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RISK MEASURES ON ORLICZ HEART SPACES

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Abstract. In this paper, we are interested to find a robust representation of the risk measure which is defined via a convex Young function. We consider convex risk measures defined on Orlicz heart spaces with Banach lattice values and their dual representation.

1. Introduction

The concept of risk measures has been studied by many authors. Artzner et al. [2] introduced the notion of coherent risk measure which is understood to be a measure of initial capital requirements that investors and managers should provide in order to overcome negative evolutions of the market. Delbaen [9, 10] extended this risk measure to more general settings. Fölmer and Schied [13] generalized the notion of coherent risk measure on $L(\Omega)$ spaces. Fritelli and Rosazza Gianin [16] established the more general concepts of convex and monetary risk measures. Cheridito and Li [8] give a new result about convex risk measures on Orlicz heart spaces with real values. Jouini et al. [18] were the first to introduce set-valued coherent risk measures. Hamel extended the approach of Jouini et al. to define set-valued convex risk measures on the space $L^p(\mathbb{R}^d, P)$ of Bochner p-integrable functions with valued in $\mathbb{R}^d$.

Offwood [20] considered the connection between utility functions and real valued Orlicz spaces as was noted by Fritelli and obtained a representation on set-valued convex risk measures on Orlicz heart spaces.

This paper is organized as follows: in section 1, we introduce some notations required in the text and give preliminaries on Banach lattices E endowed with an ordering relation $\leq$. Then, we define the Orlicz space $L^\varphi(\Omega, E, \mu)$, its dual $H_\varphi(\Omega, E, \mu)$ (which is called the Orlicz heart space) and their associated norms. In the third part, we introduce a risk measure ρ from an Orlicz heart space to a Banach lattice E and we define its acceptance set $\mathcal{A}$ containing all acceptable financial positions. Then, we give a robust representation of our measure ρ. Finally, we give some examples of this risk measure, namely value at risk $V@R$ and average value at risk $AV@R$ which are convex risk measures.

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2. Orlicz Spaces and Orlicz Heart Spaces

Consider a real vector space E which is ordered by some order relation $\leq$. This space E is called a vector lattice if any two element $x, y \in E$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x, y)$ and satisfy the following properties:

$(L1)$ $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in E$.

$(L2)$ $0 \leq x$ implies $0 \leq tx$ for all $x \in E$ and $t \in \mathbb{R}_+$.

We denote by $E_+ := \{x \in E \,|\, 0 \leq x\}$ the positive cone of E. For $x \in E$, let:

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \quad \text{and} \quad |x| := x \vee (-x)$$

be the positive part, the negative part and the absolute value of x, respectively. A norm $\|.\|$ on a vector lattice E is called a lattice norm if:

$$|x| \leq |y| \Rightarrow \|x\| \leq \|y\| \quad \text{for} \quad x, y \in E. \tag{2.1}$$

In particular, a Banach lattice is a real Banach space E endowed with an ordering $\leq$ such that $(E, \leq)$ is a vector lattice and the norm on E is a lattice norm.

Let E be a Banach lattice and E_+ be its non-negative cone. Its dual Banach lattice and its cone are denoted by E^* and E^*_+ respectively and $< x, x^* >$ will be the usual duality product.

Definition 2.1. A map $\varphi : [0, \infty] \to \overline{\mathbb{R}}$ is called a *Young function* if φ is left continuous, increasing, convex, $\varphi(0) = 0$ and $\lim_{r \to +\infty} \frac{\varphi(r)}{r} = \infty$.

Also, φ is continuous except possibly at a single point, where it jumps to ∞. So, the assumption of left-continuous is needed only at that one point. The conjugate

$$\varphi^*(y) = \sup_{x \geq 0}\{xy - \varphi(x)\} \quad ; \quad y \geq 0$$

is a Young function.

Definition 2.2. A map $\alpha : \mathbb{R} \to \overline{\mathbb{R}}$ is called a *utility function* if α is increasing, concave and continuous on $\mathbb{R}$.

We use concavity to minimize the risk and the growth to maximize the wealth.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, E a Banach lattice and consider the following space for $p \geq 1$

$$L^p(\Omega, E, \mu) = \{f : \Omega \to E \,|\, \|f(x)\|_p^p := \int_\Omega \|f(x)\|^p d\mu(x) < \infty\}.$$

Then the dual space of $L^p(\Omega, E, \mu)$ is

$$(L^p(\Omega, E, \mu))^* = L^q(\Omega, E, \mu) \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

Let $L^\infty(\Omega, E, \mu)$ be a Banach space with values in E of essentially bounded and measurable functions, with norm defined by:

$$\|y\|_\infty = \operatorname{ess\,sup}\{\|y(\omega)\|, \ \omega \in \Omega\}.$$

Definition 2.3. Let φ be a Young function and $c > 0$. Consider the space

$$L_c^{\varphi}(\Omega, E, \mu) := \{X : \Omega \to E \,|\, X \text{is continuous and } \int_{\Omega} \varphi(c\|X(\omega)\|)d\mu(\omega) < \infty\}, \tag{2.2}$$

