1 - \rho^2}\Phi^{-1}(\alpha)\right)\right).$$

□

*Remark 3.10.* If we set in Equation (3.14)  $\rho = 0$  (i.e. we assume that the financial institution  $i$  and the financial system  $s$  are not correlated) we obtain

$$g^{-1}(\alpha, u) = \alpha, \quad u \in [0, 1].$$

Hence  $CoVaR_\alpha^{s|L^i=VaR^i} = l$  for all  $l \in \mathbb{R}$ . Consequently we have that  $\Delta CoVaR_\alpha^{s|i}$  is equal to Zero (i.e. there is no systemic risk contribution from  $i$  to  $s$ ) if the  $i$  and  $s$  are uncorrelated.

*Remark 3.11.* In the context of Remark 3.3, we have

$$\tilde{\alpha} = \Phi\left(\rho\Phi^{-1}(F_i(l)) + \sqrt{1 - \rho^2}\Phi^{-1}(\alpha)\right). \quad (3.15)$$

By Corollary 3.5 we have

$$\begin{aligned} \Delta CoVaR_\alpha^{s|i} &= F_s^{-1}\left(\Phi\left(\rho\Phi^{-1}(\alpha) + \sqrt{1 - \rho^2}\Phi^{-1}(\alpha)\right)\right) \\ &\quad - F_s^{-1}\left(\Phi\left(\rho\Phi^{-1}(F_i(\mu_i)) + \sqrt{1 - \rho^2}\Phi^{-1}(\alpha)\right)\right) \end{aligned}$$

and if  $F_i$  is symmetric, then

$$\begin{aligned} \Delta CoVaR_\alpha^{s|i} &= F_s^{-1} \left( \Phi \left( \rho \Phi^{-1}(\alpha) + \sqrt{1 - \rho^2} \Phi^{-1}(\alpha) \right) \right) \\ &\quad - F_s^{-1} \left( \Phi \left( \rho \Phi^{-1}(0.5) + \sqrt{1 - \rho^2} \Phi^{-1}(\alpha) \right) \right) \\ &= F_s^{-1} \left( \Phi \left( \rho \Phi^{-1}(\alpha) + \sqrt{1 - \rho^2} \Phi^{-1}(\alpha) \right) \right) - F_s^{-1} \left( \Phi \left( \sqrt{1 - \rho^2} \Phi^{-1}(\alpha) \right) \right). \end{aligned}$$

It is important to remark that the distributions functions  $F_i$  and  $F_s$  can be assumed to be any type of univariate distribution function satisfying Assumption 1.3.

Let us consider in the rest of this section the particular case where  $L^i$  and  $L^s$  are both an univariate normal distributed with expected values  $\mu_i$ ,  $\mu_s$  and standard deviation  $\sigma_i$ ,  $\sigma_s$  respectively. Let  $N_i$  and  $N_s$  be the distribution function of  $L^i$  and  $L^s$  respectively i.e.  $N_i := N(\mu_i, \sigma_i^2)$  and  $N_s := N(\mu_s, \sigma_s^2)$ . The formula for  $CoVaR_\alpha^{s|L^i=l}$  is given by the following corollary.

**Corollary 3.12.**

$$CoVaR_\alpha^{s|L^i=l} = \rho \frac{\sigma_s}{\sigma_i} (l - \mu_i) + \sqrt{1 - \rho^2} \sigma_s \Phi^{-1}(\alpha) + \mu_s. \quad (3.16)$$

*Proof.* By Theorem 3.2 we have:

$$\begin{aligned} CoVaR_\alpha^{s|L^i=l} &= N_s^{-1} \left( \Phi \left( \rho \Phi^{-1}(N_i(l)) + \sqrt{1 - \rho^2} \Phi^{-1}(\alpha) \right) \right) \\ &= N_s^{-1} \left( N_s \left( \sigma_s \rho \Phi^{-1}(N_i(l)) + \sigma_s \sqrt{1 - \rho^2} \Phi^{-1}(\alpha) + \mu_s \right) \right) \\ &= \rho \frac{\sigma_s}{\sigma_i} (l - \mu_i) + \sqrt{1 - \rho^2} \sigma_s \Phi^{-1}(\alpha) + \mu_s. \end{aligned}$$

□

*Remark 3.13.* The last case considered above was a combination of a bivariate Gaussian copula with two univariate Gaussian distributed margins. This case was already analysed by Jäger-Ambrożewicz in [12]. Differently from the method provided here, Jäger-Ambrożewicz derived a closed formula for  $CoVaR_\alpha^{s|L^i=VaR_i}$  by using the expression of the conditional probability for bivariate normal distribution (cf. e.g. [8] Equation 2.6). Equation (3.16) coincides with the formula provided in [12] showing thus that the formula proposed by Jäger-Ambrożewicz is a particular case of the formula provided here in Theorem 3.2. We will not further consider this particular case here and remark that for this case also the expressions of  $CoVaR_\alpha^{s|L^i=l}$  and  $\Delta CoVaR$  can be derived from Property 3.4 b).

#### 4. Tail Events and Systemic Crisis

Recall that the main idea of the measurement of systemic risk contribution through  $CoVaR$  method is to capture the potential for the spreading of financial distress across financial institutions by estimating the increase in tail co-movement (cf. [2]). Hence, in the context of the analysis and the measurement of systemic risk the dependence between the financial institution  $i$  and the financial system  $s$  have to be considered in the tail of their joint distribution. It is thus important to quantify the extreme (or tail) dependence of  $i$  and  $s$  when the systemic risk

contribution of  $i$  is analysed. This can be done using the so called tail dependence coefficients.

**Definition 4.1** (cf. [15] Definition 5.30). Let  $(X, Y)$  be a bivariate random variable with marginal distribution functions  $F$  and  $G$ , respectively. The upper tail dependence coefficient of  $X$  and  $Y$  is the limit (if it exists) of the conditional probability that  $Y$  is greater than the  $100\alpha - th$  percentile of  $G$  given that  $X$  is greater than the  $100\alpha - th$  percentile of  $F$  as  $\alpha$  approaches 1, i.e.

$$\lambda_u := \lim_{\alpha \rightarrow 1^-} \lambda_u(\alpha), \text{ with } \lambda_u(\alpha) := Pr(Y > G^{-1}(\alpha) | X > F^{-1}(\alpha)). \quad (4.1)$$

If  $\lambda_u \in (0, 1]$ , then  $(X, Y)$  is said to show upper tail dependence or extremal dependence in the upper tail; if  $\lambda_u = 0$ , they are asymptotically independent in the upper tail. Similarly, the lower tail dependence coefficient  $\lambda_l$  is the limit (if it exists) of the conditional probability that  $Y$  is less than or equal to the  $100\alpha - th$  percentile of  $G$  given that  $X$  is less than or equal to the  $100\alpha - th$  percentile of  $F$  as  $\alpha$  approaches 0, i.e.

$$\lambda_l := \lim_{\alpha \rightarrow 0^+} \lambda_l(\alpha), \text{ with } \lambda_l(\alpha) := \lim_{\alpha \rightarrow 0^+} Pr(Y \leq G^{-1}(\alpha) | X \leq F^{-1}(\alpha)). \quad (4.2)$$

Note that  $\lambda_u$  measures the probability that  $Y$  exceeds the threshold  $G^{-1}(\alpha)$ , conditional on that  $X$  exceeds the threshold  $F^{-1}(\alpha)$ . In other words,  $\lambda_u$  measures the tendency for extreme events to occur simultaneously.

