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STRUCTURE AND DECOMPOSITIONS OF THE LINEAR SPAN OF GENERALIZED STOCHASTIC MATRICES

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Abstract. We study the topological properties of the Lie group of invertible constant row sum matrices and the structure and Levi decomposition of the derived Lie algebra of constant row sum matrices and of the Lie algebra of constant, and in particular zero, row sum matrices. The Peirce decomposition of constant row sum matrices with respect to the usual and Jordan matrix product is obtained. The form of automorphisms on constant and zero row sum matrices and, in particular, on constant row sum matrices with nonnegative/nonpositive entries, viewed as cones, is also considered.

1. Introduction and Notation

Constant row sum square matrices are the closed linear span of generalized row stochastic matrices, i.e., matrices with real entries and row sums equal to one. In particular, zero row sum matrices can be viewed as a special kind of limit points of constant (non-zero) row sum matrices. Laplacian matrices are an important example of zero row sum matrices [5]. In this paper we study the Lie structure and decompositions of these matrices as stated in the abstract. We will use the following notations:

- $\hat{A}$: The invertible elements of a matrix set A.
- $\mathrm{Aut}(\Omega)$: The automorphism group of a semigroup Ω.
- $S_\lambda(n,\mathbb{R})$: $(n\times n)$ matrices with real entries and with all row sums equal to $\lambda \neq 0$.
- $\mathfrak{zrs}(n,\mathbb{R})$: $(n\times n)$ matrices with real entries and with all row sums equal to 0.
- $\mathfrak{zcs}(n,\mathbb{R})$: $(n\times n)$ matrices with real entries and with all column sums equal to 0.
- $\mathfrak{zrcs}(n,\mathbb{R}) = \mathfrak{zrs}(n,\mathbb{R}) \cap \mathfrak{zcs}(n,\mathbb{R})$: $(n\times n)$ matrices with real entries and with all row and column sums equal to 0.
- $\mathfrak{crs}(n,\mathbb{R}) = \mathfrak{zrs}(n,\mathbb{R}) \cup_{\lambda\in\mathbb{R}-\{0\}} S_\lambda(n,\mathbb{R})$: $(n\times n)$ matrices with real entries and with all row sums equal to some constant.
- $\mathfrak{crs}_+(n,\mathbb{R})$: $(n\times n)$ matrices with nonnegative real entries and with all row sums equal to some positive constant.

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• $\mathfrak{crs}_-(n,\mathbb{R})$: $(n\times n)$ matrices with non-positive real entries and with all row sums equal to some negative constant.

• $\mathfrak{gl}(n,\mathbb{R})$: $(n\times n)$ matrices with real entries.

• $GL(n,\mathbb{R})$: invertible $(n\times n)$ matrices with real entries.

• $[X,Y] := XY - YX$ for $X,Y\in\mathfrak{gl}(n,\mathbb{R})$.

• $e_1=(1,0,...,0)$, $e_2=(0,1,0,...,0)$, ...,$e_n=(0,0,...,0,1)$: the standard orthonormal basis for $\mathbb{R}^n$.

• $E_i(n)$: the $(n\times n)$ matrix with all rows equal to e_i. $E_{i,j}$: the $(n\times n)$ matrix that has a 1 in position i,j and zeroes everywhere else.

• I_n: the $(n\times n)$ identity matrix.

• J_n: the $(n\times n)$ matrix all of whose entries are equal to 1.

• A^T: the transpose of A.

• $\mathfrak{a}=e\mathfrak{a}e\oplus e\mathfrak{a}(1-e)\oplus(1-e)\mathfrak{a}(1-e)\oplus(1-e)\mathfrak{a}e$: the Peirce decomposition [6] of an algebra $\mathfrak{a}$ with respect to its idempotent ($e^2=e$) element e.

• $\mathfrak{l}=\mathfrak{s}\oplus_s\mathfrak{r}$: the Levi decomposition of the Lie algebra $\mathfrak{l}$.

• $\mathfrak{sl}(n,\mathbb{R})$: the Lie algebra of traceless $(n\times n)$ matrices with real entries.

• $x\star y := \frac{xy+yx}{2}$: Jordan algebra (non-associative) product.

• $\mathfrak{J}=\mathfrak{J}_1\oplus\mathfrak{J}_{1/2}\oplus\mathfrak{J}_0$: the Peirce decomposition [6] of a Jordan algebra $\mathfrak{J}$ with respect to its idempotent ($e\star e=e$) element e where $\mathfrak{J}_i=\{x_i\in\mathfrak{J}: x_i\star e = ix_i\}$, $i=1,1/2,0$.

2. Lie Structure of $\hat{S}_1(n,\mathbb{R})$

Proposition 2.1. *Let $A\in\mathfrak{gl}(n,\mathbb{R})$. Then: $A\in\mathfrak{zrs}(n,\mathbb{R})$ iff $AJ_n=\mathbf{0}$ and $A\in\mathfrak{crs}(n,\mathbb{R})$ iff $AJ_n=\lambda_A J_n$ where $\lambda_A\in\mathbb{R}$ the sum of each row of A. Moreover, zero row sum matrices are not invertible.*

Proof. Direct computation. For invertibility, if $A\in\mathfrak{zrs}(n,\mathbb{R})$ were invertible then $AJ_n=\mathbf{0}$ would imply $J_n=\mathbf{0}$. □

We notice that for each $\lambda\neq 0$, $S_\lambda(n,\mathbb{R})=\lambda\, S_1(n,\mathbb{R})$. Thus it suffices to study $S_1(n,\mathbb{R})$.