The *Orlicz space* $L^{\varphi}(\Omega, E, \mu)$ is given by:

$$L^{\varphi}(\Omega, E, \mu) = \bigcup_{c>0} L_c^{\varphi}(\Omega, E, \mu). \tag{2.3}$$

and the norm associated to $L^{\varphi}(\Omega, E, \mu)$ is given by

$$\|X\|_{\varphi} := \inf\{\lambda > 0 \,|\, \mathbb{E}(\varphi(\|\frac{X}{\lambda}\|)) \leq 1\}$$

The *Orlicz heart space* corresponding to φ is defined as:

$$H_{\varphi}(\Omega, E, \mu) = \bigcap_{c>0} L_c^{\varphi}(\Omega, E, \mu). \tag{2.4}$$

and the norm of an element $Z \in L^{\varphi^*}(\Omega, E, \mu)$ associated to $H_{\varphi}(\Omega, E, \mu)$ is

$$\|Z\|_{\varphi^*}^* := \sup\{\mathbb{E} < Z, X > \,| \quad \|X\|_{\varphi^*} \leq 1\} \ , \ X \in L^{\varphi^*}(\Omega, E, \mu)$$

2.1. (Δ_2) condition. [23], [11]

Definition 2.4. A Young function φ satisfies the (Δ_2) condition if there exists a constant $M > 0$ such that

$$\varphi(2u) \leq M\varphi(u) \ , \ \forall \ u \geq 0 \tag{2.5}$$

Using this relation, one can show that

$$(H_{\varphi}(\Omega, E, \mu))^* = L^{\varphi^*}(\Omega, E^*, \mu),$$

(see [21] p 132 - 138).

3. Generalized Risk Measure With Banach Lattice Values

In this section, we examine risk measures with values in a Banach lattice. Assuming that there exists $x_0 \in E$ strictly positive i.e, $x_0 \wedge x \in E_+ \setminus \{0\}$ for all $x \in E_+ \setminus \{0\}$.

Definition 3.1. A map $\rho : H_{\varphi}(\Omega, E, \mu) \to E$ is called a *risk measure* if ρ satisfies:

(a) Monotonicity: for $X, Y \in H_{\varphi}(\Omega, E, \mu)$, if $Y \geq X$, then $\rho(Y) \leq \rho(X)$.

(b) Cash invariance: for any $x_0 \in E$ and all $X \in H_{\varphi}(\Omega, E, \mu)$

$$\rho(X + 1_{\Omega}x_0) = \rho(X) - x_0$$

If, in addition, ρ has the properties:

(c) Positive homogeneity: for any $\beta \geq 0$ and $X \in H_{\varphi}(\Omega, E, \mu)$:

$$\rho(\beta X) = \beta\rho(X)$$

(d) Sub-additive:

$$\rho(X + Y) \leq \rho(X) + \rho(Y), \quad \forall \ X, Y \in H_{\varphi}(\Omega, E, \mu)$$

(e) Convexity: for any $X, Y \in H_{\varphi}(\Omega, E, \mu)$ and $\lambda \in [0, 1]$,

$$\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$$

then ρ is called a *coherent measure.*

Remark that the measure $\rho(X)$ is understood as a capital requirement for X. In every state of the world the monotonicity means that the capital requirement for X will be smaller than Y μ-almost sure.

The cash invariance means that if we add an amount of money x_0 to a position X should reduce the capital requirement for X by x_0. The positive homogeneity says that the capital requirement scale linearly when net worths are multiplied with non-negative constants. The sub-additivity means that the capital requirement for an aggregated discounted net worth $X+Y$ should not exceed the sum of the capital requirement for X plus the capital requirement for Y. We define

$$\mathcal{A} = \{X \in H_\varphi(\Omega, E, \mu) \,|\quad \rho(X) \leq 0\}. \tag{3.1}$$

as the set of acceptable positions. The following propositions summarize the relations between risk measures and their acceptance sets. We use the convention $\inf \emptyset = \infty$.

Proposition 3.2. *Let $H_\varphi(\Omega, E, \mu)$ be an Orlicz heart space and $\rho : H_\varphi(\Omega, E, \mu) \to E$ be a risk measure with an acceptance set $\mathcal{A}$. Then:*

(i) *Let $X \in \mathcal{A}$, $Y \in H_\varphi(\Omega, E, \mu)$. If $Y \geq X$ then $Y \in \mathcal{A}$.*

(ii)

$$\rho(X) = \inf\{x_0 \in E \,|\, X + \mathbf{1}_\Omega x_0 \in \mathcal{A}\},\ \forall\ X \in H_\varphi(\Omega, E, \mu) \tag{3.2}$$

(iii)

$$\inf\{x_0 \in E \,|\ 1_\Omega x_0 \geq Z \textit{ for some } Z \in \mathcal{A}\} \in E. \tag{3.3}$$

Proof. $\rho : H_\varphi(\Omega, E, \mu) \to E$, $\mathcal{A}$ is its acceptance set.