If  $(L^i, L^s)$  does not show tail dependence ( $\lambda_u = \lambda_l = 0$ ), the extreme events of  $L^i$  and  $L^s$  appear to occur independently in each margin. This means that there is no systemic risk contribution between  $i$  and  $s$ .

**Proposition 4.2.** *cf. [6] Providing that they exist the upper and lower tail dependence coefficient can be expressed in term of copula as follows:*

$$\lambda_u = \lim_{u \rightarrow 1^-} \frac{1 - 2u + C(u, u)}{1 - u} \quad (4.3)$$

and

$$\lambda_l = \lim_{u \rightarrow 0^+} \frac{C(u, u)}{u}. \quad (4.4)$$

The tail dependence coefficient of the Gaussian Copula is given by (cf.[6])

$$\begin{aligned} \lambda_u &= 2 \lim_{\alpha \rightarrow 1^-} \left[ 1 - \Phi \left( \frac{\Phi^{-1}(\alpha) - \rho \Phi^{-1}(\alpha)}{\sqrt{1 - \rho^2}} \right) \right] \\ &= 2 \lim_{\alpha \rightarrow 1^-} \left[ 1 - \Phi \left( \frac{\Phi^{-1}(\alpha) \sqrt{1 - \rho}}{\sqrt{1 + \rho}} \right) \right]. \end{aligned}$$

It follows that

$$\lambda_u = \begin{cases} 0 & \text{if } \rho < 1 \\ 1 & \text{if } \rho = 1. \end{cases}$$

Therefore if we assume the bivariate Gaussian copula as the dependence model for  $(L^i, L^s)$ , then, regardless of how high a correlation we choose, if we go far enough into the tail, extreme events appear to occur independently in  $L^i$  and  $L^s$ .

This means that the Gaussian copula is related to the independence in the tail and hence does not capture tail co-movements. This presents a big gap since tail events especially tail co-movements are the main features of systemic financial crisis (cf. [1]). This is the reason why we connect the *CoVaR* concept to copula's theory in order to develop an analytical formula for  $CoVaR_\alpha^{s|L^i=l}$  allowing the analysis and the computation of systemic risk contribution for a more general stochastic setting than only the bivariate Gaussian setting. Our formula in Theorem 3.2 allows to consider other dependence models, especially those which are appropriate for the modelling of the simultaneous tail behaviour of losses during a financial crisis. It is also more flexible in the sense that it allows each margin independently of other to take a large class of distributions functions (for example we can assume that  $L^i$  is  $t$ -distributed and that  $L^s$  is normal distributed).

## 5. Applications to Non-Gaussian Copulas

In this section we apply the formula provided in Theorem 3.2 to non-Gaussian copulas. Especially we consider the bivariate  $t$ -copula as special case of the class of bivariate elliptical copula. We also consider the case of Archimedean copula and the convex combination of copula.

**Elliptical copulas** are the most used copulas in modern finance and risk-management. They are derived from multivariate elliptical distributions function using the Sklar's theorem (see Corollary 2.5). The two most important elliptical copulas are the Gaussian and the  $t$  copulas (student's copula). Both have in their central part the same behaviour and properties as the multivariate normal distribution (This is one reason of their popularity), but show different behaviours in the tail.

The **Student  $t$  copula** can be considered as a generalization of the normal copula allowing the consideration of tail-dependence. It has in addition to the correlation coefficient  $\rho$  a second dependence parameter, the degree of freedom  $\nu$ , which controls the heaviness of the tails.

**Definition 5.1.** The *distribution function of a bivariate  $t$  distributed random variable* with correlation coefficient  $\rho$  is given by:

$$t_{\rho,\nu}(u, v) = \int_{-\infty}^u \int_{-\infty}^v \frac{1}{2\pi\sqrt{1-\rho^2}} \left( 1 + \frac{s^2 + t^2 - 2\rho^t st}{\nu(1-\rho^2)} \right)^{-\frac{\nu+2}{2}} dsdt,$$

where  $\nu$  denotes the number of degrees of freedom.

For  $\nu < 3$  the variance does not exist, and for  $\nu < 5$  the fourth moment does not exist. The  $t$  copula and the Gaussian copula are close to each other in their central part, and become closer and closer in their tail only when  $\nu$  increases. Especially both copulas are almost identical when  $\nu \rightarrow \infty$ .

**Definition 5.2.** The *bivariate  $t$  copula*,  $C_{\rho,\nu}^t$  is defined as

$$\begin{aligned} C_{\rho,\nu}^t(u, v) &= t_{\rho,\nu}(t_\nu^{-1}(u), t_\nu^{-1}(v)) \\ &= \int_{-\infty}^{t_\nu^{-1}(u)} \int_{-\infty}^{t_\nu^{-1}(v)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left( 1 + \frac{s^2 + t^2 - 2\rho^t st}{\nu(1-\rho^2)} \right)^{-\frac{\nu+2}{2}} dsdt, \end{aligned}$$

where  $t_\nu$  denotes the distribution function of a standard  $t$  with  $\nu$  degrees of freedom univariate distributed random variable.

The tail dependence coefficients of the  $t$  Copula  $C_{\rho,\nu}^t$  is given by ([6])

$$\lambda_l = \lambda_u = 2 - 2t_{\nu+1} \left( \left( \frac{(\nu+1)(1-\rho)}{1+\rho} \right)^{\frac{1}{2}} \right).$$

It follows that,

$$\lambda_u = \begin{cases} > 0 & \text{if } \rho > -1 \\ 0 & \text{if } \rho = -1 \end{cases}.$$

So, provided that  $\rho > -1$ , the bivariate  $t$  copula is able to capture the dependence of extreme values and is thus appropriate for the modelling and the analysis of systemic risk contribution.

The  $t$  copula  $C_{\rho,\nu}^t(u, v)$  can be expressed as follows (cf. e.g. [17] Page 299) :

$$C_{\rho,\nu}^t(u, v) = \int_0^u t_{\nu+1} \left( \left( \frac{\nu+1}{\nu + [t_\nu^{-1}(u)]^2} \right)^{1/2} \frac{t_\nu^{-1}(v) - \rho t_\nu^{-1}(t)}{\sqrt{1-\rho^2}} \right) dt. \quad (5.1)$$

Now based on Theorem 3.2 we compute the expression of  $g(v, u)$ . We obtain

$$\begin{aligned} g(v, u) &= \frac{\partial C_{\rho,\nu}^t(u, v)}{\partial u} \\ &= t_{\nu+1} \left( \left( \frac{\nu+1}{\nu + [t_\nu^{-1}(u)]^2} \right)^{1/2} \frac{t_\nu^{-1}(v) - \rho t_\nu^{-1}(u)}{\sqrt{1-\rho^2}} \right). \end{aligned}$$