Lemma 2.2. *$\mathfrak{crs}(n,\mathbb{R})$ is a matrix semigroup and $\mathfrak{zrs}(n,\mathbb{R})$, $S_1(n,\mathbb{R})$ are subsemigroups of $\mathfrak{crs}(n,\mathbb{R})$.*

Proof. The semigroup operation is matrix multiplication. If $A,B\in\mathfrak{crs}(n,\mathbb{R})$ then $AJ_n=\lambda A$ and $BJ_n=\mu B$ for some $\lambda,\mu\in\mathbb{R}$. Thus $ABJ_n=\lambda\mu J_n$ so $AB\in\mathfrak{crs}(n,\mathbb{R})$. Similarly, if $A,B\in\mathfrak{zrs}(n,\mathbb{R})\subset\mathfrak{crs}(n,\mathbb{R})$ then $AJ_n=0$ and $BJ_n=0$. Thus $ABJ_n=0$ so $AB\in\mathfrak{crs}(n,\mathbb{R})$. Also, if $A,B\in S_1(n,\mathbb{R})\subset\mathfrak{crs}(n,\mathbb{R})$ then $AJ_n=J_n$ and $BJ_n=J_n$. Thus $ABJ_n=J_n$ so $AB\in S_1(n,\mathbb{R})$. □

Proposition 2.3. *For $n\geq 3$, $\hat{S}_1(n,\mathbb{R})$, the set of invertible $S_1(n,\mathbb{R})$ matrices, is a non-compact, not connected matrix Lie group whose Lie algebra $\hat{\mathfrak{s}}_1(n,\mathbb{R})$ is equal to $\mathfrak{zrs}(n,\mathbb{R})$.*

Proof. Let $A, B \in \hat{S}_1(n,\mathbb{R})$. Then $AJ_n = J_n$ implies that $J_n = A^{-1}J_n$ so $A^{-1} \in \hat{S}_1(n,\mathbb{R})$. Moreover, $AJ_n = J_n$ and $BJ_n = J_n$ imply that $ABJ_n = J_n$ which means that $AB \in \hat{S}_1(n,\mathbb{R})$, i.e., $\hat{S}_1(n,\mathbb{R})$ is a group. The equivalence of matrix norm convergence to entry-wise convergence implies that $\hat{S}_1(n,\mathbb{R})$ is a closed matrix subgroup of $GL(n,\mathbb{R})$ and so $\hat{S}_1(n,\mathbb{R})$ is a Lie group which is not compact since it contains the matrix

$$A_n = \begin{pmatrix} n+1 & -n & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \ \|A_n\| = \max_{i,j} |A_n(i,j)| = n+1 \to \infty$$

and is not path connected since the $(n \times n)$ identity matrix $I_n \in \hat{S}_1(n,\mathbb{R})$ with $\det I_n = 1$ and the block matrix

$$Y = \left(\begin{array}{c|c} I_{n-2} & \mathbf{0} \\ \hline \mathbf{0} & Y_0 \end{array}\right) \in \hat{S}_1(n,\mathbb{R}) \ , \ Y_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \det Y = -1$$

cannot be connected with a continuous path lying entirely in $\hat{S}_1(n,\mathbb{R})$ since that would imply the existence of an invertible matrix with determinant equal to zero. To show that

$$\hat{\mathfrak{s}}_1(n,\mathbb{R}) = \mathfrak{zrs}(n,\mathbb{R})$$

we notice that each $X \in \mathfrak{zrs}(n,\mathbb{R})$ is of the form $A'(0)$, where

$$A(t) = I_n + tX \in \hat{S}_1(n,\mathbb{R})$$

with $A(0) = I$, where t is in a closed interval containing 0 in which

$$\det A(t) \neq 0.$$

Thus

$$\mathfrak{zrs}(n,\mathbb{R}) \subseteq \hat{\mathfrak{s}}_1(n,\mathbb{R}).$$

For the opposite inclusion, suppose that

$$A(t) = \left(a_{ij(t)}\right) \in \hat{S}_1(n,\mathbb{R})$$

with $A(0) = I$ and $A'(0) = X \in \mathfrak{zrs}(n,\mathbb{R})$. Then for each $i = 1, 2, ..., n$,

$$\sum_{j=1}^{n} a_{ij}(t) = 1 \implies \sum_{j=1}^{n} a'_{ij}(t) = 0 \implies \sum_{j=1}^{n} a'_{ij}(0) = 0,$$

i.e., X is a zero row sum matrix, so

$$\hat{\mathfrak{s}}_1(n,\mathbb{R}) \subseteq \mathfrak{zrs}(n,\mathbb{R}).$$

$\square$

3. The Derived Lie Algebra of $S_1(n,\mathbb{R})$

Definition 3.1. We define $S_1(n,\mathbb{R})'$ to be the set of all finite linear combinations of elements of the form $[A,B]$, where $A,B \in S_1(n,\mathbb{R})$.

