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SOME QUANTUM DYNAMICAL SEMI-GROUPS WITH QUANTUM STOCHASTIC DILATION

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Abstract. We consider the GNS Hilbert space $\mathcal{H}$ of a uniformly hyper-finite C^*- algebra and study a class of unbounded Lindbladian arises from commutators. Exploring the local structure of UHF algebra, we have shown that the associated Hudson-Parthasarathy type quantum stochastic differential equation admits a unitary solution. The vacuum expectation of homomorphic cocycle, implemented by the Hudson-Parthasarathy flow, is conservative and gives the minimal semi-group associated with the formal Lindbladian. We also associate conservative minimal semi-groups to another class of Lindbladian by solving the corresponding Evan-Hudson equation.

1. Introduction

Quantum dynamical semi-groups (QDS) appear naturally when one studies the evolution of irreversible open quantum systems. QDS are non-commutative analogue to Markov semi-groups in classical probability. For a uniformly continuous semi-groups, the generator is a bounded, conditionally completely positive (CCP) map. In [7], Lindblad proved that for hyper-finite von Neumann algebras, which includes the case of $\mathcal{B}(\mathcal{H})$, the generator $\mathcal{L}$ of uniformly continuous QDS can be written as $\mathcal{L}(X) = \phi(X) + G^*X + XG$, where ϕ is a completely positive map and $G \in \mathcal{B}(\mathcal{H})$. In [1] Christensen and Evans proved that for general C^*-algebras, the generator of a uniformly continuous QDS exhibits the similar structure.

For the case of a strongly continuous QDS, structure of the generator is not well understood, Kato [6] and Davies [2] studied some unbounded operators or forms similar to above on $\mathcal{B}(\mathcal{H})$ and gave a construction of one-parameter semi-groups, so-called minimal semi-group. Under certain assumptions, Davies in [3] showed that the unbounded generator have a similar form as for the bounded case, thus extends the Lindblad's result to strongly continuous QDS. However, these semi-groups need not preserve the identity, i.e., need not be Markov. Generally such unbounded operator or form referred as Lindbladian. Starting with a Lindbladian, a similar construction of a minimal semi-group was done for any von Neumann algebra in [11].

In this article, we have considered Hudson-Parthasarathy (HP) quantum stochastic differential equation associated with a model of unbounded Lindbladian and construct the QDS by taking vacuum expectation. There are various attempts

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to study HP quantum stochastic differential equation with unbounded coefficients, for example see [4, 11] and references therein.

In section 2, we discuss briefly QDS and some results of quantum stochastic calculus and quantum stochastic differential equations (QSDE) with bounded operator coefficients. In section 3, a class of unbounded Lindblad form are defined on the GNS space of UHF C^*-algebra and properties of structure maps are studied. Finally, exploring the local structure of UHF algebra, it is shown that the associated HP equation admits a unitary solution. This implies that the expectation semi-group of the homomorphic co-cycle implemented by this unitary is conservative and therefore the unique (also minimal) C_0-contraction semi-group associated with the given form. The model is very special and hence simple enough to allow the construction of the minimal semi-group, without any of the machineries of the abstract theories, mentioned earlier in the introduction.

2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{B}(\mathcal{H})$ denotes the von Neumann algebra of bounded linear operators on $\mathcal{H}$.

Definition 2.1. A *quantum dynamical semi-group* on a von Neumann algebra $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a semi-group $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of completely positive maps on $\mathcal{A}$ with the following properties:

(*i*) $\mathcal{T}_t(I) \leq I$, for all $t \geq 0$.
(*ii*) $\mathcal{T}_t$ is a ultra-weakly continuous operator i.e. normal for all $t \geq 0$.
(*iii*) for each $a \in \mathcal{A}$, the map $t \to \mathcal{T}_t(a)$ is continuous with respect to the ultra-weak topology on $\mathcal{A}$.

A QDS is called *Markov* or *Conservative* if $\mathcal{T}_t(I) = I$ for every t.

Theorem 2.2. [7, 1, 10, 11] *A bounded map $\mathcal{L}$ on the von Neumann algebra $\mathcal{B}(\mathcal{H})$ is the infinitesimal generator of a uniformly continuous QDS $(\mathcal{T}_t)_{t\geq 0}$ if and only if it can be written as*

$$\mathcal{L}(X) = \sum_{n=1}^{\infty} L_n^* X L_n + G^* X + XG, \text{ for all } X \in \mathcal{B}(\mathcal{H}),$$

where L_n's and G are in $\mathcal{B}(\mathcal{H})$ and the series on the right side converges strongly, with G generator of a contraction semi-group in $\mathcal{H}$. The QDS is Markov if and only if