The function  $g$  is invertible and its inverse is obtained by solving the equation  $g(v, u) = \alpha$  for  $v$ . This leads to,

$$v = g^{-1}(\alpha, u) = t_\nu \left( \rho t_\nu^{-1}(u) + \sqrt{\frac{(1-\rho^2)(\nu + [t_\nu^{-1}(u)]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right).$$

From this, we obtain the following formula for  $CoVaR_\alpha^{s|L^i=L^l}$  and  $CoVaR_\alpha^\beta$

**Proposition 5.3.** *Let the  $t$  copula be the copula of  $(L^i, L^s)$ , then for every  $l \in \mathbb{R}$*

$$\begin{aligned} CoVaR_\alpha^{s|L^i=L^l} &= \\ &F_s^{-1} \left( t_\nu \left( \rho t_\nu^{-1}(F_i(l)) + \sqrt{\frac{(1-\rho^2)(\nu + [t_\nu^{-1}(F_i(l))]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) \right) \end{aligned}$$

and

$$CoVaR_\alpha^\beta = F_s^{-1} \left( t_\nu \left( \rho t_\nu^{-1}(\beta) + \sqrt{\frac{(1-\rho^2)(\nu + [t_\nu^{-1}(\beta)]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) \right),$$

where  $F_i$  and  $F_s$  represent the univariate distribution function of  $L^i$  and  $L^s$  respectively and  $\beta$  denotes the regulatory risk level of the financial institution  $i$ .

**Corollary 5.4.** *Assume that  $L^i$  and  $L^s$  each follow an univariate standard  $t$  distribution with  $\nu$  degrees of freedom, then for every  $l \in \mathbb{R}$*

$$\begin{aligned} CoVaR_\alpha^{s|L^i=l} &= \rho l + \sqrt{\frac{(1-\rho^2)(\nu+l^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \\ \Delta CoVaR_\alpha^{s|i}(l_1, l_2) &= \rho VaR_\alpha^i + \sqrt{\frac{(1-\rho^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) [\nu + VaR_\alpha^i - \nu] \\ &= VaR_\alpha^i \left[ \rho + \sqrt{\frac{(1-\rho^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right]. \end{aligned}$$

We use here the fact that the standard  $t$  distribution with  $\nu$  degree of freedom has a mean equal to zero (and a variance equal to  $\frac{\nu}{\nu-2}$ ).

The standard  $t$  distribution can be extended through linear transformation of the form

$$X := a + bZ, \quad Z \sim t_\nu.$$

The distribution of  $X$  is called generalized  $t$  distribution ( $X \sim T(a, b^2, \nu)$ ). The mean of  $X$  is equal to  $a$  ( $E[X] = a$ ) and its variance  $V[X]$  is given by

$$V[X] = b^2 V[Z] = b^2 \frac{\nu}{\nu-2}.$$

The corresponding density  $f_T$  is obtained using the Transformation formula for density (cf. e.g. [14], theorem. 1.101). Let  $f_t$  be the density function of standard  $t$  distribution, then

$$f_T(x) = f_t(g(z)) = \frac{f_t(g^{-1}(z))}{|g'(g^{-1}(z))|}, \quad \text{with } g(z) = a + bz$$

such that

$$f_T(x) = f_t\left(\frac{x-a}{b}\right) \left|\frac{1}{b}\right|, \quad b \neq 0.$$

We have that

$$T(x) = Pr(X \leq x) = Pr(a + bZ \leq x) = t_\nu\left(\frac{x-a}{b}\right).$$

$$\begin{aligned} T(x) = \alpha &\Leftrightarrow \frac{x-a}{b} = t_\nu^{-1}(\alpha) \\ &\Leftrightarrow x = bt_\nu^{-1}(\alpha) + a \\ &\Leftrightarrow T^{-1}(\alpha) = bt_\nu^{-1}(\alpha) + a. \end{aligned} \tag{5.2}$$

Let us now consider the following cases:

- (1) Let  $\mu_s := E(L^s)$ . If  $L^s \sim T(\mu_s, \sigma_s^{*2}, \nu)$ , i.e.  $\left(\frac{L^s - \mu_s}{\sigma_s^*}\right)$  follows an univariate standard  $t$  distribution with  $\nu$  degrees, then

$$\begin{aligned} CoVaR_\alpha^{s|i}(l) &= \\ F_s^{-1} \left( t_\nu \left( \rho t_\nu^{-1}(F_i(l)) + \sqrt{\frac{(1-\rho^2)(\nu + [t_\nu^{-1}(F_i(l))]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) \right) &= \\ \sigma_s^* t_\nu^{-1} \left( t_\nu \left( \rho t_\nu^{-1}(F_i(l)) + \sqrt{\frac{(1-\rho^2)(\nu + [t_\nu^{-1}(F_i(l))]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) \right) + \mu_s &= \\ \sigma_s^* \left( \rho t_\nu^{-1}(F_i(l)) + \sqrt{\frac{(1-\rho^2)(\nu + [t_\nu^{-1}(F_i(l))]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) + \mu_s & \end{aligned}$$

- (2) Let  $\mu_i := E(L^i)$  and  $\mu_s := E(L^s)$ . If

$$L^i \sim T(\mu_i, \sigma_i^{*2}, \nu), \quad L^s \sim T(\mu_s, \sigma_s^{*2}, \nu),$$

i.e.  $\left(\frac{L^i - \mu_i}{\sigma_i^*}\right)$  and  $\left(\frac{L^s - \mu_s}{\sigma_s^*}\right)$  each follows an univariate standard  $t$  distribution with  $\nu$  degrees of freedom, then

$$\begin{aligned} CoVaR_\alpha^{s|i}(l) &= \\ \sigma_s^* \left( \rho t_\nu^{-1}(F_i(l)) + \sqrt{\frac{(1-\rho^2)(\nu + [t_\nu^{-1}(F_i(l))]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) + \mu_s &= \\ \sigma_s^* \left( \rho t_\nu^{-1} \left( t_\nu \left( \frac{l - \mu_i}{\sigma_i^*} \right) \right) + \sqrt{\frac{(1-\rho^2)(\nu + [t_\nu^{-1}(t_\nu(\frac{l - \mu_i}{\sigma_i^*}))]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) & \\ + \mu_s = \sigma_s^* \left( \rho \left( \frac{l - \mu_i}{\sigma_i^*} \right) + \sqrt{\frac{(1-\rho^2)(\nu + (\frac{l - \mu_i}{\sigma_i^*})^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) & \\ + \mu_s = \frac{\sigma_s^* \rho}{\sigma_i^*} (l - \mu_i) + \sigma_s^* t_{\nu+1}^{-1}(\alpha) \sqrt{\frac{(1-\rho^2)(\nu + (\frac{l - \mu_i}{\sigma_i^*})^2)}{\nu+1}} + \mu_s. & \end{aligned}$$



By (1.8) we have that for given  $l_1, l_2 \in \mathbb{R}$

$$\begin{aligned}
& \Delta CoVaR_{\alpha}^{s|i}(l_1, l_2) \\
&= CoVaR_{\alpha}^{s|i}(l_1) - CoVaR_{\alpha}^{s|i}(l_2) \\
&= F_s^{-1} \left( t_{\nu} \left( \rho t_{\nu}^{-1}(F_i(l_1)) + \sqrt{\frac{(1-\rho^2)(\nu + [t_{\nu}^{-1}(F_i(l_1))]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) \right) \\
&\quad - \left[ F_s^{-1} \left( t_{\nu} \left( \rho t_{\nu}^{-1}(F_i(l_2)) + \sqrt{\frac{(1-\rho^2)(\nu + [t_{\nu}^{-1}(F_i(l_2))]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) \right) \right].
\end{aligned}$$

