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# SOME NEW CONNECTIONS BETWEEN BROWNIAN BRIDGES AND BROWNIAN MOTIONS

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Abstract. The main message in this paper is that there are surprisingly many different Brownian bridges, some of them familiar, some of them less familiar. Many of these Brownian bridges are very close to Brownian motions. Somewhat loosely speaking, we show that all the bridges can be conveniently mapped onto each other, and, hence, to one "standard" bridge. The unitary operators play in these mappings the central role. The results are extended to Kiefer processes.

## 1. Introduction

Consider a standard Brownian motion $w(t), t \in [0,1]$, and for a function $\phi \in L_2[0,1]$, consider the Wiener stochastic integral

$$w(\phi) = \int_0^1 \phi(s) w(ds).$$

As we know, each $w(\phi)$ is a Gaussian random variable with expected value 0 and variance

$$Ew^2(\phi) = \int_0^1 \phi^2(s) ds = \|\phi\|^2;$$

if $w(\phi)$ and $w(\psi)$ are two such integrals, then they are jointly Gaussian, with expected values and variances as above and covariance

$$Ew(\phi)w(\psi) = \int_0^1 \phi(s)\psi(s) ds = \langle \phi, \psi \rangle.$$

A family $\{w(\phi), \phi \in \Phi \subseteq L_2[0,1]\}$ is called a *function-parametric Brownian motion* (on $\Phi$).

For the standard Brownian bridge, defined as a projection of the standard Brownian motion,

$$v(t) = w(t) - tw(1), \; t \in [0,1], \tag{1.1}$$

we can also consider its function-parametric version

$$v(\phi) = \int_0^1 \phi(s) v(ds) = \int_0^1 \phi(s) w(ds) - \int \phi(s) ds \, w(1)$$

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which one can re-write in Hilbert space notations as

$$v(\phi) = w(\phi) - \langle \phi, q \rangle w(q), \tag{1.2}$$

where $q$ denotes, here and everywhere below, the function identically equal to 1 on its domain, in the present case – on the interval $[0,1]$. The family $\{v(\phi), \phi \in \Phi \subseteq L_2[0,1]\}$ is called a *function-parametric Brownian bridge.*

Since in this paper we are not concerned with continuity properties of our function-parametric processes in $f$, we can take $\Phi = L_2[0,1]$ and call "function-parametric Brownian motion" or "function-parametric Brownian bridge" the families of zero-mean Gaussian linear functionals in $\phi$ with the appropriate covariance. For the theory of Gaussian linear functionals on Hilbert space we refer to [13].

The construction we discuss and exemplify here is this: if $U$ stands for a unitary operator in $L_2[0,1]$, then the processes we want to consider are obtained as the transformation

$$w(U\phi) \quad \text{and} \quad v(U\phi).$$

What kind of processes can one obtain in this way? Will they have useful and interesting properties? Will new connections between, say $v$ and $w$ arise in this way?

The answer, that the transformed processes will be all Gaussian processes with correlation operators, unitarily equivalent to those of $w$ and $v$, would be far too general and not very interesting. (It also would not be quite correct – see Proposition 2.1.) The description of trajectory-by-trajectory transformations, with specific choice of $U$, may, however, assume an unusual and interesting form. Can, for example, $v$ be transformed into $w$, in such a simple form as Proposition 2.1 suggests? Is there one Brownian bridge, defined by (1.1) and (1.2) and so very dominant in the current theory; or there are many other bridges, not less useful and interesting, and connected with $w$ via unitary transformations? Although these questions have been considered in [11], a somewhat broader review and some simple and, therefore, useful examples have not been given there. To some extent, we remedy this omission here.

## 2. Some Previous Results Revisited

In the introduction of [7] the following transformation was considered as an example (of more general case, which then followed): if $v$ is the standard Brownian bridge, then the process $b$, defined as

$$v(dt) = -v(t)\frac{dt}{1-t} + b(dt), \tag{2.1}$$

is standard Brownian motion. The first application of this fact in a statistical context, so far as this author is aware, can be found in [2]. The usual heuristic explanation behind (2.1) is that

$$E[v(dt)|v(s), s \le t] = -v(t)\frac{dt}{1-t},$$

and hence (2.1) is Doob-Meier decomposition of $v(t), t \in [0,1]$, and $b$ is the innovation martingale of $v$. At the same time, multiplying both sides of (2.1) by $f$ and integrating leads to its function-parametric equivalent. Namely, let $\Phi_1 = \{\phi \in L_2[0,1] : \int_0^1 \phi(s)ds = 0\}$ and define the operator $U$ by

$$U\phi(t) = \phi(t) - \int_0^t \phi(s)\frac{ds}{1-s}, \quad \phi \in L_2[0,1].$$