Proposition 3.2. *$S_1(n,\mathbb{R})'$ is a Lie algebra.*

Proof. Clearly, $S_1(n,\mathbb{R})'$ is a vector space. To see that it is closed under the Lie bracket operation, let $[A_1,B_1]$ and $[A_2,B_2]$ be in $S_1(n,\mathbb{R})'$. Then

$$[[A_1,B_1],[A_2,B_2]] = [A_1B_1 - B_1A_1, A_2B_2 - B_2A_2]$$

$$= [A_1B_1, A_2B_2] - [A_1B_1, B_2A_2] - [B_1A_1, A_2B_2] + [B_1A_1, B_2A_2] \in S_1(n,\mathbb{R})'$$

since for all $i,j \in \{1,2\}$, $A_iB_j, B_jA_i \in S_1(n,\mathbb{R})$ because $S_1(n,\mathbb{R})$ is a semigroup. □

Since commutators have zero trace, the derived set is always contained in $\mathfrak{sl}(n,\mathbb{R})$.

Proposition 3.3. *For $n \geq 3$, the n^2-n-1 dimensional derived Lie algebra $S_1(n,\mathbb{R})'$ admits the Levi decomposition*

$$S_1(n,\mathbb{R})' = \mathfrak{s} \oplus_s \mathfrak{r},$$

where $\mathfrak{r}$ is the $(n-1)$-dimensional abelian matrix Lie algebra generated by the matrices

$$R_i := [E_n(n), E_i(n)] = E_i(n) - E_n(n)\ ,\ i = 1,...,n-1$$

and $\mathfrak{s}$ is the (n^2-2n) -dimensional matrix Lie algebra

$$\mathfrak{s} = \mathfrak{zrcs}(n,\mathbb{R}) \cap \mathfrak{sl}(n,\mathbb{R}).$$

Moreover, $S_1(n,\mathbb{R})'$ is a not semi-simple (thus not simple), not solvable and not nilpotent Lie subalgebra of $\mathfrak{zrs}(n,\mathbb{R})$.

Proof. Since the R_i's are linearly independent $\mathfrak{r}$ is $(n-1)$-dimensional and, using the fact that

$$E_k(n)E_m(n) = E_m(n)\ ,\ k,m = 1,2,...,n$$

we obtain

$$[R_i,R_j] = [E_n(n) - E_i(n), E_n(n) - E_j(n)] = E_n(n) - E_j(n) - E_n(n) + E_j(n) = 0$$

so $\mathfrak{r}$ is abelian. To prove the direct sum decomposition, let $A,B \in S_1(n,\mathbb{R})$. Then by Lemma 2.2, AB and BA have row sums equal to 1 so $[A,B]$ is a zero row sum matrix. Defining the matrix

$$C = (c_{ij})_{i,j=1}^n \in \mathfrak{r},$$

where

$$c_{i,j} = \begin{cases} \lambda_i, & \text{if } j \neq n, \\ \\ -\sum_{i=1}^{n-1} \lambda_i, & \text{if } j = n, \end{cases}$$

and the λ_i's are chosen so that for each $i = 1,2,...,n-1$

$$n \cdot \lambda_i = i - \text{th column sum of } [A,B]$$

we find that $[A,B]-C$ is a zero row and column sum, traceless matrix. Thus

$$S_1(n,\mathbb{R})' \subseteq \mathfrak{s} \oplus \mathfrak{r},$$

where the directness of the sum is proved by noticing that if

$$C = (c_{ij})_{i,j=1}^n \in \mathfrak{r}$$

then the c_{ij}'s must be as above but with the λ_i's arbitrary. Such a matrix C will also be an element of $\mathfrak{r}$ if and only if all λ_i's are equal to zero. To prove that the two sets $S_1(n,\mathbb{R})'$ and $\mathfrak{s}\oplus\mathfrak{r}$ are actually equal we will show that the orthogonal complement of $\mathfrak{r}$ with respect to the trace inner product in $S_1(n,\mathbb{R})'$ is $\mathfrak{s}$. In other words we will show that if $X \in S_1(n,\mathbb{R})'$ then: $X \perp R_i$ for all $i = 1, 2, ...$ if and only if $X \in \mathfrak{zrcs}(n,\mathbb{R}) \cap \mathfrak{sl}(n,\mathbb{R})$. It suffices to consider the case $X = [A,B]$. Then, being a commutator, using properties of the trace functional we see that X is traceless. Moreover, being in $S_1(n,\mathbb{R})'$, X is a zero row sum matrix. For each $i = 1, 2, ...$, $\mathrm{Tr}(\mathrm{R_i X^T}) = 0$ is equivalent to saying that the $i-th$ column sum of X is equal to the n-th column sum of X. Equivalently, all column sums of X are equal. That, combined with the fact that X is a zero row sum matrix, is equivalent to saying that X is a zero column sum matrix as well. If $X = (x_{ij})_{i,j=1,2,...,n} \in S_1(n,\mathbb{R})'$ then for each one of the basis matrices R_i of $\mathfrak{r}$

$$[R_i, X] = \sum_{j=1}^{n} (x_{nj} - x_{ij}) E_j \in \mathfrak{r}.$$

It follows that

$$[\mathfrak{r}, S_1(n,\mathbb{R})'] \subseteq \mathfrak{r}$$

so $\mathfrak{r}$ is a solvable (being abelian) ideal of $S_1(n,\mathbb{R})'$ and in particular