Consider again the previous two cases

(1) If  $L^s \sim T(\mu_s, \sigma_s^{*2}, \nu)$ , then

$$\begin{aligned}
& \Delta CoVaR_{\alpha}^{s|i}(l_1, l_2) = \\
& \sigma_s^* \left( \rho t_{\nu}^{-1}(F_i(l_1)) + \sqrt{\frac{(1-\rho^2)(\nu + [t_{\nu}^{-1}(F_i(l_1))]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) + \mu_s \\
& - \left[ \sigma_s^* \left( \rho t_{\nu}^{-1}(F_i(l_2)) + \sqrt{\frac{(1-\rho^2)(\nu + [t_{\nu}^{-1}(F_i(l_2))]^2)}{\nu+1}} t_{\nu+1}^{-1}(\alpha) \right) + \mu_s \right] \\
&= \sigma_s^* \rho (t_{\nu}^{-1}(F_i(l_1)) - t_{\nu}^{-1}(F_i(l_2))) \\
& \quad + \frac{\sigma_s^* t_{\nu+1}^{-1}(\alpha) \sqrt{1-\rho^2}}{\sqrt{\nu+1}} ([t_{\nu}^{-1}(F_i(l_1))]^2 - [t_{\nu}^{-1}(F_i(l_2))]^2) \\
&= \sigma_s^* (t_{\nu}^{-1}(F_i(l_1)) - t_{\nu}^{-1}(F_i(l_2))) \\
& \quad \times \left[ \rho + \frac{t_{\nu+1}^{-1}(\alpha) \sqrt{1-\rho^2}}{\sqrt{\nu+1}} (t_{\nu}^{-1}(F_i(l_1)) + t_{\nu}^{-1}(F_i(l_2))) \right].
\end{aligned}$$

(2) If  $L^i \sim T(\mu_i, \sigma_i^{*2}, \nu)$  and  $L^s \sim T(\mu_s, \sigma_s^{*2}, \nu)$ , then

$$\begin{aligned}
\Delta \text{CoVaR}_\alpha^{s|i}(l_1, l_2) &= \frac{\sigma_s^* \rho}{\sigma_i^*} (l - \mu_i) + \sigma_s^* t_{\nu+1}^{-1}(\alpha) \sqrt{\frac{(1-\rho^2) \left( \nu + \left( \frac{l-\mu_i}{\sigma_i^*} \right)^2 \right)}{\nu+1}} + \mu_s \\
&\quad - \left[ \frac{\sigma_s^* \rho}{\sigma_i^*} (l - \mu_i) + \sigma_s^* t_{\nu+1}^{-1}(\alpha) \sqrt{\frac{(1-\rho^2) \left( \nu + \left( \frac{l-\mu_i}{\sigma_i^*} \right)^2 \right)}{\nu+1}} + \mu_s \right] \\
&= \frac{\sigma_s^* \rho}{\sigma_i^*} (l_1 - l_2) + \frac{\sigma_s^* t_{\nu+1}^{-1}(\alpha) \sqrt{1-\rho^2}}{\sqrt{\nu+1}} \left( \left( \frac{l_1 - \mu_i}{\sigma_i^*} \right)^2 - \left( \frac{l_2 - \mu_i}{\sigma_i^*} \right)^2 \right) \\
&= \frac{\sigma_s^* \rho}{\sigma_i^*} (l_1 - l_2) + \frac{\sigma_s^* t_{\nu+1}^{-1}(\alpha) \sqrt{1-\rho^2}}{\sqrt{\nu+1}} \left( \frac{l_1 - l_2}{\sigma_i^*} \right) \left( \frac{l_1 + l_2 - 2\mu_i}{\sigma_i^*} \right) \\
&= \frac{\sigma_s^*}{\sigma_i^*} (l_1 - l_2) \left[ \rho + \frac{t_{\nu+1}^{-1}(\alpha) \sqrt{1-\rho^2}}{\sqrt{\nu+1}} \left( \frac{l_1 + l_2 - 2\mu_i}{\sigma_i^*} \right) \right]. \tag{5.3}
\end{aligned}$$

**Corollary 5.5.** Let  $\mu_i := E(L^i)$  and  $\mu_s := E(L^s)$ . If  $L^i \sim T(\mu_i, \sigma_i^{*2}, \nu)$  and  $L^s \sim T(\mu_s, \sigma_s^{*2}, \nu)$ , i.e.  $\left( \frac{L^i - \mu_i}{\sigma_i^*} \right)$  and  $\left( \frac{L^s - \mu_s}{\sigma_s^*} \right)$  each follows an univariate standard  $t$  distribution with  $\nu$  degrees of freedom, then

$$\Delta \text{CoVaR}_\alpha^{s|L^i=l} = \sigma_s^* t_\nu^{-1}(\alpha) \left[ \rho + t_\nu^{-1}(\alpha) \frac{t_{\nu+1}^{-1}(\alpha) \sqrt{1-\rho^2}}{\sqrt{\nu+1}} \right]. \tag{5.4}$$

*Proof.* By (5.3) we have that

$$\begin{aligned}
\Delta \text{CoVaR}_\alpha^{s|L^i=l} &= \Delta \text{CoVaR}_\alpha^{s|i}(VaR_\alpha^i, \mu_i) \\
&= \frac{\sigma_s^*}{\sigma_i^*} (VaR_\alpha^i - \mu_i) \left[ \rho + \frac{t_{\nu+1}^{-1}(\alpha) \sqrt{1-\rho^2}}{\sqrt{\nu+1}} \left( \frac{VaR_\alpha^i + \mu_i - 2\mu_i}{\sigma_i^*} \right) \right].
\end{aligned}$$

Since the Value-at-Risk can be expressed in terms of a quantile (see (1.2)), it follows from (5.2) that

$$VaR_\alpha^i = \mu_i + \sigma_i^* t_\nu^{-1}(\alpha).$$

$$\begin{aligned}
\Delta \text{CoVaR}_\alpha^{s|L^i=l} &= \frac{\sigma_s^*}{\sigma_i^*} (\mu_i + \sigma_i^* t_\nu^{-1}(\alpha) - \mu_i) \\
&\quad \times \left[ \rho + \frac{t_{\nu+1}^{-1}(\alpha) \sqrt{1-\rho^2}}{\sqrt{\nu+1}} \left( \frac{\mu_i + \sigma_i^* t_\nu^{-1}(\alpha) + \mu_i - 2\mu_i}{\sigma_i^*} \right) \right] \\
&= \sigma_s^* t_\nu^{-1}(\alpha) \left[ \rho + t_\nu^{-1}(\alpha) \frac{t_{\nu+1}^{-1}(\alpha) \sqrt{1-\rho^2}}{\sqrt{\nu+1}} \right].
\end{aligned}$$

□

Note that the dependence in the Gaussian and t-copulas setting are essentially determined by the correlation coefficient  $\rho$  (elliptical copula). The correlation coefficient is often considered as being a poor tool for describing dependence when the margins are non-normal (cf. [15]). This motivates the use of **Archimedean copula**.