(i) $\forall\ X \in \mathcal{A}$, $Y \in H_\varphi(\Omega, E, \mu)$. If $Y \geq X$ implies $\rho(Y) \leq \rho(X) \leq 0$, then $Y \in \mathcal{A}$.

(ii)

$$\begin{aligned} \inf\{x_0 \in E \,|\, X + 1_\Omega x_0 \in \mathcal{A}\} &= \inf\{x_0 \in E \,|\, \rho(X + 1_\Omega x_0) \leq 0\} \\ &= \inf\{x_0 \in E \,|\, \rho(X) - x_0 \leq 0\} \\ &= \inf\{x_0 \in E \,|\, \rho(X) \leq x_0\} \\ &= \rho(X). \end{aligned}$$

(iii) (3.3) is a consequence of monotonicity.

It implies that for all $X \in H_\varphi(\Omega, E, \mu)$ we obtain:

$$\begin{aligned} \inf\{x_0 \in E \,|\, \mathbf{1}_\Omega x_0 \geq Z\ ,\ Z \in \mathcal{A}\} &= \inf\{x_0 \in E \,|\, \rho(\mathbf{1}_\Omega x_0) \leq \rho(Z) \leq 0\} \\ &= \inf\{x_0 \in E \,|\, \rho(0) - x_0 \leq 0\} \\ &= \inf\{x_0 \in E \,|\, \rho(0) \leq x_0\} \\ &= \rho(0) \in E. \end{aligned}$$

□

Corollary 3.3. *ρ is a risk measure on $H_\varphi(\Omega, E, \mu)$ with value in a Banach lattice. Then,*

(i) *If ρ is convex, then $\mathcal{A}$ is convex.*

(ii) *If ρ is coherent, then $\mathcal{A}$ is a convex cone.*
(iii) *$\mathcal{A}$ has the following property:*

$$\forall X \in H_\varphi(\Omega, E, \mu), \quad \inf\{x_0 \in E \,|\, X + 1_\Omega x_0 \geq Z \ , \ Z \in \mathcal{A}\} \in E. \tag{3.4}$$

Proof. (i) If ρ is convex, then for all $X_1, X_2 \in H_\varphi(\Omega, E, \mu)$ and $\lambda \in [0,1]$ such that

$$\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2)$$

Let X_1 and $X_2 \in \mathcal{A}$, we have $\rho(\lambda X_1 + (1-\lambda)X_2) \leq \lambda\rho(X_1) + (1-\lambda)\rho(X_2) \leq 0$ because $\rho(X_1) \leq 0$ and $\rho(X_2) \leq 0$ for $\lambda \in [0,1]$, then $\lambda X_1 + (1-\lambda)X_2 \in \mathcal{A}$. So $\mathcal{A}$ is convex.

(ii) ρ is coherent implies ρ is positive homogenous.

$$\forall X \in \mathcal{A}, \quad \rho(\lambda X) = \lambda\rho(X) \leq 0, \quad \forall\lambda \geq 0$$

then $\rho(\lambda X) \leq 0$ implies $\lambda X \in \mathcal{A}$. Therefore $\mathcal{A}$ is a cone and the convexity follows as in (i).

(iii)

$$\begin{aligned}
\inf_{Z\in\mathcal{A}}\{x_0 \in E \,|\, X + \mathbf{1}_\Omega x_0 \geq Z\} &= \inf_{Z\in\mathcal{A}}\{x_0 \in E \,|\, \rho(X + \mathbf{1}_\Omega x_0) \leq \rho(Z) \leq 0\} \\
&= \inf\{x_0 \in E \,|\, \rho(X) - x_0 \leq 0\} \\
&= \inf\{x_0 \in E \,|\, \rho(X) \leq x_0\} \\
&= \rho(X) \in E.
\end{aligned}$$

Then the result holds.

□

Proposition 3.4. *Let $H_\varphi(\Omega, E, \mu)$ be the Orlicz heart space and $\mathcal{B}$ a subset of $H_\varphi(\Omega, E, \mu)$ with the property (iii) of Proposition 3.2. Then*

$$\rho(X) = \inf\{x_0 \in E \,| \quad X + 1_\Omega x_0 \geq Z \quad \textit{for some}\, Z \in \mathcal{B}\}. \tag{3.5}$$

this application ρ defines a risk measure on $H_\varphi(\Omega, E, \mu)$ whose acceptance set $\mathcal{A}$ is the smallest subset of $H_\varphi(\Omega, E, \mu)$ that contains $\mathcal{B}$ and satisfies (i) of Proposition 3.2. If $\mathcal{B}$ is convex, then so is ρ. If $\mathcal{B}$ is a convex cone, then ρ is coherent. If $\mathcal{B}$ satisfies (3.4), then ρ is included in E.

Proof. ρ is a risk measure and $\mathcal{B}$ is contained in $\mathcal{A}$. Moreover, for each $X \in \mathcal{A}$ and $n \geq 1$, there exists $Z^n \in \mathcal{B}$ such that $X + 1_\Omega \frac{1}{n} \geq Z^n$.