$$[\mathfrak{r}, \mathfrak{s}] \subseteq \mathfrak{r}.$$

To see that $\mathfrak{r}$ is the maximal solvable ideal of $S_1(n,\mathbb{R})'$, i.e., to show that the radical $\mathfrak{r}'$ of $S_1(n,\mathbb{R})'$ is $\mathfrak{r}$, suppose that $\mathfrak{r}' = \mathfrak{r} \oplus \mathfrak{r}_0$ where $\mathfrak{r}_0 \subseteq \mathfrak{s}$. Since $[\mathfrak{r}_0^{(k)}, \mathfrak{r}_0^{(k)}] \subseteq [\mathfrak{r}'^{(k)}, \mathfrak{r}'^{(k)}]$ and $\mathfrak{r}'$ is solvable it follows that $\mathfrak{r}_0$ is solvable. Moreover, since $[\mathfrak{r}_0, \mathfrak{s}] \subseteq [\mathfrak{r}', \mathfrak{s}] \subseteq [\mathfrak{r}', S_1(n,\mathbb{R})'] \subseteq \mathfrak{r}'$ and $[\mathfrak{r}_0, \mathfrak{s}] \subseteq [\mathfrak{s}, \mathfrak{s}] \subseteq \mathfrak{s}$ it follows that $[\mathfrak{r}_0, \mathfrak{s}] \subseteq \mathfrak{r}_0$, i.e., $\mathfrak{r}_0$ is an ideal of $\mathfrak{s}$. But $\mathfrak{s}$ is semi-simple therefore its only solvable ideal is $\{\mathbf{0}\}$. Thus $\mathfrak{r}_0 = \{\mathbf{0}\}$ and so $\mathfrak{r}' = \mathfrak{r}$. Thus

$$S_1(n,\mathbb{R})' = \mathfrak{s} \oplus_s \mathfrak{r}.$$

To show that $\mathfrak{s}$ is semi-simple we observe that if $\mathfrak{s}$ had a solvable ideal $\mathfrak{h}$ different from $\{\mathbf{0}\}$ then $\mathfrak{h}$ would also be an ideal of $S_1(n,\mathbb{R})'$ and by the maximality of $\mathfrak{r}$ we would obtain that $\mathfrak{h} \subseteq \mathfrak{r}$. But $\mathfrak{h} \subseteq \mathfrak{s}$ and since $\mathfrak{s} \cap \mathfrak{r} = \{\mathbf{0}\}$ we would have $\mathfrak{h} = \{\mathbf{0}\}$. Since $S_1(n,\mathbb{R})'$ contains an abelian (thus solvable) proper ideal $\mathfrak{r}$ other than the trivial one it follows that $S_1(n,\mathbb{R})'$ is not semi-simple thus not simple either. Moreover, since the maximal solvable ideal $\mathfrak{r}$ of $S_1(n,\mathbb{R})'$ is not $S_1(n,\mathbb{R})'$ itself, it follows that $S_1(n,\mathbb{R})'$ is not solvable. Since every nilpotent algebra is solvable, $S_1(n,\mathbb{R})'$ is not nilpotent either. □

A general basis for $S_1(n,\mathbb{R})'$, $n \geq 3$, is provided by the n^2-n-1 matrices

$$X_{i,j} := E_{i,j} - E_{i,n} \ , \ i,j = 1,2,...,n-1 \ , \ i \neq j$$

$$Y_{i,n-1} := E_{n,i} - E_{n,n-1}\ ,\ i = 1, 2, ..., n-2$$

$$Z_{i,n} := X_{i,i} + X_{n,n-1}\ ,\ i = 1, 2, ..., n-1$$

where $E_{i,j}$ is the $(n \times n)$ matrix that has a 1 in position i, j and zeros everywhere else, corresponding to the natural basis for $\mathfrak{gl}(n, \mathbb{R})$. It is easy to see that the above matrices are linearly independent and so (being the right number) they form a basis for $S_1(n, \mathbb{R})'$. For example in the case $n = 3$

$$X_{12} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\ ,\ X_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\ ,\ Y_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

$$Z_{13} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}\ ,\ Z_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

For $n = 4$ the basis matrices are

$$X_{12} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X_{13} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{21} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$X_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, X_{32} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$Y_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}, Y_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

$$Z_{14} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, Z_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$Z_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

4. Lie Structure of $\mathfrak{crs}(n,\mathbb{R})$ and $\mathfrak{zrs}(n,\mathbb{R})$ for $n \geq 3$

Proposition 4.1. *For $n \geq 3$ $\mathfrak{zrs}(n,\mathbb{R})$ and $\mathfrak{crs}(n,\mathbb{R})$ are not semi-simple, not nilpotent, not solvable Lie subalgebras of $\mathfrak{gl}(n,\mathbb{R})$ admitting the Levi decompositions*

$$\mathfrak{zrs}(n,\mathbb{R}) = \mathfrak{s} \oplus_s \mathfrak{r}$$

and

$$\mathfrak{crs}(n,\mathbb{R}) = \mathfrak{s} \oplus_s \mathfrak{r}_0,$$

where

$$\mathfrak{s} = \mathfrak{zrcs}(n,\mathbb{R}) \cap \mathfrak{sl}(n,\mathbb{R})$$

and $\mathfrak{r}$, $\mathfrak{r}_0$ are respectively the linear spaces generated by $R_i := [E_n(n), E_i(n)]$, $i = 1, ..., n-1$ and $Z_n := I_n - \frac{1}{n}J_n$ for $\mathfrak{r}$ and by R_i, $i = 1, ..., n-1$, Z_n and I_n for $\mathfrak{r}_0$. Moreover, $\mathfrak{crs}(n,\mathbb{R})' = S_1(n,\mathbb{R})'$.