Unlike as in the elliptical copulas, the dependence in a bivariate Archimedean copula is not controlled by a constant (the correlation parameter) but by a function  $\varphi$  called generator. This gives to Archimedean copulas good analytical properties, and the ability to reproduce a large spectrum of dependence structures.

**Theorem 5.6.** [[16], Theorem 4.1.4] *Let  $\varphi$  be a continuous, strictly decreasing function from  $[0, 1]$  to  $[0, \infty]$  such that  $\varphi(1) = 0$ , and let  $\varphi^{[-1]}(t)$  be the pseudo-inverse of  $\varphi$  defined by*

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t) & \text{if } 0 \leq t \leq \varphi(0) \\ 0 & \text{if } \varphi(0) < t \leq \infty. \end{cases},$$

then the function  $C$  from  $[0, 1]^2$  to  $[0, 1]$  given by

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)). \quad (5.5)$$

is a copula if and only if  $\varphi$  is convex.

Note that the composition of the pseudo-inverse with the generator gives the identity i.e.

$$\varphi^{[-1]}(\varphi(t)) = t. \quad \forall t \in [0, \infty].$$

If  $\varphi(0) = \infty$  the generator is said to be strict and its pseudo-inverse  $\varphi^{[-1]}$  coincide with the ordinary functional inverse  $\varphi^{-1}$  (cf. [16] Definition 4.1.1).

**Definition 5.7.** A function  $\phi$  satisfying the conditions in Theorem 5.6 is called *generator of a copula*. A copula constructed through a generator is called Archimedean copula.

The lower and upper tail dependence coefficient of an Archimedean copula can be computed using the following corollary.

**Corollary 5.8** ([16] Corollary. 5.4.3). *Let  $C$  be an Archimedean copula with a continuous, strictly, decreasing and convex generator  $\varphi$ , then*

$$\lambda_u = 2 - \lim_{x \rightarrow 0^+} \frac{1 - \varphi^{-1}(2x)}{1 - \varphi^{-1}(x)} \quad \text{and} \quad \lambda_l = \lim_{x \rightarrow \infty} \frac{1 - \varphi^{-1}(2x)}{1 - \varphi^{-1}(x)}$$

In the context of systemic risk analysis, we are interested by Archimedean copulas showing positive (upper or lower) tail dependence (e.g. Gumbel and Clayton copula).

*Remark 5.9.* In the case we assume copula with positive upper (lower) tail dependence, the loss have to be defined as a positive (negative) number (cf. [15]).

**Example 5.10** (Gumbel Copula). The generator of the Gumbel copula is defined by

$$\varphi_\theta(t) = (-\ln(t))^\theta \quad \text{for } \theta \geq 1. \quad (5.6)$$

It holds  $\varphi_\theta(0) = \infty$ , i.e.  $\varphi_\theta$  is strict and its inverse is  $\varphi_\theta^{-1}(t) = \exp\left(-t^{\frac{1}{\theta}}\right)$ . The Gumbel copula is then according to 5.5 given by:

$$C_\theta^{Gu}(u, v) = \exp\left(-\left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{\frac{1}{\theta}}\right), \quad 1 \leq \theta < \infty,$$

where  $\theta$  represents the strength of dependence. By Corollary 5.8, the tail dependence coefficients of the Gumbel copula are given by:

$$\lambda_u = 2 - 2^{\frac{1}{\theta}} \quad \text{and} \quad \lambda_l = 0.$$

The Gumbel copula is thus able to model contagion effect and is therefore a good alternative model for the analysis of systemic risk contribution.

According to Theorem 3.2 the corresponding function  $g$  is given by:

$$g_{Gu}(v, u) := \frac{\partial C_\theta^{Gu}(u, v)}{\partial u} = \exp\left(-\left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{\frac{1}{\theta}}\right) \\ \times \left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{-\frac{\theta-1}{\theta}} \cdot \frac{(-\ln(u))^{\theta-1}}{u}.$$

The function  $g$  is for  $u \in (0, 1)$  and for all  $\theta > 1$  strictly increasing with respect to  $v$  and therefore invertible.

So according to Theorem 3.2 we can compute  $CoVaR_\alpha^{s|L^i=l}$  by

$$CoVaR_\alpha^{s|L^i=l} = F_s^{-1}\left(g_{Gu}^{-1}(\alpha, F_i(l))\right). \quad (5.7)$$

By imposing some conditions to the generator  $\varphi$  of an Archimedean Copula, we can derive, using Theorem 3.2, an explicit expression of  $CoVaR_\alpha^{s|L^i=l}$  in terms of  $\varphi$ .

**Proposition 5.11.** *Assume that the copula  $C$  associated to the joint distribution of  $(L^i, L^s)$  is a bivariate Archimedean copula with generator  $\varphi$ . If  $\varphi$  is strict and its derivative  $\varphi'$  is invertible, then the explicit formula for  $CoVaR_\alpha^{s|L^i=l}$  for a given level  $\alpha, \alpha \in (0, 1)$  is given by*

$$CoVaR_\alpha^{s|L^i=l} = F_s^{-1}\left(\varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(F_i(l))}{\alpha}\right)\right) - \varphi(F_i(l))\right)\right). \quad (5.8)$$

*Proof.* In fact, let  $C$  be an Archimedean copula with a strict generator  $\varphi$  such that

$$C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v))$$

and it holds

$$\varphi(C(u, v)) = \varphi(u) + \varphi(v). \quad (5.9)$$

Hence,

$$\frac{\partial[\varphi(C(u, v))]}{\partial u} = \frac{\partial[\varphi(u) + \varphi(v)]}{\partial u}$$

i.e.

$$\frac{\partial C(u, v)}{\partial u} \cdot \varphi'(C(u, v)) = \frac{\partial \varphi(u)}{\partial u} = \varphi'(u)$$

it follows that

$$\frac{\partial C(u, v)}{\partial u} = \frac{\varphi'(u)}{\varphi'(C(u, v))} = \frac{\varphi'(u)}{\varphi'(\varphi^{-1}[\varphi(u) + \varphi(v)])}.$$

We have thus

$$g(v, u) = \frac{\partial C(u, v)}{\partial u} = \frac{\varphi'(u)}{\varphi'(\varphi^{-1}[\varphi(u) + \varphi(v)])}.$$

Now, set  $g(v, u) = \alpha$  and solve for  $v$ . If  $\varphi'$  is invertible we obtain

$$g^{-1}(\alpha, u) = \varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(u)}{\alpha}\right)\right) - \varphi(u)\right),$$

and by Theorem 3.2 we have

$$\begin{aligned} CoVaR_{\alpha}^{s|L^i=l} &= F_s^{-1}\left(g^{-1}(\alpha, F_i(l))\right) \\ &= F_s^{-1}\left(\varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(F_i(l))}{\alpha}\right)\right) - \varphi(F_i(l))\right)\right). \end{aligned}$$

□

**Corollary 5.12.**

$$CoVaR_{\alpha}^{\beta} = F_s^{-1}\left(\varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(\beta)}{\alpha}\right)\right) - \varphi(\beta)\right)\right).$$