Then

$$\begin{aligned} v(\phi) &= -\int_0^1 \phi(t)v(t)\frac{dt}{1-t} + b(\phi) \\ &= \int_0^1 \Big[\int_0^t \phi(s)\frac{ds}{1-s}\Big] v(dt) + b(\phi), \end{aligned}$$

so that (2.1) turns into

$$v(U\phi) = b(\phi). \tag{2.2}$$

**Proposition 2.1.** *Operator The $U$ defined above is a unitary operator from $L_2[0,1]$ to $\Phi_1$ and the process $b(\phi)$ is the function-parametric Brownian motion.*

Concerns about the singularity at $t = 1$ in the representation (2.1) and, hence, in the definition of $U$, have been expressed in the literature for a long time, almost forming a tradition. The tradition persisted, it seems, because a direct proof that

$$\int_0^t \phi(s)\frac{ds}{1-s} \in L_2[0,1] \quad \text{for all } \phi \in L_2[0,1],$$

was not forthcoming with usual easiness of such statements. Therefore we stress that $U$ is a bounded operator on all of $L_2[0,1]$, and not only on, say, $L_2[0,1) = \cup C_n$, where

$$C_n = \{\phi \in L_2[0,1] : \phi(t) = 0, t \in [1-1/n, 1]\}.$$

The proof we can give here consists of three short steps: the Cauchy- Schwarz inequality shows that

$$\phi \in L_2[0,1] \implies \psi = U(\phi) \in L_1[0,1];$$

on any $C_n$ the operator $U$ has $L_2$-norm 1; and finally, if $\phi_n \in C_n$ and $\phi_n \to \phi$, then $\psi_n$ forms a Cauchy sequence and hence converges to some $\psi'$, which, then, has to be $\psi$.

What was realized in [9] was that construction similar to (2.2) is possible in the case of $d$-dimensional time. Namely, choose now our functions $\phi$ from $L_2[0,1]^d$ and introduce in $[0,1]^d$ what we called a *scanning family* $\mathcal{A} = \{A_t, 0 \le t \le 1\}$ of Borel subsets, such that

$$\begin{aligned} &A_t \subset A_{t'} \text{ for } t < t', \\ &\mu(A_0) = 0,\ \mu(A_1) = 1, \\ &\mu(A_t) \text{ is absolutely continuous in } t. \end{aligned}$$

Here $\mu$ denotes Lebesgue measure on $[0,1]^d$. We associate with $\mathcal{A}$ the family of projectors $\Pi_t, 0 \le t \le 1$, along with their complements:

$$\Pi_t\phi(x) = I(x \in A_t)\phi(x), \quad \Pi_t^c\phi(x) = \phi(x) - \Pi_t\phi(x),$$

where $I(x \in A)$ is indicator function of the set $A$ . Consider $v(\Pi_t \phi)$, which now has two "times": a functional time $\phi \in L_2[0,1]^d$ and the more or less artificially inserted time $t \in [0,1]$. We call the family $\{v(\Pi_t \psi), \psi \in L_2[0,1]^d\}$, the "past" of $v$ up to time $t$. Then, as one can see in [9],

$$E[v(\Pi_{dt}\phi)|v(\Pi_t\psi), \psi \in L_2[0,1]^d] = -\int_{A_{dt}} \phi(y)\, \frac{d\mu(y)}{1-\mu(A_t)}\, v(\Pi_t q) = -\frac{\langle \Pi_{dt}\phi, q\rangle}{1-\mu(A_t)}\, v(\Pi_t q).$$

It is, therefore, obvious that the process in $t$ defined as

$$v(\Pi_t\phi) + \int_0^t \int_{A_{ds}} \phi(y)\, \frac{d\mu(y)}{1-\mu(A_s)}\, v(\Pi_s q) = b(\phi, t)$$

is, for every $\phi$, a Gaussian martingale in $t$. What is true, however, is that $b(\phi, t)$ has independent increments in $\phi$, so that $b(\phi, 1)$ is the function-parametric Brownian motion.