Proof. By Proposition 3.3 all elements of the subalgebra $S_1(n,\mathbb{R})'$ of $\mathfrak{zrs}(n,\mathbb{R})$ are trace zero. Since $\operatorname{Tr} Z_n = \operatorname{Tr}\left(I_n - \frac{1}{n}J_n\right) = n-1 \neq 0$ for $n > 1$ it follows that $Z_n \notin S_1(n,\mathbb{R})'$. Since , by Proposition 3.3, $S_1(n,\mathbb{R})'$ has dimension n^2-n-1 and $Z_n \in \mathfrak{zrs}(n,\mathbb{R})$ with the dimension of $\mathfrak{zrs}(n,\mathbb{R})$ equal to n^2-n, it follows that Z_n is the sole generator of the missing non-zero trace elements of $\mathfrak{zrs}(n,\mathbb{R})$. We thus obtain the postulated direct sum decomposition. By Proposition 3.3

$$[\mathfrak{r}, \mathfrak{s}] \subseteq \mathfrak{r}$$

thus proving the postulated Levi decomposition. The proof for $\mathfrak{crs}(n,\mathbb{R})$ follows easily from the fact that each element of $\mathfrak{crs}(n,\mathbb{R})$ is equal to an element of $\mathfrak{zrs}(n,\mathbb{R})$ plus a multiple of the identity. Similarly, since the elements of $\mathfrak{crs}(n,\mathbb{R})$ are multiples of elements of $S_1(n,\mathbb{R})$, it follows that $\mathfrak{crs}(n,\mathbb{R})' = S_1(n,\mathbb{R})'$. Since the non-trivial proper subalgebra $S_1(n,\mathbb{R})'$ of $\mathfrak{zrs}(n,\mathbb{R})$ and $\mathfrak{crs}(n,\mathbb{R})$ is not solvable they are not solvable either. Since every nilpotent algebra is solvable, $\mathfrak{zrs}(n,\mathbb{R})$ and $\mathfrak{crs}(n,\mathbb{R})$ are not nilpotent. Moreover, $\mathfrak{zrs}(n,\mathbb{R})$ and $\mathfrak{crs}(n,\mathbb{R})$ are not semi-simple since they contain the non-trivial abelian ideal generated by the R_i's. □

5. The Case $n = 2$

In the (2×2) case, $\hat{S}_1(2,\mathbb{R})$ is a non-compact, not connected matrix Lie group whose Lie algebra $\hat{\mathfrak{s}}_1(2,\mathbb{R})$ is equal to $\mathfrak{zrs}(2,\mathbb{R})$ just as in the case $n \geq 3$. However,

$$S_1(2,\mathbb{R})' := [S_1(2,\mathbb{R}), S_1(2,\mathbb{R})]$$
$$= \{X \in \mathfrak{zrs}(2,\mathbb{R}) \mid X = x \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \; x \in \mathbb{R}\}$$

is an one-dimensional abelian, simple, not semi-simple, solvable and nilpotent Lie subalgebra of $\mathfrak{zrs}(2,\mathbb{R})$ having a trivial Levi decomposition

$$S_1(2,\mathbb{R})' = \{\mathbf{0}\} \oplus_s S_1(2,\mathbb{R})'.$$

Thus $S_1(2,\mathbb{R})'$ is an example of a Lie algebra where simplicity does not imply semi-simplicity due to the fact that $\dim L = 1$. We also have that

$$\mathfrak{crs}(2,\mathbb{R}) = \{\begin{pmatrix} a & \lambda - a \\ b & \lambda - b \end{pmatrix} \mid \lambda, a, b \in \mathbb{R}\}$$

is a three-dimensional Lie subalgebra of $\mathfrak{gl}(2,\mathbb{R})$ with generators

$$A := \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \ , \ B := \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \ , \ E_2(2) := \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfying the commutation relations

$$[B,A] = B + A \ , \ [A, E_2(2)] = 0 \ , \ [E_2(2), B] = B + A.$$

In particular A and B are the generators of $\mathfrak{zrs}(2,\mathbb{R})$. Moreover, $\mathfrak{crs}(2,\mathbb{R})' = S_1(2,\mathbb{R})'$, and $\mathfrak{zrs}(2,\mathbb{R})$, $\mathfrak{crs}(2,\mathbb{R})$ are solvable, not simple, not semi-simple, not nilpotent Lie subalgebras of $\mathfrak{gl}(2,\mathbb{R})$ having the trivial Levi decomposition

$$\mathfrak{zrs}(2,\mathbb{R}) = \{\mathbf{0}\} \oplus_s \mathfrak{zrs}(2,\mathbb{R}), \quad \mathfrak{crs}(2,\mathbb{R}) = \{\mathbf{0}\} \oplus_s \mathfrak{crs}(2,\mathbb{R}).$$

6. Peirce Decomposition

Proposition 6.1. *The matrix*

$$e = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & 1-\sum_{i=1}^{n-1} x_i \\ x_1 & x_2 & \cdots & x_{n-1} & 1-\sum_{i=1}^{n-1} x_i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & \cdots & x_{n-1} & 1-\sum_{i=1}^{n-1} x_i \end{pmatrix}$$
$$= E_n(n) - \sum_{i=1}^{n-1} x_i R_i \in S_1(n,\mathbb{R})$$

is an idempotent element of $\mathfrak{crs}(n,\mathbb{R})$, $n = 2,3,...$, *containing the matrices* $E_i(n)$, $i = 1,...,n$, *where the* R_i*'s are as in Proposition 3.3.*