$$Re(G) = -\frac{1}{2} \sum_{n=1}^{\infty} L_n^* L_n.$$

For more general QDS, the generator can be understood as one coming from a similarly defined quadratic form on $\mathcal{H}$, e.g., for $X \in \mathcal{B}(\mathcal{H})$,

$$\langle u, \mathcal{L}(X)v\rangle \equiv \langle u, XGv\rangle + \langle Gu, Xv\rangle + \sum_{n=1}^{\infty} \langle L_n u, X L_n v\rangle \tag{2.1}$$

where these L_n and G are unbounded operators, G is the generator of a C_0-contraction semi-group in $\mathcal{H}$ such that $\mathrm{Dom}(G) \subseteq \mathrm{Dom}(L_n)$, for each n and

$$\langle u, \mathcal{L}(I)v\rangle \equiv \langle u, Gv\rangle + \langle Gu, v\rangle + \sum_{n=1}^{\infty} \langle L_n u, L_n v\rangle = 0, \tag{2.2}$$

for all $u, v \in \mathrm{Dom}(G)$.

Conversely, let G be the generator (not necessarily bounded) of a C_0-contraction semi-group in $\mathcal{H}$ and L_n be a family of closed densely defined linear operators in $\mathcal{H}$ with $\mathrm{Dom}(G) \subseteq \mathrm{Dom}(L_n)$ and let $\mathcal{L}$ define formally by (2.1) satisfies (2.2). Then the aim is to construct a canonical (minimal) semigroup associated with the formal Lindbladian $\mathcal{L}$, for some results in this direction see [2, 6, 11].

We conclude this section with a brief discussion of Quantum stochastic calculus developed by Hudson and Parthasarathy. We state a result of existence and uniqueness of unitary solution for QSDE. For detail see [10, 11].

For a separable Hilbert space $\mathcal{H}$, let $\Gamma_{sym}(\mathcal{H})$ denotes the symmetric Fock space over $\mathcal{H}$. For any $u \in \mathcal{H}$, we denote by $e(u)$, the exponential vector in $\Gamma_{sym}(\mathcal{H})$ associated with u:

$$e(u) = \bigoplus_{n\geq 0} \frac{1}{\sqrt{n!}} u^{\otimes n}.$$

Given a contraction T on $\mathcal{H}$, the second quantization $\Gamma(T)$ on $\Gamma_{sym}(\mathcal{H})$ is defined by $\Gamma(T)e(u) = e(Tu)$ and extends to a contraction on $\Gamma_{sym}(\mathcal{H})$. Moreover, if T is an isometry (respectively unitary), then so is $\Gamma(T)$.

Let us write Γ_{sym} for the symmetric Fock space $\Gamma_{sym}(L^2(\mathbb{R}_+, \mathbf{k}))$, where $\mathbf{k}$ is a Hilbert space with an orthonormal basis $\{e_l : 1 \leq l \leq m\}$. The following result provides a nice criterion for the existence of a unitary solution for an HP type QSDE with bounded coefficients.

Theorem 2.3. [10] *Let $H, \{L_i\ ;\ 1 \leq i \leq m\}, \{S^i_j\ ;\ 1 \leq i, j \leq m\}$ are bounded operators in $\mathcal{H}$ satisfying the following conditions:*

(i) H is self-adjoint.
(ii) $\sum_{1\leq i,j\leq m} S^i_j \otimes |e_i\rangle\langle e_j|$ is a unitary operator in $\mathcal{H} \otimes \mathbf{k}$.

Define

$$L^i_j = \begin{cases} S^i_j - \delta^i_j, & \text{if } 1 \leq i, j \leq m; \\ L_i, & \text{if } 1 \leq i \leq m, j = 0; \\ -\sum_{1\leq k\leq m} L^*_k S^k_j, & \text{if } 1 \leq j \leq m, i = 0; \\ -(\iota H + \frac{1}{2}\sum_{1\leq k\leq m} L^*_k L_k), & \text{if } i = j = 0; \end{cases} \tag{2.3}$$

where δ^i_j is the Kronecker's delta function. Then there exists a unique unitary process U_t satisfying the QSDE on $\mathcal{H} \otimes \Gamma_{sym}$

$$U_t = I + \sum_{i,j=0}^{m} \int_0^t U_s L^i_j \Lambda^j_i(ds), \tag{2.4}$$

where Λ_0^0 is time, for $i,j \geq 1, \Lambda_i^j$ is conservation, Λ_i^0 is creation and Λ_0^j is annihilation processes.