**Example 5.13** (Clayton Copula). The generator of the the Clayton Copula

$$\varphi(t) = \frac{1}{\theta}(t^{-\theta} - 1). \quad \theta \in [-1, \infty) - \{0\}$$

and is strict for  $\theta > 0$ . The Clayton copula can be thus expressed in this case as follows

$$C_{\theta}^{Cl}(u, v) = (u^{-\theta} + v^{-\theta} - 1)^{-\frac{1}{\theta}}, \quad u, v \in (0, 1). \quad (5.10)$$

and we have  $\varphi_{\theta}^{-1}(s) = (1 + \theta s)^{-\frac{1}{\theta}}$ ,  $\varphi'_{\theta}(t) = -t^{-\theta-1}$ ,  $\varphi'^{-1}(z) = -z^{-\frac{1}{\theta+1}}$ . And by Proposition 5.11, we have that

$$\begin{aligned} CoVaR_{\alpha}^{s|L^i=l} &= F_s^{-1}\left(\varphi^{-1}\left(\varphi\left(\varphi'^{-1}\left(\frac{\varphi'(F_i(l))}{\alpha}\right)\right) - \varphi(F_i(l))\right)\right) \\ &= F_s^{-1}\left(\left(1 + F_i(l)^{-\theta}\left(\alpha^{-\frac{\theta}{\theta+1}} - 1\right)\right)^{-\frac{1}{\theta}}\right). \end{aligned}$$

The description of tail dependence structures arising from real financial data is very important for an effective estimation of systemic risk contribution. For this purpose convex combination of copulas are more appropriate than single copula (such as elliptical copula and Archimedean). The **convex combination of copulas** provides more flexibility by the description of tail dependence structures. In fact it is possible to describe a set of different tail dependence structures by combining two or more copulas.

As a bivariate copula can be seen as a specific bivariate distribution, it is clear that the convex linear combination of two copulas is again a copula (see e.g. [16],

Chapter 2). Formally, let  $C_1$  and  $C_2$  be two copulas. Then the function  $C$  defined by

$$C(u, v) := \alpha C_1(u, v) + (1 - \alpha) C_2(u, v), \quad u, v, \alpha \in (0, 1) \text{ is a copula.}$$

The following remark specifies the effect of tail dependence of the underlying copulas on that of their convex combination.

*Remark 5.14.* Let  $C$  be a convex combination of two bivariate copulas  $C_1$  and  $C_2$ . Denote by  $\lambda_u^1$  ( $\lambda_l^1$ ),  $\lambda_u^2$  ( $\lambda_l^2$ ) and  $\lambda_u$  ( $\lambda_l$ ) the upper (lower) tail dependence coefficients of  $C_1$ ,  $C_2$ ,  $C$  respectively, then

$$\lambda_u = \alpha \lambda_u^1 + (1 - \alpha) \lambda_u^2 \quad \text{and} \quad \lambda_l = \alpha \lambda_l^1 + (1 - \alpha) \lambda_l^2. \quad (5.11)$$

In fact, as  $C$  is a copula, its tail dependence coefficients can be computed using Equation (4.3) and (4.4) respectively.

*Remark 5.15.* Let  $C_1$  and  $C_2$  be two copulas satisfying Assumption 1.3. If we assume that the copula  $C$  associated to the joint distribution of  $(L^i, L^s)$  is a convex combination of  $C_1$  and  $C_2$ , then the function  $g(v, u) := \alpha g_1(v, u) + (1 - \alpha) g_2(v, u)$  (where  $g_i(v, u) := \frac{\partial C_i(u, v)}{\partial u}$ ,  $i \in \{1, 2\}$ ) is invertible with respect to the parameter  $v$ , and for all  $l \in \mathbb{R}$  and a given  $\alpha \in (0, 1)$

$$CoVaR_\alpha^{s|L=l} = F_s^{-1}(g^{-1}(\alpha, F_i(l))). \quad (5.12)$$

In fact under Assumption 1.3  $g_1$  and  $g_2$  are each strictly increasing with respect to  $v$  (see Remark 3.1). This implies, that  $g(v, u)$  is also strictly increasing with respect to  $v$  and thus invertible. (5.12) is then obtained by applying Theorem 3.2.

**Example 5.16** (Convex Combination of Clayton and Gumbel Copula). For the Clayton copula  $C_{\theta_1}^{Cl}(u, v) = (u^{-\theta_1} + v^{-\theta_1} - 1)^{-\frac{1}{\theta_1}}$  we have:

$$g_1 := \frac{\partial C_{\theta_1}^{Cl}(u, v)}{\partial u} = u^{-\theta_1 - 1} (u^{-\theta_1} + v^{-\theta_1} - 1)^{-\frac{\theta_1 + 1}{\theta_1}}, \quad \lambda_u = 0 \quad \text{and} \quad \lambda_l = 2^{-\frac{1}{\theta_1}}.$$

Denote by  $C$  the convex combination of the Clayton Copula  $C_{\theta_1}^{Cl}$  and the Gumbel copula  $C_{\theta_2}^{Gu}$

$$C(u, v) := \alpha C_{\theta_1}^{Cl}(u, v) + (1 - \alpha) C_{\theta_2}^{Gu}(u, v), \quad \alpha \in (0, 1).$$

Then by Lemma 5.14 the upper and the lower tail dependence coefficient of  $C$  are given by

$$\lambda_u = \alpha \cdot 0 + (1 - \alpha) \left(2 - 2^{\frac{1}{\theta_2}}\right) = (1 - \alpha) \left(2 - 2^{\frac{1}{\theta_2}}\right).$$

The copula  $C$  has thus positive upper tail dependence coefficient and is hence appropriate for the analysis of systemic risk contribution.

We have that

$$\begin{aligned} \frac{\partial C(u, v)}{\partial u} &= \alpha \left( u^{-\theta_1 - 1} (u^{-\theta_1} + v^{-\theta_1} - 1)^{-\frac{\theta_1 + 1}{\theta_1}} \right) + (1 - \alpha) \times \\ &\left[ e^{\left( -((-\ln(u))^{\theta_2} + (-\ln(v))^{\theta_2})^{\frac{1}{\theta_2}} \right)} \left( (-\ln(u))^{\theta_2} + (-\ln(v))^{\theta_2} \right)^{-\frac{\theta_2 - 1}{\theta_2}} \frac{(-\ln(u))^{\theta_2 - 1}}{u} \right] \\ &= g(v, u). \end{aligned} \quad (5.13)$$

The function  $g(v, u)$  is strictly increasing with respect to  $v$  and hence invertible. Based on this we derive the following Corollary of Theorem 3.2.

**Corollary 5.17.** *If the copula  $C$  of  $(L^i, L^s)$  is a convex combination of the Clayton and the Gumbel Copula, namely*

$$C(u, v) := \alpha C_{\theta_1}^{Cl}(u, v) + (1 - \alpha) C_{\theta_2}^{Gu}(u, v), \quad \alpha \in (0, 1), \quad \theta_1, \theta_2 > 0.$$

Then for a given  $l \in \mathbb{R}$

$$CoVaR_{\alpha}^{s|L=l} = F_s^{-1}(\tilde{\alpha}), \quad (5.14)$$

where  $\tilde{\alpha}$  is the solution of the equation  $g(\tilde{\alpha}, F_i^{-1}(l)) = \alpha$  and  $g$  is given by (5.13).