Proof. Let $s = \sum_{i=1}^{n-1} x_i$. Then the proof that $e^2 = e$ follows from the fact that

$$(x_1 \ x_2 \ \cdots x_{n-1} \ 1-s) \cdot (x_j \ x_j \ \cdots x_j \ x_j)^T = x_j \quad , \quad j = 1,...,n-1$$

and

$$(x_1 \ x_2 \ \cdots x_{n-1} \ 1-s) \cdot (1-s \ 1-s \ \cdots 1-s \ 1-s)^T = 1-s$$

To show that $E_i(n)$ is included just take $x_i = 1$ and $x_j = 0$ for $j \neq i$. □

Lemma 6.2. *Idempotent elements with respect to the usual matrix product* xy *coincide with idempotent elements with respect to the Jordan product* $x \star y = \frac{1}{2}(xy+yx)$.

Proof. The proof follows from the fact that

$$e = e \star e = \frac{ee+ee}{2} = \frac{2e}{2} = e.$$

□

Proposition 6.3. *Corresponding to the idempotent element*

$$e = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & 1-\sum_{i=1}^{n-1} x_i \\ x_1 & x_2 & \cdots & x_{n-1} & 1-\sum_{i=1}^{n-1} x_i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & \cdots & x_{n-1} & 1-\sum_{i=1}^{n-1} x_i \end{pmatrix}$$

of Proposition 6.1 and the usual matrix product the algebra $\mathfrak{a} := \mathfrak{crs}(n, \mathbb{R})$ *admits the Peirce decomposition*

$$\mathfrak{a} = e\mathfrak{a}e \oplus e\mathfrak{a}(\mathbf{I} - e) \oplus (\mathbf{I} - e)\mathfrak{a}(\mathbf{I} - e) \oplus (\mathbf{I} - e)\mathfrak{a}e,$$

where

$$e\mathfrak{a}e = \{\lambda e \,:\, \lambda \in \mathbb{R}\}$$

$$e\mathfrak{a}(\mathbf{I} - e) = \{B \in \mathfrak{zrs}(n, \mathbb{R}) \,:\, B = \sum_{i=1}^{n-1} \lambda_i R_i\,,\, \lambda_i \in \mathbb{R}\}$$

i.e., $e\mathfrak{a}(\mathbf{I} - e)$ *consists of zero row sum matrices with all rows the same,*

$$(\mathbf{I} - e)\mathfrak{a}(\mathbf{I} - e) = \{\mathbf{0}\} \cup \{B \in \mathfrak{zrs}(n, \mathbb{R}) \,:\, B \neq \sum_{i=1}^{n-1} \lambda_i R_i\,,\, \lambda_i \in \mathbb{R}\}$$

i.e., with the exception of the zero matrix, $(\mathbf{I} - e)\mathfrak{a}(\mathbf{I} - e)$ *consists of zero row sum matrices with not all rows the same, and*

$$(\mathbf{I} - e)\mathfrak{a}e = \{\mathbf{0}\}.$$

Proof. Direct computation shows that the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & \lambda - \sum_{j=1}^{n-1} a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} & \lambda - \sum_{j=1}^{n-1} a_{2j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(n-1)} & \lambda - \sum_{j=1}^{n-1} a_{nj} \end{pmatrix} \in \mathfrak{crs}(n, \mathbb{R})$$

admits the Peirce decomposition

$$A = eAe + eA(\mathbf{I} - e) + (\mathbf{I} - e)A(1 - e) + (\mathbf{I} - e)Ae,$$

where $eAe = \lambda e$, $eA(\mathbf{I} - e)$ is the $(n \times n)$ matrix with all entries of its jth-column, $j = 1, ..., n-1$, equal to

$$\sum_{i=1}^{n-1} x_i\,(a_{ij} - a_{nj}) + a_{nj} - \lambda x_j$$

and all entries of the n-th column equal to

$$\sum_{j=1}^{n-1} \left(a_{nj} \left(\sum_{i=1}^{n-1} x_i - 1 \right) + x_j \left(\lambda - \sum_{i=1}^{n-1} a_{ji} \right) \right)$$

$(\mathbf{I} - e)A(\mathbf{I} - e)$ is the $(n \times n)$ matrix whose ij-th entry is

$$a_{ij} - a_{nj} + \sum_{k=1}^{n-1} x_k\,(a_{nj} - a_{kj})$$

for $j = 1, ..., n-1$ while for $j = n$ it is

$$\sum_{k=1}^{n-1} \left((a_{nk} - a_{ik}) + x_k \sum_{m=1}^{n-1} (a_{km} - a_{nm}) \right)$$

and $(\mathbf{I} - e)Ae = \mathbf{0}$. □

For example, for the stochastic matrix A and the idempotent element e below

$$A=\begin{pmatrix} 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad e=\begin{pmatrix} 1 & -2 & 3 & 0 & -1 \\ 1 & -2 & 3 & 0 & -1 \\ 1 & -2 & 3 & 0 & -1 \\ 1 & -2 & 3 & 0 & -1 \\ 1 & -2 & 3 & 0 & -1 \end{pmatrix}$$