Let $(U_t)_{t\geq 0}$ be a unitary process satisfying (2.4). Then the family of homomorphisms $\{J_t : t \geq 0\}$ defined by

$$J_t(X) = U_t^*(X \otimes I)U_t, \ X \in \mathcal{B}(\mathcal{H}).$$

satisfies the QSDE

$$J_t(X) = X \otimes I + \sum_{i,j=0}^{m} \int_0^t J_s \theta_j^i(X) \Lambda_i^j(ds) \tag{2.5}$$

where

$$\theta_j^i(X) = XL_j^i + (L_i^j)^* X + \sum_{k=1}^{m} (L_i^k)^* X L_j^k, \ \forall\, i,j \geq 0.$$

In particular θ_0^0 is given by,

$$\theta_0^0(X) = \sum_{k=0}^{m} L_k^* X L_k + XL_0^0 + L_0^{0*} X, \tag{2.6}$$

is the generator of a QDS $(\mathcal{T}_t)_{t\geq 0}$ and the homomorphic co-cycle J_t dilates $\mathcal{T}_t$ in the sense that

$$\langle ue(0), U_t^*(X \otimes I)U_t ve(0)\rangle = \langle u, \mathcal{T}_t(X)v\rangle, \forall u, v \in \mathcal{H} \text{ and } X \in \mathcal{B}(\mathcal{H}). \tag{2.7}$$

The QDS $\mathcal{T}_t$ is called the vacuum expectation of J_t. This homomorphic co-cycle J_t, implemented by the HP flow U_t, is known as an HP dilation of the QDS $\mathcal{T}_t$.

Consider the time reversal operator R_t on $L^2(\mathbb{R}_+, \mathbf{k})$ defined by

$$R_t(f)(s) := \begin{cases} f(t-s) & \text{if } \ s \leq t; \\ f(s) & \text{if } \ s > t. \end{cases} \tag{2.8}$$

Observe that R_t is a self-adjoint unitary. Thus the second quantization $\Gamma(R_t)$ is so. For a bounded process U_t, define the dual process $\tilde{U_t}$ by

$$\tilde{U_t} := (1 \otimes \Gamma(R_t)) {U_t}^* (1 \otimes \Gamma(R_t)).$$

Proposition 2.4. [11] *Let U_t be a bounded process satisfying the QSDE* (2.4)*. Then the dual process $\tilde{U_t}$ will satisfy the QSDE of the similar form given by,*

$$\tilde{U_t} = I + \sum_{i,j=0}^{m} \int_0^t \tilde{U_s} {L_i^j}^* \Lambda_i^j(ds).$$

3. Examples of Quantum Dynamical Semi-groups

In this section, we have constructed a class of formal Lindbladian on the GNS space of a UHF C^*-algebra $\mathcal{A}$. In Theorem 3.5 we will show that the associated HP equation admits a unitary solution.

Let us consider the UHF C^*-algebra $\mathcal{A}$ as the C^*-inductive limit of the infinite tensor product of the matrix algebra $M_N(\mathbb{C})$,

$$\mathcal{A} = \overline{\bigotimes_{j \in \mathbb{Z}^d} M_N(\mathbb{C})}^{c^*}.$$

The algebra $\mathcal{A}$ can be interpreted as inductive limit of full matrix algebras. For $x \in M_N(\mathbb{C})$ and $j \in \mathbb{Z}^d$, $x^{(j)}$ denotes an element of $\mathcal{A}$ with x in the j^{th} component and identity everywhere else. We shall call the elements of the form $\prod_{i \geq 1} x_i^{(j_i)}$ to be simple tensor elements in $\mathcal{A}$. For a simple tensor element x in $\mathcal{A}$, let $x_{(j)}$ be the j^{th} component of x. Support '$supp(x)$' of x is defined to be the subset $\{j \in \mathbb{Z}^d ; x_{(j)} \neq I\}$. For a general element $x \in \mathcal{A}$ such that $x = \sum_{n=1}^{\infty} c_n x_n$ with simple tensor elements x_n and complex coefficients c_n, define $supp(x) = \bigcup_{n \geq 1} supp(x_n)$. For any $\Delta \subset \mathbb{Z}^d$, let $\mathcal{A}_\Delta$ denotes the $*$-sub algebra generated by the elements of $\mathcal{A}$ with support in Δ. For $j = (j_1, j_2, \cdots, j_d) \in \mathbb{Z}^d$, define $\|j\| = max\{|j_i| \ ; \ 1 \leq i \leq d\}$ and set $\Delta_n = \{j \in \mathbb{Z}^d ; \|j\| \leq n\}$, $\partial\Delta_n = \{j \in \mathbb{Z}^d ; \|j\| = n\}$. We say an element $x \in \mathcal{A}$ is local if $x \in \mathcal{A}_{\Delta_p}$ for some $p \geq 1$. The unique normalized trace tr on $\mathcal{A}$ is given by $tr(x) = \frac{1}{N^n} Tr(x)$, for $x \in M_{N^n}(\mathbb{C})$, where Tr denotes the matrix trace. The trace tr is a faithful normal state on $\mathcal{A}$. The algebra $\mathcal{A}$ can be represented as vectors in the Hilbert space $\mathcal{H} = L^2(\mathcal{A}, tr)$, the GNS Hilbert space for $(\mathcal{A}, tr)$, and as an element of $\mathcal{B}(\mathcal{H})$ by left multiplication. We write $\mathcal{A}_{loc}$ for the dense $*$-algebra generated by local elements.