To justify this claim and show that we are again dealing with a transformation like (2.2) it is useful to make some re-arrangement: replace $v(\Pi_s q)$ by $-v(\Pi_s^c q)$, where

$$v(\Pi_s^c q) = \int_{A_s^c} v(dx)$$

and introduce notation

$$t(x) = \inf\{t : A_t \ni x\}.$$

Then using integration by parts, the double integral above can be written as

$$-\int \int_{A_{t(x)}} \phi(y)\, \frac{\mu(dy)}{1-\mu(A_{t(y)})} dv(x).$$

Now define the operator $U_{\mathcal{A}}$ as

$$U_{\mathcal{A}}\phi(x) = \phi(x) - \int_{A_{t(x)}} \phi(y)\, \frac{\mu(dy)}{1-\mu(A_{t(y)})} = \phi(x) - \int_{s \le t(x)} \frac{\langle \Pi_{ds}\phi, q\rangle}{1-\mu(A_s)}.$$

With no risk of misunderstanding, denote again

$$\Phi_1 = \Big\{\phi \in L_2[0,1]^d : \int_{[0,1]^d} \phi(x)\mu(dx) = 0\Big\}.$$

**Proposition 2.2.** *For any scanning family $\mathcal{A}$, the process $b$, defined as*

$$v(dx) = v(A_{t(x)}^c)\frac{\mu(dx)}{\mu(A_{t(x)})} + b(dx), \quad x \in [0,1]^d, \tag{2.3}$$

*is a standard Brownian motion on $[0,1]^d$. The operator $U_{\mathcal{A}}$ is a unitary operator from $L_2[0,1]^d$ to $\Phi_1$ and the process*

$$b(\phi) = b(\phi, 1) = v(U_{\mathcal{A}}\phi),\ \phi \in L_2[0,1]^d,$$

*is the corresponding function-parametric Brownian motion.*

We see that (2.3) is the $d$-dimensional analogue of (2.1) while the display above is the analogue of (2.2).

What is interesting here is not only that the proposition is true for any finite dimension $d$, but also that it is true for *any* scanning family $\mathcal{A}$. One can extend this result further – from the Brownian bridge (1.2) to the case where not one but several restrictions are imposed on $w$ to produce the extended notion of a "bridge". Namely, suppose $h_1, \dots, h_m$ are orthonormal functions in $L_2([0,1]^d)$ and consider

$$\hat{v}(\phi) = w(\phi) - \sum_{j=1}^{m} \langle \phi, h_j \rangle w(h_j). \tag{2.4}$$

Processes of this type are of great importance in asymptotic problems of statistics. It is clear that $\hat{v}(h_j) = 0$ for all $j = 1, \dots, m$, but, contrary to the usual tradition, the function $q$ does not have to be one of $h_j$. An example of this situation, arising in the martingale theory of point processes, was recently discussed in [17]. The process $\hat{v}$ has been called in [11] the *h- projected Brownian motion.* Using vector notations $h = (h_1, \dots, h_m)^T$, $v(h) = (v(h_1), \dots, v(h_m))^T$, etc., we can formulate the following generalization of Proposition 2.2.

In place of $\Phi_1$ consider now $\Phi_h = \{\phi \in L_2[0,1]^d : \int_{[0,1]^d} \phi(x)h(x)\mu(dx) = 0\}$. Define the operator

$$U_{\mathcal{A},h}\phi(x) = \phi(x) - \int_{s \le t(x)} \langle \Pi_{ds}\phi, h^T \rangle (I - \langle \Pi_s h, \Pi_s h^T \rangle)^{-1} h(x),$$

where $I$ denotes $m$-dimensional identity matrix. We see that operator $U_{\mathcal{A},h}$ indeed is the $m$-dimensional extension of the operator $U_{\mathcal{A}}$.

**Proposition 2.3.** *For any scanning family, the operator $U_{\mathcal{A},h}$ is a unitary operator from $L_2[0,1]^d$ to $\Phi_h$ and the process*

$$b(\phi) = v(U_{\mathcal{A},h}\phi), \ \phi \in L_2[0,1]^d$$

*is a function-parametric Brownian motion.*

This proposition is one of the key statements of [9] and is more or less known to statistical communities because of the consequences it has for goodness of fit theory (see, e.g., [12]). It is less known to specialists in stochastic analysis, in particular to those interested in martingales in multi-dimensional time. To the best of my knowledge, the case of countably many $h_i$s (countably many restrictions) have not been considered in sufficient generality. However, we show below that there are interesting cases when "continually constrained" Brownian motion can be naturally mapped back into Brownian motion. In this case, however, the statement will not be "for any scanning family" but that "there exists an appropriate scanning family", which will be described explicitly. There are, however, infinitely many "appropriate" scanning families.