This shows that $\mathcal{A}$ is contained in every subset of $H_\varphi(\Omega, E, \mu)$ containing $\mathcal{B}$ and satisfying (iii) of Proposition 3.2. That ρ is convex when $\mathcal{B}$ is so, coherent when $\mathcal{B}$ is a convex cone, and with value in E when $\mathcal{B}$ satisfies (3.4). □

4. Robust Representation

In this part, we give the robust representation for a coherent and a convex risk measure on Orlicz heart spaces that are Banach lattices valued. This also exploited for the representation of convex functionals in Cheridito et al. [6], Biagini and Fritelli [16] and Delbaen [7].

The core of A, denoted by $core(A)$, is the algebraic interior of a subset A of $H_\varphi(\Omega, E, \mu)$ which consists of all $x \in A$ that has an algebraic neighborhood contained in A; i.e.,

$$core(A) := \{x_0 \in A \,|\, \forall x \in E, \quad \exists t_x > 0, \forall t \in [0, t_x]\ ,\ x_0 + t.x \in A\}$$

In general $core(core(A)) \neq core(A)$. If A is a convex set, then $core(core(A)) = core(A)$.

$int(A)$ is the interior of a subset A of a topological vector space which is contained in its algebraic interior $core(A)$. $int(A)$ is the union of all open set contained in A.

Lemma 4.1. *If $f : E \to E$ is an increasing function, then $core(domf) = int(domf)$.*

Proof. It is easy to check that $int(domf) \subset core(domf)$.

To prove the second inclusion, we assume that $f : E \to E$ where E is a Banach lattice on an algebraic neighborhood of $x \in E$ but not on a neighborhood of x, then there exists a sequence $(z_n) \in E$ such that $\|z_n\| \leq \frac{1}{4^n}$ and $f(x + z_n) \in E$ for all $x \in E$.

(z_n) can be written as $z_n = z_n^+ - z_n^-$, then $\|z_n^+\| \leq \frac{1}{4^n}$ and $f(x + z_n^+) \in E$.

Let define $z := \sum_{n \geq 1} 2^n z_n^+$. By assumption, there exists an $\varepsilon > 0$ such that $f(x + \varepsilon z)$ is finite. For all n with $n2^n \geq 1$, we have

$$\infty > f(x + \varepsilon z) \geq f(x + \varepsilon 2^n z_n^+) \geq f(x + z_n^+) \in E$$

which is not true, then the result holds. □

Definition 4.2. A function $f : E \to \mathbb{R}$ is called a *proper function* if $f(x) < \infty$ for at least one x and $f(x) > -\infty$ for every $x \in E$.

For a proper function f from a Banach lattice E to $\mathbb{R}$, we have the following properties:

- **(a)** Every continuous map from a compact space to a Hausdorff space is both proper and closed.
- **(b)** A proper function satisfies the inequality of Young:

$$xb \leq f(x) + f^*(b),\ \forall\ x, b \in E$$

The function f is called *sub-differentiable* at $x \in E$ if $f(x) \in \mathbb{R}$ and there exists an element $x^* \in E^*$, the dual topological of E, such that

$$x^*(z) \leq f(x + z) - f(x), \quad \forall z \in E$$

We define the conjugate of f:

$$f^*(x^*) := \sup_{x \in E}\{x^*(x) - f(x)\}$$

f^* is $\sigma(E^*, E)$ lower semi-continuous and convex on E^*. From the definition of f^*, we have

$$f(x) \geq f^{**}(x) := \sup_{x^* \in E^*}\{x^*(x) - f^*(x^*)\}, \quad \forall x \in E$$

Moreover,

$$f(x) = \max_{x^* \in E^*} \{x^*(x) - f^*(x^*)\}, \quad \forall x \in E \tag{4.1}$$

where f is sub-differentiable.

let

$$\triangle(X) := \{f : E \to \mathbb{R} \mid f \text{ is a proper convex function}\}$$

Theorem 4.3 (Theorem 2.4.9, [25]). *Let $f \in \triangle(X)$. If f is continuous at $\overline{x} \in domf$, then $Df(\overline{x})$ is non empty and ω^*-compact. Furthermore, $f^{'}(\overline{x}, .)$ is continuous and*

$$\forall u \in E, \quad f_{\varepsilon}^{'}(\overline{x}, u) = \max\{< u, x^* > \mid x^* \in Df(\overline{x})\} \tag{4.2}$$

Theorem 4.4. *Let f be an increasing and a convex function in E. Then for all $x \in dom(f)$ the following holds:*

(i) *f is Lipschitz and continuous in $\vartheta(x)$ with respect to the norm on E.*
(ii) *f is sub-differentiable at x.*
(iii) *f is given by $f(x) = \max_{x^* \in E^*} \{x^*(x) - f^*(x^*)\}$.*

Proof. (i) It follows from Corollary 2.2.12 [25].

(ii) We have $core(domf) = int(dom(f)$, every $x \in core(domf)$ has a neighborhood U such that $U \subset dom(f)$. By Proposition 3.1 [24] it follows that f is continuous and sub-differentiable at x.