Consider a formal element of the type

$$r := \sum_{n=1}^{\infty} W_n \text{ such that } \sum_{n=1}^{\infty} \|W_n\| = \infty,$$

where each W_n belongs to $\mathcal{A}_{\partial\Delta_n}$. Let us denote formally

$$\sum_{n=1}^{\infty} W_n^* \text{ by } r^*.$$

Now, if we set $\mathcal{C}_r(x) = [r, x] = \sum_{n=1}^{\infty} [W_n, x]$ for $x \in \mathcal{A}_{loc}$, clearly it is well defined since $[W_n, x] = 0$ for all $n > m$ when x is in finite dimensional algebra $\mathcal{A}_{\Delta_m} \subseteq \mathcal{A}_{loc}$. Thus we have a densely defined linear operator $(\mathcal{C}_r, \mathcal{A}_{loc})$ in $\mathcal{H}$.

Lemma 3.1. *Let r be as above and $n \geq 1$. Consider the element $r_n = \sum_{k=1}^{n} W_k$ in $\mathcal{A}$ and define a bounded operator $\mathcal{C}_r^{(n)}$ on $\mathcal{H}$ by setting $\mathcal{C}_r^{(n)}(x) = [r_n, x] = \sum_{k=1}^{n} [W_k, x]$ for $x \in \mathcal{A}_{loc}$. Then for each $n \geq 1$, $\mathcal{A}_{\Delta_n}$ is an invariant subspace for $\mathcal{C}_r$ and $\mathcal{C}_r^{(n)}$.*

Also for $m \geq p$,

$$\mathcal{C}_r|_{\mathcal{A}_{\Delta_p}} = \mathcal{C}_r^{(m)}|_{\mathcal{A}_{\Delta_p}} = \mathcal{C}_r^{(p)}|_{\mathcal{A}_{\Delta_p}}. \tag{3.1}$$

Proof. For x is in $\mathcal{A}_{\Delta_n}, [W_k, x] = 0$ for $k > n$. Thus $[r, x] = [r_n, x] \in \mathcal{A}_{\Delta_n}$ and $\mathcal{A}_{\Delta_n}$ is an invariant subspace under $\mathcal{C}_r$ and $\mathcal{C}_r^{(n)}$. Now for $x \in \mathcal{A}_{\Delta_p}$ and $m \geq p$, it is easy to see that $\mathcal{C}_r(x) = \mathcal{C}_r^{(m)}(x) = \mathcal{C}_r^{(p)}(x)$. □

Proposition 3.2. *The operator $(\mathcal{C}_r, \mathcal{A}_{loc})$ is closable.*

Proof. We shall show that $\mathcal{A}_{loc} \subseteq \mathrm{Dom}(\mathcal{C}_r^*)$ and for $x \in \mathcal{A}_{loc}$, $\mathcal{C}_r^*(x) = \mathcal{C}_{r^*}(x) = [r^*, x]$, thereby showing that the operator $\mathcal{C}_r^*$ is densely defined and therefore $(\mathcal{C}_r, \mathcal{A}_{loc})$ is closable. Indeed for $x \in \mathcal{A}_{loc}$, there exists $p \geq 1$ such that $x \in \mathcal{A}_{\Delta_p}$. Define $\Phi_x(y) := \langle x, \mathcal{C}_r y\rangle \, \forall y \in \mathcal{A}_{loc}$. For each $y \in \mathcal{A}_{loc}$, there exists m such that $y \in \mathcal{A}_{\Delta_m}$. As $\{\mathcal{A}_{\Delta_n}\}$ is an increasing family of algebras, with no loss of generality, let us assume $m \geq p$. Then by definition and property of trace and Lemma 3.1,

$$\Phi_x(y) = tr(x^*\mathcal{C}_r y) = tr(x^*\mathcal{C}_r^{(m)} y) = tr((\mathcal{C}_{r^*}^{(m)} x)^* y) = \langle \mathcal{C}_{r^*}^{(p)} x, y\rangle = \langle \mathcal{C}_{r^*} x, y\rangle,$$

and thus

$$|\Phi_x(y)| \leq \|\mathcal{C}_{r^*} x\| \|y\|, \; \forall \; y \in \mathcal{A}_{loc}.$$