we find $eAe=e$, $(\mathbf{I}-e)Ae=\mathbf{0}$

$$eA(\mathbf{I}-e)=\begin{pmatrix} -1/4 & 11/4 & -9/2 & 3/4 & 5/4 \\ -1/4 & 11/4 & -9/2 & 3/4 & 5/4 \\ -1/4 & 11/4 & -9/2 & 3/4 & 5/4 \\ -1/4 & 11/4 & -9/2 & 3/4 & 5/4 \\ -1/4 & 11/4 & -9/2 & 3/4 & 5/4 \end{pmatrix}$$

$$(\mathbf{I}-e)A(\mathbf{I}-e)=\begin{pmatrix} -3/4 & -3/4 & 2 & -3/4 & 1/4 \\ -3/4 & -3/4 & 5/2 & -3/4 & -1/4 \\ -1/2 & -1/2 & 3/2 & -1/2 & 0 \\ -3/4 & -3/4 & 2 & -3/4 & 1/4 \\ -3/4 & -3/4 & 3/2 & -3/4 & 3/4 \end{pmatrix}$$

Proposition 6.4. *With respect to the Jordan algebra product $x \star y = \frac{1}{2}(xy+yx)$ and the idempotent element*

$$e=\begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & 1-\sum_{i=1}^{n-1} x_i \\ x_1 & x_2 & \cdots & x_{n-1} & 1-\sum_{i=1}^{n-1} x_i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1 & x_2 & \cdots & x_{n-1} & 1-\sum_{i=1}^{n-1} x_i \end{pmatrix}$$

of Proposition 6.1, the Jordan algebra $\mathfrak{J} := \mathfrak{crs}(n,\mathbb{R})$ admits the Peirce decomposition

$$\mathfrak{crs}(n,\mathbb{R}) = \mathfrak{J}_1 \oplus \mathfrak{J}_{1/2} \oplus \mathfrak{J}_0,$$

where

$$\mathfrak{J}_1 = \{\lambda e \,:\, \lambda \in \mathbb{R}\}$$

$$\mathfrak{J}_{1/2} = \{A \in \mathfrak{zrs}(n,\mathbb{R}) \,:\, A = \sum_{i=1}^{n-1} \lambda_i R_i \ ,\ \lambda_i \in \mathbb{R}\},$$

where the R_i's are as in Proposition 3.3, i.e., $\mathfrak{J}_{1/2}$ consists of $(n \times n)$ zero row sum matrices with identical rows. For $\mathfrak{J}_0$: if $x_i = 0$, $i = 1, ..., n-1$, then

$$\mathfrak{J}_0 = \{A = (a_{ij}) \in \mathfrak{zrs}(n,\mathbb{R}) \,:\, a_{nj} = 0\,,\ j = 1, ..., n\}$$

while if $x_1 = ... = x_{i_0-1} = 0$ and $x_{i_0} \neq 0$, for some $i_0 \in \{1, 2, ..., n-2\}$, then

$$\mathfrak{J}_0 = \{A = (a_{ij}) \in \mathfrak{zrs}(n,\mathbb{R}) \,:\,$$
$$a_{i_0 j} = -\frac{\sum_{i=i_0+1}^{n-1} a_{ij} x_i + a_{nj}(1-\sum_{i=i_0}^{n-1} x_i)}{x_{i_0}} \ ,\ j = 1, ..., n-1\}$$

i.e., depending on the entries of e, $\mathfrak{J}_0$ consists of zero row sum matrices with all but one rows arbitrary.

Proof. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1(n-1)} & \lambda - \sum_{j=1}^{n-1} a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2(n-1)} & \lambda - \sum_{j=1}^{n-1} a_{2j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(n-1)} & \lambda - \sum_{j=1}^{n-1} a_{nj} \end{pmatrix} \in \mathfrak{crs}(n,\mathbb{R})$$

Solving the equations $A \star e = iA$ for $i = 1, 1/2, 0$, setting the sum of the row one entries of $A \star e - iA$ equal o zero, we find that: for $i = 0$ and $i = 1/2$ we must have $\lambda = 0$. For $i = 1$ we can have $\lambda \in \mathbb{R}$ be arbitrary and we obtain the stated form of the solution matrices A. □

For example, for the stochastic matrix A and the idempotent element e below

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix} \quad e = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$$

we have the Jordan algebra Peirce decomposition

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 1/6 & -4/3 & 7/6 \\ 1/6 & -4/3 & 7/6 \\ 1/6 & -4/3 & 7/6 \end{pmatrix} + \begin{pmatrix} -1/6 & 1/3 & -1/6 \\ -2/3 & 1/3 & 1/3 \\ -5/6 & 2/3 & 1/6 \end{pmatrix}$$

7. Automorphisms on $\mathfrak{crs}(n,\mathbb{R})$ and $\mathfrak{zrs}(n,\mathbb{R})$

Definition 7.1. Let $\mathfrak{a}$ be a matrix semigroup. An *automorphism* $\phi : \mathfrak{a} \mapsto \mathfrak{a}$ is a one-to-one and onto mapping satisfying $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in \mathfrak{a}$. We denote by $\mathrm{Aut}(\mathfrak{a})$ the group of all such ϕ's and by $\mathrm{Inn}(\mathfrak{a})$ the *inner automorphisms* of the form $\phi(X) = M^{-1}XM$ for some invertible matrix $M \in \mathfrak{a}$, i.e., $M \in \hat{\mathfrak{a}}$ provided that $\hat{\mathfrak{a}}$ is nonempty.