(iii) (i) and (4.1) implies that f is continuous at x and there exists a neighborhood of x on which f is bounded.

□

Definition 4.5. A probability measure $\mathbb{Q}$ in $(\Omega, \mathcal{F})$ is called *absolutely continuous* with respect to μ if

$$\mathbb{Q}(A) = 0 \Rightarrow \mu(A) = 0.$$

In the following, we identify a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which is absolutely continuous with respect to μ with its Radon-Nikodym derivative $\xi = \frac{d\mathbb{Q}}{d\mu} \in L^1(\Omega, E, \mu)$. Define

$$\mathfrak{B} := \{\xi \in L^1(\Omega, E, \mu) | \xi \geq 0, \ \mathbb{E}_\mu(\xi) = 1\}$$

which represents the set of all probability measure on $(\Omega, \mathcal{F}, \mu)$ that are absolutely continuous with respect to μ.

We denote $\mathfrak{B}^{\varphi^*}$ by $\mathfrak{B}^{\varphi^*} = \mathfrak{B} \cap \mathrm{L}^{\varphi^*}(\Omega, \mathrm{E}, \mu)$.

Definition 4.6. A mapping $\alpha : \mathfrak{B}^{\varphi^*} \to E$ is called a *penalty function* if it is bounded from below and not identically equal to ∞.

We said that α satisfies the growth condition if there exist a and $b \in E$ such that

$$\alpha(\mathbb{Q}) \geq a + b\|\mathbb{Q}\|_\varphi^* \ , \ \forall \quad \mathbb{Q} \in \mathfrak{B}^{\varphi^*} \tag{4.3}$$

and for any penalty function α on $\mathfrak{B}^{\varphi^*}$, we give the robust representation of ρ_α:

$$\rho_\alpha(X) := \sup_{\mathbb{Q} \in \mathfrak{B}^{\varphi^*}} \mathbb{E}_\mathbb{Q}(-X) - \alpha(\mathbb{Q}) \ , \ X \in \mathrm{H}_\varphi(\Omega, E, \mu)$$

ρ_α is a risk measure on $\mathrm{H}_\varphi(\Omega, E, \mu)$ which is lower semi-continuous and convex. Now, let's recall the Hahn-Banach theorem:

Theorem 4.7. *(Hahn Banach) Suppose that $\mathfrak{B}$ and $\mathfrak{C}$ are two non-empty, disjoint and convex subsets of a locally convex space E. Then, if $\mathfrak{B}$ is compact and $\mathfrak{C}$ is closed, there exists a continuous linear functional ℓ on E such that*

$$\sup_{x\in\mathfrak{C}} \ell(x) < \inf_{y\in\mathfrak{B}} \ell(y)$$

Theorem 4.8. *Let l^φ be an Orlicz sequence space with a norm given by*

$$\|X\| = \inf\{\lambda > 0 \,|\, \sum_{i=1}^{\infty} \varphi(\frac{t_i}{\lambda}) \leq 1\}$$

for $X = \{t_i\} \in l^\varphi$. We say that a sequence of balls with centres $\{X_i\}$ and redias r can be packed into the unit ball of a space E if $\{X_i\}$ and r satisfy the following properties:

(a) $\|X_i\| \leq 1 - r\ ,\ i = 1, 2, ...,$
(b) $\|X_i - X_j\| \leq 2r\ ,\ i \neq j\ ,\ i, j = 1, 2....$

The ball-packing constant of E is defined as $\Lambda(E) = \sup\{r > 0 \,|\,$ there exists $\{X_i\}_{i=1}^{\infty} \subset E$ such that $\|X_i\| \leq 1 - r,\ \|X_i - X_j\| \leq 2r\ (i \neq j)\}$. Then

$$\Lambda(E) \leq \frac{1}{2} \tag{4.4}$$

Proof. In fact, if $\Lambda(E) > \frac{1}{2}$, then there exist r and $\{X_i\}_{i=1}^{\infty} \subset E$ satisfying $\Lambda(E) > r > \frac{1}{2}$ such that

$$\|X_i\| \leq 1 - r < \frac{1}{2}$$

$$\|X_i - X_j\| \geq 2r > 1\ (i \neq j)\}$$

so when $\neq j$ we have

$$\frac{1}{2} > \|X_i\| \leq \|X_i - X_j\| > 1 - \frac{1}{2} = \frac{1}{2}$$

with is a contradiction. □

we have the following result:

Theorem 4.9. *Let α be a penalty function on $\mathfrak{B}^{\varphi^*}$. Then there is an equivalence between the conditions:*

(i) *α satisfies (4.3).*
(ii) *$core(dom\rho_\alpha)$ is a non empty set.*
(iii) *ρ_α is with value in a Banach lattice and every $X \in H_\varphi(\Omega, E, \mu)$ has a neighborhood on which the risk measure is Lipschitz-continuous with respect to $\|.\|_\varphi$.*

Proof. $(iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii)$

- $(iii) \Rightarrow (ii)$ is trivial.