Thus $x \in \mathrm{Dom}(\mathcal{C}_r^*)$ and

$$\mathcal{C}_r^*(x) = \mathcal{C}_{r^*}(x), \forall x \in \mathcal{A}_{loc}. \tag{3.2}$$

□

We denote by $\bar{\mathcal{C}}_r$, the closure of a densely defined, closable operator $\mathcal{C}_r$. Note here that for a operator T on $\mathcal{H}$, $T^* = \bar{T}^*$, if T is closable. Then by standard theorem of von Neumann, $\mathcal{C}_r^*\bar{\mathcal{C}}_r$ is a positive self-adjoint operator in $\mathcal{H}$ and $\mathrm{Dom}(\mathcal{C}_r^*\bar{\mathcal{C}}_r)$ is a core for $\bar{\mathcal{C}}_r$. Furthermore, the operator $G := -\frac{1}{2}\mathcal{C}_r^*\bar{\mathcal{C}}_r$ generates a C_0-contraction semi-group $\mathcal{S}_t$ in $\mathcal{H}$.

Proposition 3.3. *For $n \geq 1$, define the bounded operator $G^{(n)}$ on $\mathcal{H}$ by*

$$G^{(n)} := -\frac{1}{2}\mathcal{C}_{r^*}^{(n)}\mathcal{C}_r^{(n)}.$$

Then each $\mathcal{A}_{\Delta_n}$ is an invariant under $G^{(n)}$. Furthermore, for $m \geq p$,

$$G^{(m)}|_{\mathcal{A}_{\Delta_p}} = G^{(p)}|_{\mathcal{A}_{\Delta_p}} = G|_{\mathcal{A}_{\Delta_p}}. \tag{3.3}$$

Proof. By Lemma 3.1, we have $\mathcal{A}_{\Delta_n}$ invariant under $\bar{\mathcal{C}}_r$ and $\mathcal{C}_r^{(n)}$ and for $m \geq p$, the identity $\mathcal{C}_r|\mathcal{A}_{\Delta_p} = \mathcal{C}_r^{(m)}|\mathcal{A}_{\Delta_p} = \mathcal{C}_r^{(p)}|\mathcal{A}_{\Delta_p}$ holds. As $\mathcal{C}_r^*(x) = \mathcal{C}_{r^*}(x), \; \forall \; x \in \mathcal{A}_{loc}$, we have $\mathcal{C}_r^*|\mathcal{A}_{\Delta_p} = \mathcal{C}_{r^*}^{(m)}|_{\mathcal{A}_{\Delta_p}} = \mathcal{C}_{r^*}^{(p)}|_{\mathcal{A}_{\Delta_p}}$ and hence result follows. □

Proposition 3.4. *The subspace $\mathcal{A}_{loc}$ is a core for the operator G.*

Proof. It is enough to prove that the subspace $\mathcal{A}_{loc}$ is invariant under the semigroup $\mathcal{S}_t$. For a vector $x \in \mathcal{A}_{loc}$, there exists $n \geq 1$, such that $x \in \mathcal{A}_{\Delta_n}$. Now by Lemma 3.1, for any $k \geq 0$, $G^k(x) = {G^{(n)}}^k(x) \in \mathcal{A}_{\Delta_n}$ and it follows that the series $\sum_{k\geq 0} \frac{t^k G^k x}{k!}$ converges strongly in $\mathcal{A}_{\Delta_n}$. Therefore, we have, $\mathcal{S}_t x = \mathcal{S}_t^{(n)} x \equiv e^{tG^{(n)}}$

for $x \in \mathcal{A}_{\Delta_n}$. Thus, $\mathcal{S}_t$ leaves $\mathcal{A}_{loc}$ invariant and by Nelson's theorem [9], the core property follows. □

Now consider the sesquilinear form, Lindbladian, $\mathcal{L}(X)$ with the domain $\mathcal{A}_{loc} \times \mathcal{A}_{loc} \subseteq \mathrm{Dom}(G) \times \mathrm{Dom}(G)$ given by

$$\langle u, \mathcal{L}(X)v\rangle \equiv \langle u, XGv\rangle + \langle Gu, Xv\rangle + \langle \bar{\mathcal{C}}_r u, X\bar{\mathcal{C}}_r v\rangle. \tag{3.4}$$

By definition of G, it is clear that $\langle u, \mathcal{L}(I)v\rangle = \langle u, Gv\rangle + \langle Gu, v\rangle + \langle \bar{\mathcal{C}}_r u, \bar{\mathcal{C}}_r v\rangle = 0$.

Let $\mathcal{A}_{loc} \otimes \mathcal{E}$ be the linear span of $\{x \otimes e(f) \ : \ x \in \mathcal{A}_{loc}, \ f \in L^2(\mathbb{R}_+, \mathbb{C})\}$. Then the set $\mathcal{A}_{loc} \otimes \mathcal{E}$ is a dense subspace of $\mathcal{H} \otimes \Gamma_{sym}$.