The processes we have in mind occur when Brownian motion in multi-dimensional time is constrained, or projected, in one of its arguments, but remains free in its other arguments. To illustrate the situation, it is sufficient to consider Brownian motion on the unit square and consider the process

$$v(t,s) = w(t,s) - tw(1,s), \ (t,s) \in [0,1]^2. \tag{2.5}$$

Speaking again heuristically, we can look at $v(t, ds)$ in each "narrow" strip $[0,1] \times ds$ as projection of the Brownian motion $w(t, ds)$. In particular, it is clear that $v(1, s) = 0$ for all $s$. In $s$, the increments $v(t, ds)$ are independent, can thus be "glued together" into a Brownian motion. The process $v(t, s)$ is called the Kiefer process and occurs, for example, in statistical problems with one-sided copulas and in the theory of "sequential" empirical processes with the sample size as another variable (see, e.g., [18] and [16], respectively).

To map the whole of $v(t, s)$ into Brownian motion we can map $v(t, ds)$, in each strip, into Brownian motion in $t$ in the same way as we did in (2.1) and Proposition 2.1. Namely, consider the subspace $\Phi_{\sqcup} = \{\phi \in L_2[0,1]^2 : \int_0^1 \phi(\tau, s) d\tau = 0\}$ and define the operator $V$ as

$$V\phi(t, s) = \phi(t, s) - \int_0^t \phi(\tau, s) \frac{d\tau}{1-\tau}, \ \phi \in L_2[0,1]^2.$$

This is unitary operator, which maps $L_2[0,1]^2$ into $\Phi_{\sqcup}$, and the process

$$v(V\phi) = b(\phi), \quad \phi \in L_2[0,1]^2,$$

is function-parametric Brownian motion on $L_2[0,1]^2$.

The "sequential" empirical processes we mentioned earlier are often based on multi-dimensional observations, which in asymptotic analysis will lead to Kiefer processes with multidimensional $t$. However, one can adopt the use of scanning families here. As we have briefly seen above, even for one-dimensional $t$, the idea of using the scanning families may be sensible. For this, and for the sake of simplicity, let us still assume below one-dimensional $t$. Generalization to the multi-dimensional case can be carried out without change of notations.

Consider the family of Borel subsets of the form: $\mathcal{B} = \{B_t = A_t \times [0,1], 0 \le t \le 1\}$, where $\{A_t, 0 \le t \le 1\}$ is a scanning family in $[0,1]$. Associate with this $\mathcal{B}$ the operator

$$\begin{aligned} V_{\mathcal{B}}\phi(x, y) &= \phi(x, y) - \int_{t(x',y')<t(x,y)} \phi(x', y') \frac{dx'dy'}{1 - \mu(B_{t(x',y')})} \\ &= \phi(x, y) - \int_{t<t(x,y)} \Big(\int_{B_{dt}} \phi(x', y') dx'dy'\Big) \frac{1}{1-\mu(B_t)}. \end{aligned}$$

This is also a unitary operator which maps $L_2[0,1]^2$ on $\Phi_{\sqcup}$. If $A_t = [0,t]$ i.e. if $B_t = [0,t] \times [0,1]$, then what we get will be the operator $V$ defined above.

**Proposition 2.4.** *With the process $v$ defined in* (2.5) *the process*

$$v(V_{\mathcal{B}}\phi) = b(\phi)$$

*is a standard Brownian motion.*

We believe that the proposition is true for wider choice of scanning families: instead of the sets $B_t = A_t \times [0,1]$, which are cylindrical in $s$, one can use a family $\mathcal{B} = \{B_{t,s}\}$ such that for each $s$ the "section" of $B_{t,s}$, i.e. $\{t' : (t', s) \in B_{t,s}\}$ form a scanning family in $t \in [0,1]$. Note, in sharp contrast, that the family $[0,1] \times [0,s]$ is not "scanning" in $t$ and indeed will be useless in application to our Kiefer processes. Note, at the same time, that the freedom in the choice of different scanning families for different $s$ may lead to geometric pathologies, which we do not investigate here.