- $(ii) \Rightarrow (i)$

  Now, assume that ρ_α is with value in a Banach lattice space on an algebraic neighborhood of $X \in H_\varphi(\Omega, E, \mu)$. Since $Y \mapsto \rho_\alpha(-Y)$ is increasing and by $core(domf) = int(domf)$ that there exists an $\varepsilon > 0$ such that ρ_α is a Banach lattice valued on the closed ball $B_\varepsilon(X)$ with redias ε around X. Therefore $L^\infty(\Omega, E, \mu)$ is $\|.\|_\varphi$ dense in $H_\varphi(\Omega, E, \mu)$ implies there exists a sequence $(Y^n)_{n\geq 1}$ of bounded random variables such that

$$\|Y^n - X\|_\varphi \leq \varepsilon 2^{-n-2}$$

  If α does not satisfy (4.3), then there exists a sequence of probability measure $(\mathbb{Q}^n)_{n\geq 1}$ in $\mathcal{B}^{\varphi^*}$ such that

$$\alpha(\mathbb{Q}^n) < -n - \|Y^n\|_\infty + \varepsilon 2^{-n-2}\|\mathbb{Q}^n\|_\varphi^* \quad for \quad all \quad n \geq 1$$

  Since $(L^{\varphi^*}, \|.\|_\varphi^*)$ is the dual norm of $H_\varphi(\Omega, E, \mu)$, there exists for every $n \geq 1$, $Z^n \in H_\varphi(\Omega, E, \mu)$ such that $Z^n \leq 0$, $\|Z^n\|_\varphi \leq 1$ and $\mathbb{E}_{\mathbb{Q}^n}(-Z^n) \geq \frac{1}{2}\|\mathbb{Q}^n\|_\varphi^*$.

  The random variable $Z := \varepsilon \sum_{n\geq 1} 2^{-n} Z^n$ is included in $H_\varphi(\Omega, E, \mu)$ and $\|Z\|_\varphi \leq \varepsilon$, then

$$\begin{aligned}
\rho_\alpha(X+Z) &\geq \rho_\alpha(X + \varepsilon 2^{-n} Z^n) \\
&\geq \mathbb{E}_{\mathbb{Q}^n}(-X - \varepsilon 2^{-n} Z^n) - \alpha(\mathbb{Q}^n) \\
&\geq \mathbb{E}_{\mathbb{Q}^n}(-Y^n) + \mathbb{E}_{\mathbb{Q}^n}(Y^n - X) + \varepsilon 2^{-n}\mathbb{E}_{\mathbb{Q}^n}(-Z^n) - \alpha(\mathbb{Q}^n) \\
&\geq -\|Y^n - X\|_\varphi \|\mathbb{Q}^n\|_\varphi^* + \varepsilon 2^{-n-1}\|\mathbb{Q}^n\|_\varphi^* + n - \varepsilon 2^{-n-2}\|\mathbb{Q}^n\|_\varphi^* \\
&\geq n \quad \forall \quad n \geq 1,
\end{aligned}$$

  which is not true because of $\rho_\alpha(0) \in E$ on $B_\varepsilon(X)$. Therefore, α must fulfill (4.3) and (i) is proved.

- $(i) \Rightarrow (iii)$ We assume that there exist a and $b \in E$ such that

$$\alpha(\mathbb{Q}) \geq a + b\|\mathbb{Q}\|_\varphi^*, \quad \forall \mathbb{Q} \in \mathfrak{B}^{\varphi^*}$$

  We can choose $X \in H_\varphi(\Omega, E, \mu)$. It exists $\widetilde{X} \in L^\varphi(\Omega, E, \mu)$ such that $\|X - \widetilde{X}\|_\varphi \leq b$ and we obtain

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(-X) - \alpha(\mathbb{Q}) &= \mathbb{E}_{\mathbb{Q}}(-\widetilde{X}) + \mathbb{E}_{\mathbb{Q}}(\widetilde{X} - X) - \alpha(\mathbb{Q}) \\
&\leq \|\widetilde{X}\|_\varphi \|\mathbb{Q}\|_\varphi^* + \|\widetilde{X} - X\|_\varphi \|\mathbb{Q}\|_\varphi^* - a - b\|\mathbb{Q}\|_\varphi^* \\
&\leq \|\widetilde{X}\|_\varphi \|\mathbb{Q}\|_\varphi^* - a, \quad \forall \quad \mathbb{Q} \in \mathfrak{B}^{\varphi^*},
\end{aligned}$$

  which shows that $\rho_\alpha(X) \leq \|\widetilde{X}\|_\varphi \|\mathbb{Q}\|_\varphi^* - a \in E$. Hence, ρ_α is a Banach lattice valued and the rest follows from Theorem 4.4 (i).