Theorem 3.5. *Consider the HP type QSDE in* $\mathcal{A}_{loc} \otimes \mathcal{E}$

$$U_t = I + \int_0^t U_s G ds + \int_0^t U_s \bar{\mathcal{C}}_r a^\dagger(ds) - \int_0^t U_s \mathcal{C}_r^* a(ds), \tag{3.5}$$

where $a^\dagger, a$ *are creation and annihilation processes respectively. The QSDE* (3.5) *admits a unitary solution* U_t. *Moreover, the expectation semi-group* $(\mathcal{T}_t)_{t\geq 0}$ *of the homomorphic co-cycle* $J_t(X) = U_t^*(X \otimes I)U_t$ *is the unique (minimal) semi-group associated with the formal Lindbladian* $\mathcal{L}$ *in* (3.4) *and is conservative.*

Proof. Recall that the UHF algebra $\mathcal{A}$ can be approximated by finite dimensional algebras, namely $A_{\Delta_n} = \prod_{\|j\|\leq n} M_N(\mathbb{C})$ and $\mathcal{A}_{loc} = \bigcup_{n=0}^{\infty} \mathcal{A}_{\Delta_n}$. For $n \geq 0$, consider the following QSDE in $\mathcal{A}_{loc} \otimes \mathcal{E}$,

$$U_t^{(n)} = I + \int_0^t U_s^{(n)} G^{(n)} ds + \int_0^t U_s^{(n)} \mathcal{C}_r^{(n)} a^\dagger(ds) - \int_0^t U_s^{(n)} {\mathcal{C}_r^{(n)}}^* a(ds). \tag{3.6}$$

By Theorem 2.3, the QSDE 3.6 admits a unitary solution $U_t^{(n)}$ on $\mathcal{H} \otimes \Gamma_{sym}$.

We now show that the operators $U_t^{(n)}$ satisfy some compatibility condition, that is for $n \geq m$,

$$U_t^{(n)}|_{\mathcal{A}_{\Delta_m}} = U_t^{(m)}|_{\mathcal{A}_{\Delta_m}}. \tag{3.7}$$

Here the symbol $T|_{\mathcal{A}_{\Delta_m}}$ means the restriction of T to the subspace $A_{\Delta_m} \otimes \Gamma_{sym}$.

Since these operators $\mathcal{C}_r^{(m)}, {\mathcal{C}_r^{(m)}}^*$ and $G^{(m)}$ leave $\mathcal{A}_{\Delta_m}$ invariant, the restriction $U_t^{(m)}|_{\mathcal{A}_{\Delta_m}}$ satisfies the following QSDE in $A_{\Delta_m} \otimes \mathcal{E}$,

$$U_t^{(m)}|_{\mathcal{A}_{\Delta_m}} = I|_{\mathcal{A}_{\Delta_m}} + \int_0^t U_s^{(m)}|_{\mathcal{A}_{\Delta_m}} G^{(m)}|_{\mathcal{A}_{\Delta_m}} ds \tag{3.8}$$

$$+ \int_0^t U_s^{(m)}|_{\mathcal{A}_{\Delta_m}} \mathcal{C}_r^{(m)}|_{\mathcal{A}_{\Delta_m}} a^\dagger(ds) - \int_0^t U_s^{(m)}|_{\mathcal{A}_{\Delta_m}} {\mathcal{C}_r^{(m)}}^*|_{\mathcal{A}_{\Delta_m}} a(ds).$$

For $n \geq m$, consider the QSDE in $A_{\Delta_m} \otimes \mathcal{E}$,

$$U_t^{(n)}|_{\mathcal{A}_{\Delta_m}} = I|_{\mathcal{A}_{\Delta_m}} + \int_0^t U_s^{(n)}|_{\mathcal{A}_{\Delta_m}} G^{(n)}|_{\mathcal{A}_{\Delta_m}} ds \tag{3.9}$$

$$+\int_0^t U_s^{(n)}|_{\mathcal{A}_{\Delta_m}}\mathcal{C}_r^{(n)}|_{\mathcal{A}_{\Delta_m}}a^\dagger(ds)-\int_0^t U_s^{(n)}|_{\mathcal{A}_{\Delta_m}}\mathcal{C}_r^{(n)^*}|_{\mathcal{A}_{\Delta_m}}a(ds).$$

With reference to Lemma 3.1, equation (3.3) and Theorem 2.3, the unitary processes $U_t^{(n)}|_{\mathcal{A}_{\Delta_m}}$ and $U_t^{(m)}|_{\mathcal{A}_{\Delta_m}}$ satisfy the same QSDE in $A_{\Delta_m}\otimes\mathcal{E}$. Therefore, by uniqueness of solution in Theorem 2.3, (3.7) follows.