Proposition 7.2. *For each matrix $M \in \hat{\mathfrak{crs}}(n,\mathbb{R})$, the mappings $\phi : A \mapsto M^{-1}AM$ are inner automorphisms of $\mathfrak{crs}(n,\mathbb{R})$ and $\mathfrak{zrs}(n,\mathbb{R})$.*

Proof. We will use Proposition 2.1. If $A \in \mathfrak{zrs}(n,\mathbb{R})$ and $M \in \hat{\mathfrak{crs}}(n,\mathbb{R})$ then $M^{-1}AMJ_n = \lambda_M M^{-1}AJ_n = \mathbf{0}$, where $\lambda_M \in \mathbb{R}$ the row sum of M. Thus $M^{-1}AM \in \mathfrak{zrs}(n,\mathbb{R})$ and every $B \in \mathfrak{zrs}(n,\mathbb{R})$ is of the form $M^{-1}AM$ for $A = MBM^{-1} \in \mathfrak{zrs}(n,\mathbb{R})$. Similarly, for $A \in \mathfrak{crs}(n,\mathbb{R})$ and $M \in \hat{\mathfrak{crs}}(n,\mathbb{R})$ we have $M^{-1}AMJ_n = \lambda_M M^{-1}AJ_n = \lambda_M\lambda_A M^{-1}J_n = \lambda_M\lambda_A{\lambda_M}^{-1}J_n = \lambda_A J_n$. Thus $M^{-1}AM \in \mathfrak{crs}(n,\mathbb{R})$ and every $B \in \mathfrak{crs}(n,\mathbb{R})$ is of the form $M^{-1}AM$ for $A = MBM^{-1} \in \mathfrak{crs}(n,\mathbb{R})$. The inverse map is in each case $A \mapsto MAM^{-1}$. Thus the maps $A \mapsto M^{-1}AM$, where $M \in \hat{\mathfrak{crs}}(n,\mathbb{R})$, are in $\mathrm{Inn}(\mathfrak{crs}(\mathrm{n},\mathbb{R}))$ and $\mathrm{Inn}(\mathfrak{zrs}(\mathrm{n},\mathbb{R}))$. □

Proposition 7.3. *$\mathfrak{zrs}(n,\mathbb{R})$ and $\mathfrak{crs}(n,\mathbb{R})$ are (not proper) convex cones while $\mathfrak{crs}_+(n,\mathbb{R})$ and $\mathfrak{crs}_-(n,\mathbb{R})$ are proper convex cones.*

Proof. If $A, B \in \mathfrak{zrs}(n,\mathbb{R})$ and $\lambda, \mu > 0$ then $(\lambda A + \mu B)J_n = \lambda AJ_n + \mu BJ_n = 0$ implies that $\mathfrak{zrs}(n,\mathbb{R})$ is a cone. Similarly, for $A, B \in \mathfrak{crs}(n,\mathbb{R})$, $(\lambda A + \mu B)J_n = \lambda AJ_n + \mu BJ_n = (\lambda\lambda_A + \mu\lambda_B)J_n$ implies that $\mathfrak{crs}(n,\mathbb{R})$ is a cone. Since in any norm topology on $\mathfrak{gl}(n,\mathbb{R})$, $\overline{\mathfrak{zrs}(n,\mathbb{R})} \cap (\overline{-\mathfrak{zrs}(n,\mathbb{R})}) = \overline{\mathfrak{zrs}(n,\mathbb{R})} = \mathfrak{zrs}(n,\mathbb{R}) \neq \{\mathbf{0}\}$, where the bar denotes topological closure, it follows that $\mathfrak{zrs}(n, \mathbb{R})$ is not a proper cone. An exact same reasoning shows that $\mathfrak{crs}(n, \mathbb{R})$ is not a proper cone either. The inclusion of a plus or minus sign in $\mathfrak{crs}_+(n, \mathbb{R})$ and $\mathfrak{crs}_-(n, \mathbb{R})$ shows that the above intersections of closures are in that case $\{\mathbf{0}\}$. Thus $\mathfrak{crs}_+(n, \mathbb{R})$ and $\mathfrak{crs}_-(n, \mathbb{R})$ are proper convex cones. □

Definition 7.4. Let Ω be a cone. An *automorphism* $\phi : \Omega \mapsto \Omega$ is a linear one-to-one and onto mapping [2]. We denote by $\mathrm{Aut}(\Omega)$ the group of all such ϕ's.

Proposition 7.5. *The automorphisms of* $\mathfrak{crs}_+(n, \mathbb{R})$ *and* $\mathfrak{crs}_-(n, \mathbb{R})$ *are of the form* $T(A) = Q\,P(A)$ *where* Q *is an* $(n \times n)$ *permutation (thus invertible) matrix and* $P : A \mapsto P(A)$ *is the one-to-one transformation that maps the matrix* A *to a matrix* $P(A)$ *each of whose rows is a permutation of the corresponding row of* A.

Proof. The underlying operation is matrix addition. By Theorem 3.4 of [4], since every matrix in $\mathfrak{crs}_+(n, \mathbb{R})$ is a positive constant multiple of a row stochastic matrix, all linear preservers of $\mathfrak{crs}_+(n, \mathbb{R})$ are of the form $T(A) = Q\,P(A)$ where Q is an $(n \times n)$ permutation matrix and each row of the $(n \times n)$ matrix $P(A)$ is a permutation of the corresponding row of A. In order for the linear preserver to be one-to-one Q and P must be as in the statement of this Lemma. □

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