□

5. Examples

5.1. Value at risk. A common approach to the problem of measuring the risk of a financial position X consists in specifying a quantile of X under the given probability measure μ. For $\lambda \in (0,1)$, a λ-quantile of a random variable X on $H_\varphi(\Omega, E, \mu)$ is any q with the property

$$\mu[X \leq q] \geq \lambda \quad and \quad \mu[X < q] \leq \lambda$$

and the set of all λ-quantile of X is an interval $[q_X^-(\lambda), q_X^+(\lambda)]$, where

$$q_X^-(t) = \sup\{x \mid \mu(X < x) < t\} = \inf\{x \mid \mu(X \leq x) \geq t\}$$

is the lower and

$$q_X^+(t) = \inf\{x \mid \mu(X \leq x) > t\} = \sup\{x \mid \mu(X < x) \leq t\}$$

is the upper quantile function of X.

Definition 5.1. Fix a level $\lambda \in (0,1)$. For a financial position X, we define its *value at risk at level* λ as

$$V@R_\lambda(X) := -q_X^+(\lambda) = q_{-X}^-(1-\lambda) = \inf\{x_0 \mid \mu(X + 1_\Omega x_0 < 0) \leq \lambda\} \tag{5.1}$$

In financial terms, $V@R_\lambda(X)$ is the smallest amount of capital which, if added to X and invested in the risk-free asset, keeps the probability of a negative outcome below the level λ. However, value at risk only controls the probability of a loss, it does not capture the size of such a loss if it occurs. Clearly, $V@R_\lambda$ is a risk measure on $H_\varphi(\Omega, E, \mu)$, which is positively homogeneous. $V@R_\lambda$ is not convex and so $V@R_\lambda$ is not a convex risk measure (see Example 4.46 [14]).

Proposition 5.2. *For each* $X \in H_\varphi(\Omega, E, \mu)$ *and each* $\lambda \in (0,1)$,

$$V@R_\lambda(X) = \min\{\rho(X) \mid \rho \text{ is convex, continuous from above and } \geq V@R_\lambda\}$$

Proof. Let $q := -V@R_\lambda(X) = q_X^+(\lambda)$ so that $\mu(X < q) \leq \lambda$. Let $A \in E$. If A satisfies $\mu(A) > \lambda$, then $\mu(A \cap \{X \geq q\}) > 0$. Thus, we may define a measure $\mathbb{Q}_A$ by

$$\mathbb{Q}_A := \mu(. \mid A \cap \{X \geq q\}).$$

It follows that $\mathbb{E}_{\mathbb{Q}_A}(-X) \leq -q = V@R_\lambda(X)$.

Let $\mathcal{Q} := \{\mathbb{Q}_A \mid \mu(A) > \lambda\}$, and we use this to define a coherent risk measure ρ via

$$\rho(X) := \sup_{\mathbb{Q}_A \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}_A}(-X)$$

Then $\rho(X) \leq V@R_\lambda(X)$. Hence, the assertion shows that $\rho(X) \geq V@R_\lambda(X)$ for each $X \in H_\varphi(\Omega, E, \mu)$. Let $\varepsilon > 0$ and $A := \{X \leq -V@R_\lambda(X) + 1_\Omega \varepsilon\}$.

Clearly $\mu(A) > \lambda$, and so $\mathbb{Q}_A \in \mathcal{Q}$. Moreover, $\mathbb{Q}_A(A) = 1$, and we obtain

$$\rho(X) \geq \mathbb{E}_{\mathbb{Q}_A}(-X) \geq V@R_\lambda(X) - 1_\Omega \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

For the rest of this part, we concentrate on the following risk measure which is defined in terms of value at risk, but does not satisfy the axioms of a coherent risk measure.

Definition 5.3. The *average value at risk at level* $\lambda \in (0,1)$ of a position $X \in H_\varphi(\Omega, E, \mu)$ is given by

$$AV@R_\lambda(X) = \frac{1}{\lambda}\int_0^\lambda V@R_\gamma(X)d\gamma.$$

where γ is a level on $(0,1)$.

Sometimes, the average value at risk is also called the "conditional value at risk" or the "expected shortfall" and we write $CV@R_\lambda(X)$ or $ES_\lambda(X)$. So, we prefer the term Average Value at Risk and note that

$$AV@R_\lambda(X) = \frac{-1}{\lambda}\int_0^\lambda q_X(t)dt$$

For $\lambda = 1$ we have

$$AV@R_1(X) = -\int_0^1 q_X^+(t)dt = \mathbb{E}_\mathbb{Q}(-X)$$

Remark 5.4. For $X \in H_\varphi(\Omega, E, \mu)$, we have

$$\lim_{\lambda \downarrow 0} V@R_\lambda(X) = -ess\inf X = \inf\{x_0 \mid \mu(X + 1_\Omega x_0 < 0) \leq 0\}.$$

Hence, it makes sense to define

$$AV@R_0(X) := V@R_0(X) := -ess\inf X$$

Recall that it is continuous from above but in general not from below.