Define U_t on $\mathcal{A}_{loc}\otimes\mathcal{E}$ by setting

$$U_t(x\otimes e(f))=U_t^{(n)}(x\otimes e(f)) \text{ if } x\in\ A_{\Delta_n}$$

and extending linearly. Since the family $U_t^{(n)}$ satisfies the compatibility condition (3.7), U_t is well defined on $\mathcal{A}_{loc}\otimes\mathcal{E}$, and for $x\in\ A_{\Delta_m}$ we have

$$U_t(x\otimes e(f))=U_t^{(m)}(x\otimes e(f))=U_t^{(n)}(x\otimes e(f)),\forall n\geq m. \tag{3.10}$$

Hence $U_t^{(n)}$ converges strongly to U_t on $\mathcal{A}_{loc}\otimes\mathcal{E}$ and U_t extends to a contraction operator on $\mathcal{H}\otimes\Gamma_{sym}$. As $\mathcal{A}_{loc}\otimes\mathcal{E}$ is dense in $\mathcal{H}\otimes\Gamma_{sym}$, (3.10) gives that $U_t^{(n)}$ converges strongly to U_t on $\mathcal{H}\otimes\Gamma_{sym}$ as well and the limit U_t is an isometry.

For $U_t^{(n)}$, consider the dual process $\tilde{U_t}^{(n)}=(1\otimes\Gamma(R_t))U_t^{(n)^*}(1\otimes\Gamma(R_t))$. Then by Proposition 2.4, $\{\tilde{U_t}^{(n)}\}$ satisfies the following QSDE in $\mathcal{A}_{loc}\otimes\mathcal{E}$,

$$\tilde{U_t}^{(n)}=I+\int_0^t\tilde{U_s}^{(n)}G^{(n)^*}ds+\int_0^t\tilde{U_s}^{(n)}\mathcal{C}_r^{(n)^*}a(ds)-\int_0^t\tilde{U_s}^{(n)}\mathcal{C}_r^{(n)}a^\dagger(ds). \tag{3.11}$$

The equation (3.11) is identical to (3.6) except that $\mathcal{C}_r^{(n)}$ is replaced by $-\mathcal{C}_r^{(n)}$. So similar arguments yield that the operators $\tilde{U_t}^{(n)}$ also satisfy the compatibility condition and converge strongly to an isometry and because $\tilde{U_t}^{(n)}$ and $\Gamma(R_t)$ are unitaries, the sequence $U_t^{(n)^*}$ of unitaries converges strongly and thus it must converge to U_t^*. Hence U_t^* is an isometry, so U_t is a unitary process.

It remains to prove that U_t satisfies the QSDE (3.5). As U_t is a unitary process, the quantum stochastic integral on the right-hand side of (3.5) makes sense. Thus, it is enough to establish that integrals on the right-hand side of (3.6) converge to integrals in (3.5). For $xe(f)\in\mathcal{A}_{loc}\otimes\mathcal{E}$, we have

$$\left\|\int_0^t(U_s^{(n)}G^{(n)}-U_sG)ds(xe(f))\right\|\leq\int_0^t\|(U_s^{(n)}G^{(n)}-U_sG)(xe(f))\|ds,$$

hence by (3.3) and (3.10), it converges to 0. By estimates of quantum stochastic integrals [10], we have

$$\begin{aligned}&\left\|\int_0^t(U_s^{(n)}\mathcal{C}_r^{(n)}-U_s\mathcal{C}_r)a^\dagger(ds)(xe(f))\right\|^2\\&\qquad\leq 2e^{\int_0^t(1+|f(s)|^2)ds}\int_0^t\|(U_s^{(n)}\mathcal{C}_r^{(n)}-U_s\mathcal{C}_r)xe(f)\|^2(1+|f(s)|^2)ds.\end{aligned}$$

Therefore, by (3.1) and (3.10),

$$\lim_{n\to\infty}\left\|\int_0^t (U_s^{(n)}\mathcal{C}_r^{(n)} - U_s\mathcal{C}_r)a^\dagger(ds)(xe(f))\|^2\right\|^2 = 0.$$

Convergence of annihilation term follows from a simpler estimate and using (3.2), (3.1) and (3.10). Thus U_t is a unitary solution to the QSDE (3.5).