## 6. Alternative Model for Systemic Risk Contribution

In this section, we first argue that  $\Delta CoVaR$  as defined in [2] by Adrian and Brunnermeier (in this article Definition 1.6) is not consistent with the notion of systemic risk contribution and hence non-adequate for the analysis of systemic risk. Then by changing the way how the condition  $C(L^i)$  is defined, we define alternative financial risk measures which are consistent with the notion of systemic risk contribution and hence more appropriate for the analysis of systemic risk. The reasonable first step towards this is to introduce the notion of "distressed financial Institutions".

**Definition 6.1** (cf. [9] Definition 4.1). Let  $\mathcal{L}$  be the class of all possible losses. A mapping  $\mathcal{R} : \mathcal{L} \rightarrow \mathbb{R}$  is called a *monetary measure of risk* if it satisfies the following conditions for all  $L_1, L_2 \in \mathcal{L}$

- (1) *Monotonicity*: If  $L_1 \leq L_2$ , then  $\mathcal{R}(L_1) \leq \mathcal{R}(L_2)$
- (2) *Cash invariance*: If  $L \in \mathcal{L}$  and  $m \in \mathbb{R}$  then  $\mathcal{R}(L + m) = \mathcal{R}(L) - m$

Monotonicity property means that high losses require high risk capitals. *Cash invariance* property is motivated by the interpretation of  $\mathcal{R}(L)$  as a regulatory capital. It suggests that the regulatory capital associated to a loss  $L$  is reduced by the amount  $l > 0$  if this amount is add to  $L$ .

A loss  $L$  such that  $\mathcal{R}(L) \leq 0$  is called **acceptable**, in the sense that a financial institution whith loss  $L$  is not required by the regulator to keep any regulatory capital. The set of acceptable losses associated to a risk measure  $\mathcal{R}$  is given by

$$\mathcal{A}_{\mathcal{R}} = \{L \in \mathcal{L} \mid \mathcal{R}(L) \leq 0\}.$$

That is, a loss  $L$  is **acceptable** with respect to a risk measure  $\mathcal{R}$  if  $L \in \mathcal{A}_{\mathcal{R}}$ .

Let  $L$  be a non-acceptable loss i.e.  $L \notin \mathcal{A}_{\mathcal{R}}$ . If we add to  $L$  a cash amount of  $\mathcal{R}(L)$ , that is, we define an adjusted loss

$$\tilde{L} := L + \mathcal{R}(L),$$

then by the cash invariance property of monetary risk measure we have that

$$\mathcal{R}(\tilde{L}) = \mathcal{R}(L + \mathcal{R}(L)) = \mathcal{R}(L) - \mathcal{R}(L) = 0$$

so that  $\tilde{L} \in \mathcal{A}_{\mathcal{R}}$ . Hence one can interpret  $\mathcal{R}(L)$  as the minimum amount of capital that a financial institution with loss  $L$  should keep as regulatory capital. Formally,

$$\mathcal{R}(L) = \inf \{m \in \mathbb{R} \mid m + L \in \mathcal{A}_{\mathcal{R}}\}. \quad (6.1)$$

From a purely economic point of view, financial distress may be defined as a situation where a financial institution's operating cash flows are not sufficient to satisfy current obligations (cf. e.g. [18], A7 3.1). From a quantitative risk management perspective we can characterize a distressed financial institution as follows.

**Definition 6.2** (Distressed Financial Institutions). Let  $L$  be the loss incurred by one financial institution  $B$ . Let  $RC$  be the regulatory capital associated to the loss  $L$ . For a given time  $t$  we say that the financial institution  $B$  is in *distress* if at this time the realization  $l$  of  $L$  is greater than the associated regulatory capital  $RC$  i.e.

$$l > RC. \quad (6.2)$$

If we assume that the regulatory capital  $RC$  is determined by the Value-at-Risk, then we say that the financial institution  $B$  is in distress at the time  $t$  if

$$l > \text{Value-at-Risk}. \quad (6.3)$$

The condition  $C(L^i) = \{L^i = VaR_\alpha^i\}$  in Definition 1.6 does not fulfill the default condition 6.3. In fact a loss equal to the Value-at-Risk does not lead to a default. In fact, the financial institution  $i$  is supposed to have a regulatory capital equal to its Value-at-Risk. So, any loss smaller or equal to its Value-at-Risk is absorbed. Such losses can therefore not lead to the default of  $i$  and hence to a systemic risk contribution. It is for this reason that we say that the initial definition of  $\Delta CoVaR_\alpha^{s|i, L^i=l}$  is not consistent with the notion of systemic risk contribution. We propose in the next alternative risk measures which are consistent with the notion of systemic risk contribution.

**Definition 6.3.**

$$ECoVaR_\alpha^{s|i} := E \left[ CoVaR_\alpha^{s|i}(L^i) | L^i \geq VaR^i \right]. \quad (6.4)$$

**Proposition 6.4.**

$$ECoVaR_\alpha^{s|i} = \frac{1}{1 - F_i(VaR^i)} \int_{VaR^i}^{\infty} CoVaR_\alpha^{s|i}(l) f_i(l) dl. \quad (6.5)$$

*Proof.* From basic probability theories, (cf. e.g. [14] Def. 8.9) we have that

$$\begin{aligned} ECoVaR_\alpha^{s|i} &:= E \left( CoVaR_\alpha^{s|i}(L^i) | L^i \geq VaR^i \right) \\ &= \frac{E \left( CoVaR_\alpha^{s|i}(L^i) \mathbf{1}_{\{L^i \geq VaR^i\}} \right)}{Pr(L^i \geq VaR^i)} \\ &= \frac{1}{1 - F_i(VaR^i)} \int_{VaR^i}^{\infty} CoVaR_\alpha^{s|i}(l) f_i(l) dl. \end{aligned}$$

□

*Remark 6.5.* Assume that the confidence level for the calculation of  $VaR^i$  is  $\beta$ , then

$$ECoVaR_\alpha^{s|i} = \frac{1}{1 - \beta} \int_{VaR_\beta^i}^{\infty} CoVaR_\alpha^{s|i}(l) f_i(l) dl. \quad (6.6)$$



**Definition 6.6** (Alternative 2).

$$\Delta CoVaR_{\alpha}^{\geq} = CoVaR_{\alpha}^{s|L^i > VaR_{\alpha}^i} - CoVaR_{\alpha}^{s|L^i = VaR_{\alpha}^i}. \quad (6.7)$$

Definition 6.6 ensures that the considered region is in the distressed region of the loss  $L^i$  incurred by the financial institution  $i$  and is thus a consistent systemic risk measure.