Lemma 5.5. *For* $\lambda \in (0,1)$ *and any* $\lambda - quantile$ q *of* X*,*

$$AV@R_\lambda(X) = \frac{1}{\lambda}\mathbb{E}_\mathbb{Q}[(q-X)^+] - q = \frac{1}{\lambda}\inf_{r \in H_\varphi(\Omega,E,\mu)}\Big(\mathbb{E}_\mathbb{Q}[(r-X)^+] - \lambda r\Big) \tag{5.2}$$

Proof. Let q_X be a quantile function with $q_X(\lambda) = q$. By Lemma A.19 in [14],

$$\frac{1}{\lambda}\mathbb{E}_\mathbb{Q}[(q-X)^+] - q = \frac{1}{\lambda}\int_0^1 (q - q_X(t))^+ dt - q = -\frac{1}{\lambda}\int_0^\lambda q_X(t) = AV@R_\lambda(X)$$

This proves the first identity. The second one follows from Lemma A.22 [14]. □

References

1. Arai, T.: *Convex risk measure on Orlicz spaces: inf-convolution and shortfall*, Mathematics and Financial Economics, vol. **3**, no. 3, (2010) 73–88.
2. Artzner, P., Delbaen, F., Health, D. and Eber, J. M.: *Coherent Measures of Risk*, Mathematical Finance, vol. **9**, no. 3, (1999) 203–228.
3. Balbás, A., Balbás, B. and Balbás, R.: *Minimizing Vector Risk Measures*, Springer-Verlag, Berlin Heidelberg, 2010.
4. Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M. and Montrucchio, L.: *Complete Monotone Quasiconcave Duality*, Mathematics of operations research. vol. **36**, no. 2, (2011) 321–339.
5. Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M. and Montrucchio L.: *Risk Measures: Rationality and Diversification*, Mathematical Finance, vol. **21**, no. 4, (2011) 743–774.
6. Cheridito, P., Delbaen, F. and Kupper, M.: *Coherent and Convex Monetary Risk measures for Bounded Processes*, Stoch. Proc. Appl, vol. **112**, no. 1, (2004) 1–22.
7. Cheridito, P., Delbaen, F. and Kupper, M. : *Coherent and Convex Monetary Risk Measures for Unbounded C'adl'ag Processes*, Finance and Stochastics, vol. **10**, no. 3, (2006) 427–448.

8. Cheridito, P. and Li, T.: *Risk Measures on Orlicz Hearts*, Mathematical Finance, vol. **19**, no. 2, (2009) 189–214.
9. Delbaen, F.: *Coherent Risk measures*, Bltter der DGVFM, vol.**24**, no. 4, (2000) 733–739.
10. Delbaen, F.: *Coherent Risk measures on General Probability Spaces*, Advances in finance and stochastics.Springer Berlin Heidelberg, (2002) 1–37.
11. Eisele, K-T. and Taieb, S.: *Lattice Modules Over Rings of Bounded Random Variables*, Communications on Stochastic Analysis, vol **6**, no. 4, (2012) 525–545.
12. El Karoui, N. and Ravanelli, C.: *Cash Subadditive Risk Measures and Intereset rate Ambiguity*, Mathematical Finance, vol. **19**, no. 4, (2009) 561–590.
13. Fölmer, H. and Schied, A.: *Convex Measures of Risk and Trading Constraints*, Finance and Stochastics, vol. **6**, no. 4, (2002) 429–447.
14. Föllmer, H. and Schied, A.: *Stochastic Finance: An Introduction in Discrete Time, second revised and extended edition*, Walter de Gruyter, Berlin, 2004.
15. Föllmer, H. and Schied, A.: *Stochastic Finance: An Introduction in Discrete Time, third revised and extended edition*, Walter de Gruyter, Berlin, 2011.
16. Fritelli, M. and Gianin, E. R.: *Law Invariant Convex Risk Measures*, Advances in Mathematical Economics. Springer Tokyo, vol **7**, (2005) 42–53.
17. Fritelli, M. and Rosazza Gianin, E.: *On the penalty function and on continuity properties of risk measures*, World Scientific Publishing Company, vol. **14**, no. 1, (2011) 163–185.
18. Jouini, E. and Kallal, H.: *Arbitrage in securities markets with short sales constraints*, Mathematical Finance, vol. **5**, no. 3, (1995) 197–232.
19. Jouini, E., Schachermayer, W. and Touzi, N.: *Optimal risk sharing for law invariant monetary utility functions*, Mathematical Finance, vol. **18**, no 2, (2008) 269292.
20. Labuschagne, C. C. A. and Offwood, T.M.: *Pricing exotic options using the Wang transform*, The North American Journal of Economics and Finance, vol. **25**, (2013) 139–150.
21. Offwood, T.M.: *No free lunch and risk measures on Orlicz spaces*, Diss. University of the Witwatersrand, South Africa, 2012.
22. Orlicz, W.: *Linear Functional Analysis*, World Scientific Publishing Co. Pte. Lid, Series In Real Analysis, 1963.
23. Rao, K.C and Subramanian, N.: *The Orlicz space of entire sequences*, International Journal of Mathematics and Mathematical Sciences, vol. **2004**, no. 68, (2004) 3755–3764.
24. Ruszczyński, A. and Shapiro, A.: *Optimization of Convex Risk functions*, Mathematics of Operations Research, vol. **31**, no. 3, (2006) 433–452.
25. Zălinescu, C.: *Convex Analysis in General Vector Spaces*, World Scientific Publishing Co., Inc., River Edge, 2002.

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