Now let us consider the expectation semi-group $(\mathcal{T}_t)_{t\geq 0}$ of the homomorphic co-cycle $J_t(\cdot) = U_t^*(\cdot \otimes I)U_t$. As U_t is a unitary process, the QDS $(\mathcal{T}_t)_{t\geq 0}$ is conservative minimal semi-group associated with the form (3.4). □

We conclude by a remarks on the Lindbladian

$$\mathcal{L}(X) = \frac{1}{2}\sum_{j=1}^{\infty}\{W_j^*\delta_j(X) + \delta_j^\dagger(X)W_j\}, \quad \forall X \in \mathcal{A}_{loc} \tag{3.12}$$

where $W_j \in \mathcal{A}_{\partial\Delta_j}, \delta_j(X) = [X, W_j], \delta_j^\dagger(X) = (\delta_j(X^*))^* = [W_j^*, X]$. Though each component $W_j^*\delta_j(.) + \delta_j^\dagger(,)W_j$ are bounded maps, $\mathcal{L}$ is unbounded due to presence of infinitely many components (like in [8]). For $n \geq 1$, define a bounded map

$$\mathcal{L}^{(n)}(X) = \frac{1}{2}\sum_{j=1}^{n}\{W_j^*\delta_j(X) + \delta_j^\dagger(X)W_j\}, \forall X \in \mathcal{A}.$$

Note that for $X \in \mathcal{A}_{\Delta_n}, \delta_k(X) = \delta_k^\dagger(X) = 0$ and $\mathcal{L}^{(k)}(X) = \mathcal{L}^{(n)}(X)$ for every $k > n$.

Remark 3.6. The HP equation on $\mathcal{H} \otimes \Gamma_{sym}(L^2(\mathbb{R}_+, \mathbf{k}))$, where $\mathbf{k}$ is a separable Hilbert space with an orthonormal basis $\{e_j : j \geq 1\}$,

$$U_t = I + \int_0^t U_s(-\frac{1}{2}\sum_{j=1}^{\infty}W_j^*W_j)ds + \sum_{j=1}^{\infty}\int_0^t U_sW_ja_j^\dagger(ds) - \sum_{j=1}^{\infty}\int_0^t U_sW_j^*a_j(ds)$$

may not make sense as $-\frac{1}{2}\sum_{j=1}^{\infty}W_j^*W_j$ may have a trivial domain or not a generator of a C_0- semigroup on $\mathcal{H}$. However, there exist a homomorphic co-cycle $J_t : \mathcal{A} \to \mathcal{A}'' \otimes \mathcal{B}(\Gamma_{sym})$ satisfying the Evan-Hudson equation, for $X \in \mathcal{A}_{loc}$,

$$J_t(X) = X \otimes I + \int_0^t J_s(\mathcal{L}(X))ds + \sum_{j=1}^{\infty}\int_0^t J_s(\delta_j(X))a_j^\dagger(ds) + \sum_{j=1}^{\infty}\int_0^t J_s(\delta_j^\dagger(X))a_j(ds).$$

The expectation semi-group $(\mathcal{T}_t)_{t\geq 0}$ of the homomorphic co-cycle J_t is conservative minimal semi-group associated with the Lindbladian (3.12).

This can be seen similarly as for HP equation in theorem 3.5 by constructing J_t as a strong limit of homomorphic co-cycles $\{J_t^{(n)} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\Gamma_{sym})\}$

where $J_t^{(n)}$ satisfies the Evan-Hudson equation, for $X \in \mathcal{B}(\mathcal{H})$,

$$J_t^{(n)}(X) = X \otimes I + \int_0^t J_s^{(n)}(\mathcal{L}^{(n)}(X))ds + \sum_{j=1}^{n} \int_0^t J_s^{(n)}(\delta_j(X))a_j^\dagger(ds)$$

$$+ \sum_{j=1}^{n} \int_0^t J_s^{(n)}(\delta_j^\dagger(X))a_j(ds)$$

with bounded structure maps and finite degree of freedom (see [10]). In fact, $J_t^{(n)}$ takes $\mathcal{A}_{\Delta_n}$ to $\mathcal{A}_{\Delta_n} \otimes \mathcal{B}(\Gamma_{sym})$, and for $X \in \mathcal{A}_{\Delta_n}$,

$$J_t(X) = J_t^{(m)}(X) = J_t^{(n)}(X), \forall m \geq n.$$

Remark 3.7. The main difference of the class of Lindbladian studied here from those considered by T. Matsui in [8] is lack of translation invariance. In [8], the Lindbladian are closed densely defined operator and invariant under the natural translation coming from the discrete group $\mathbb{Z}^d$ and the invariance property is used cleverly to construct the semi-group by Hille-Yosida theory. However, the Lindbladian, we are considering here, are only given as form defined on local elements. Exploring the local structure of the UHF algebra, we manage to construct the unitary solution of the HP equation associated with 3.4 as limit of finite dimensional unitary HP flows. Here both the HP and EH dilations are possible, unlike the EH dilation of Matsui's semi-groups in [5] which makes these class of semi-groups more interesting.

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