**Definition 6.7.** Assume that  $L^i$  and  $L^s$  have density which satisfy Assumption 1.3. then for a given  $\alpha \in (0, 1)$  and for a fixed  $l$ ,  $CoVaR_{\alpha}^{s|L^i > l}$  is defined as:

$$\begin{aligned} CoVaR_{\alpha}^{s|L^i > l} &:= \inf \{h \in \mathbb{R} : Pr(L^s > h | L^i > l) \leq 1 - \alpha\} \\ &= \inf \{h \in \mathbb{R} : Pr(L^s \leq h | L^i > l) \geq \alpha\} \end{aligned}$$

Such that  $CoVaR_{\alpha}^{s|L^i > VaR_{\alpha}^i}$  is implicitly defined by

$$Pr\left(L^s \leq CoVaR_{\alpha}^{s|L^i > VaR_{\alpha}^i} | L^i > VaR_{\alpha}^i\right) = \alpha, \quad (6.8)$$

which is a special case of generalized  $CoVaR_{\alpha}^{s|C(L^i)}$  (see (1.3)) in which the conditioning event is the default of the financial institution  $i$  (i.e.  $C(L^i) = \{L^i > VaR_{\alpha}^i\}$ ).

*Remark 6.8.* Under Assumption 1.3 the function

$$J(u, v) := u - C(u, v) = C(u, 1).$$

is, for each fixed  $v \in [0, 1]$ , strong monotone increasing and hence invertible as a function of  $u$ . In fact from the boundary condition for 2-dimensional copula (see Definition 2.2), it holds

$$J(u, v) = u - C(u, v) = C(u, 1) - C(u, v). \quad (6.9)$$

Since we assume Assumption 1.3, the copula  $C$  has a strictly positive density  $c$ . By taking this into account

$$\begin{aligned} C(u, 1) - C(u, v) &= \int_0^u \int_0^1 c(x, y) dx dy - \int_0^u \int_0^v c(x, y) dx dy \\ &= \int_0^u \left( \int_v^1 c(x, y) dy \right) dx = J(u, v). \end{aligned} \quad (6.10)$$

So that, for a fixed  $v \in [0, 1]$ , the function  $J(u, v)$  is increasing and thus invertible with respect to  $u$ .

We provide here for some given copula  $C$  the function  $J$ . For this, we need to express the copula density  $c$  of the considered copula  $C$ . Recall that

$$c(u, v) = \frac{\partial^2 C(u, v)}{\partial u \partial v}. \quad (6.11)$$

**Example 6.9** (Gaussian Copula).

By (6.11) the density of the Gaussian copula is given by

$$c_\rho^{Gau}(u, v) = \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2} \left(\frac{x^2 - 2\rho xy + y^2}{1-\rho^2}\right) - x^2 - y^2\right)$$

with  $x = \Phi^{-1}(u)$  and  $y = \Phi^{-1}(v)$ . We have thus

$$J(u, v) = \int_0^u \int_v^1 \frac{1}{\sqrt{1-\rho^2}} e^{\left(-\frac{1}{2} \left(\frac{\Phi^{-1}(t)^2 - 2\rho\Phi^{-1}(t)\Phi^{-1}(s) + \Phi^{-1}(s)^2}{1-\rho^2}\right) - \Phi^{-1}(t)^2 - \Phi^{-1}(s)^2\right)} dt ds.$$

**Example 6.10** (Gumbel Copula).

By (6.11) the density of the Gumbel copula is given by

$$c_\rho^{Gu}(u, v) = [-\ln(u) - \ln(v)]^{\theta-1} \frac{\exp^{-\left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{\frac{1}{\theta}}}}{uv \left(\left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{2-\frac{1}{\theta}}\right)} \left(\left[(-\ln(u))^\theta + (-\ln(v))^\theta\right]^{\frac{1}{\theta}} + \theta - 1\right)$$

We have thus

$$J(u, v) = \int_0^u \int_v^1 dt ds \frac{\left[(-\ln(t) - \ln(s))^{\theta-1} e^{-\left[(-\ln(t))^\theta + (-\ln(s))^\theta\right]^{\frac{1}{\theta}}}\right] \left(\left[(-\ln(t))^\theta + (-\ln(s))^\theta\right]^{\frac{1}{\theta}} + \theta - 1\right)}{st \left(\left[(-\ln(t))^\theta + (-\ln(s))^\theta\right]^{2-\frac{1}{\theta}}\right)}$$

**Example 6.11** (Clayton Copula). By (6.11) the density of the Clayton copula is given by

$$c_\rho^{Cl}(u, v) = (\theta + 1) (uv)^{-(\theta+1)} (u^{-\theta} + v^{-\theta} - 1)^{-\frac{2\theta+1}{\theta}}.$$

We have thus

$$J(u, v) = \int_0^u \int_v^1 (\theta + 1) (st)^{-(\theta+1)} (s^{-\theta} + t^{-\theta} - 1)^{-\frac{2\theta+1}{\theta}} dt ds.$$

We define the function

$$j(u, v) := \frac{J(u, v)}{1-v} \tag{6.12}$$

then  $j(u, v)$  is also for each  $v$  fixed invertible as a function of  $u$ . We denote its inverse function with  $j^{-1}(u, v)$

**Theorem 6.12.**

$$CoVaR_\alpha^{L^i > l} = F_s^{-1}(j^{-1}(\alpha, F_i(l))). \tag{6.13}$$

*Proof.* Recall that for a given  $l \in \mathbb{R}$ ,  $CoVaR_\alpha^{L^i > l}$  is implicitly defined by

$$Pr\left(L^s \leq CoVaR_\alpha^{L^i > l} | L^i > l\right) = \alpha.$$

By setting  $U := F_s(L^s)$ ,  $V := F_i(L^i)$ ,  $u := F_s\left(CoVaR_\alpha^{L^i > l}\right)$  and  $v := F_i(l)$  we obtain

$$\begin{aligned} Pr\left(L^s \leq CoVaR_\alpha^{L^i > l} | L^i > l\right) &= Pr(U \leq u | V > v) \\ &= \frac{u - C(u, v)}{1 - v} = j(u, v). \end{aligned}$$

We therefore have  $Pr\left(L^s \leq CoVaR_\alpha^{L^i > l} | L^i > l\right) = j(u, v) = \alpha$ . It follows that

$$u = F_s\left(CoVaR_\alpha^{L^i > l}\right) = j^{-1}(\alpha, v).$$

Thus

$$CoVaR_\alpha^{L^i > l} = F_s^{-1}\left(j^{-1}(\alpha, F_i(l))\right). \quad (6.14)$$

□

*Remark 6.13.* Similarly to  $CoVaR_\alpha^{L^i = l}$ , we observe that  $CoVaR_\alpha^{s|L^i > l}$  is also expressed in form of a quantile of the loss distribution  $F_s$ . We have

$$CoVaR_\alpha^{s|L^i > l} = F_s^{-1}(\bar{\alpha}) = VaR_{\bar{\alpha}}^s, \quad (6.15)$$

with  $\bar{\alpha} := j^{-1}(\alpha, F_i(l))$ .

**Corollary 6.14.** *Under Assumption 1.3*

$$\begin{aligned} CoVaR_\alpha^{s|L^i > VaR_\alpha^i} &= F_s^{-1}\left(j^{-1}(\alpha, \alpha)\right) \text{ and} \\ \Delta CoVaR_\alpha^> &= F_s^{-1}\left(j^{-1}(\alpha, \alpha)\right) - F_s^{-1}\left(g^{-1}(\alpha, \alpha)\right). \end{aligned}$$

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