Consider now a somewhat more intricate form of constraint on $w$, which also leads to Kiefer-type processes. Suppose a function $h \in L_2[0,1]^2$ is such that

$$\int_{[0,1]} h^2(x,y)dx = 1 \quad \text{for all } y \in [0,1].$$

This implies, by the way, that such $h$ can not be a product $h_1(x)h_2(y)$ unless $h_2 = 1$. For any $h$ define

$$v_h(x,dy) = w(x,dy) - \int_0^x h(x',y)dx' \int h(x',y)w(dx',dy). \tag{2.6}$$

It can be immediately seen that the process $v_h(x,y)$ satisfies the boundary condition

$$\int h(x',y)v_h(dx',y) = 0 \quad \text{for all } y \in [0,1].$$

To describe its function-parametric form it is natural to introduce the operator

$$\tilde{\phi}(x,y) = \Pi_{h,y}\phi(x,y) = \phi(x,y) - \int \phi(x',y)h(x',y)dx'\, h(x,y),$$

which is a projector. Indeed

$$\Pi_{h,y}(\Pi_{h,y}\phi) = \Pi_{h,y}\phi,\ \Pi_{h,y}h = 0 \text{ and } \int \Pi_{h,y}\phi(x,y)h(x,y)dx = 0.$$

Then

$$v_h(\phi) = w(\Pi_{h,y}\phi),$$

so that in this sense $v_h$ is again a projection of $w$.

To have an intuitively clear example, let $h'(x,y) = \mathbf{I}(x<y)$ be the indicator function of event $x<y$ and let

$$h(x,y) = \frac{1}{\sqrt{y}}\mathbf{I}(x<y).$$

Then, returning to coordinates $(x,y)$,

$$v_h(x,dy) = w(x,dy) - \frac{\min(x,y)}{y} w(y,dy)$$

is the process which, in each strip $[0,1]\times dy$, must be zero for $x=y$, or, more exactly, which is a Brownian bridge in $x$ for $0 \le x \le y$, but is Brownian motion in $x$ for $x>y$. In the integral form,

$$v(x,y) = w(x,y) - \int_0^y \frac{\min(x,y')}{y'}\, w(y',dy'). \tag{2.7}$$

For each $y$, the process in (2.6) is Gaussian semi-martingale in $x$ and its innovation martingale has the form

$$v_h(dx,dy) - \frac{\int_x^1 h(x',y)v_h(dx',dy)}{\int_x^1 h^2(x',y)dx'} h(x,y)dx = b(dx,dy).$$

Multiply both sides by a function $f \in L_2[0,1]^2$ and integrate with respect to $x$. Using integration by parts we then come to the operator

$$U_h\phi(x,y) = \phi(x,y) - \int_0^x \frac{\phi(x',y)}{\int_x^{\prime 1} h^2(x'',y)dx''} dx'\, h(x,y)$$

and representation of the $b$ in the function-parametric version as

$$v_h(U_h\phi) = b(\phi).$$

## 3. Newer Results and Simpler Operators

In the previous section we reported results on mappings of projected Brownian motions, or constrained Brownian motions, or Brownian bridges, into a Brownian motion. The operators involved had the form of $Identity - Volterra$ *operator*. This is the operational analogue of innovation theory – at least, for Gaussian semi-martingales.

Here we show how various Brownian bridges can be mapped into each other. The unitary operators we will use for this are simple, but the results are unexpected. No direct probabilistic intuition seems to exist.

We shall set the notations again for readers' convenience. Let $w_F(\phi), \phi \in L_2(F)$, denote a function-parametric $F$-Brownian motion. Fix $h \in L_2(F)$, with $\|h\|_F = 1$. Then

$$v_{F,h}(\phi) = w_F(\phi) - \langle \phi, h\rangle_F w(h)$$

is $h$-projected $F$-Brownian motion. If distributions $G$ and $F$ are mutually absolutely continuous, we denote

$$l(x) = \sqrt{\frac{dG}{dF}(x)}$$

and note its role below – it leads to the isomorphism between $L_2(G)$ and $L_2(F)$: $\psi \in L_2(G)$ iff $l\psi \in L_2(F)$ and, therefore, to the isomorphism between Brownian motions: $w_F(\phi) = w_G(\psi), \phi = l\psi$. This, however, is not enough to establish an isomorphism between $v_{F,h}$ and $v_{G,r}$. To do this we use the following operator:

$$K\phi = \phi - \frac{2}{\|lr-h\|_F^2}(lr-h)\langle lr-h, \phi\rangle_F.$$

This is self-adjoint unitary operator, which maps $lr$ into $h$ and vice versa.

**Proposition 3.1.** *If $v_{F,h}$ is $h$-projected $F$-Brownian motion, then the process defined as*

$$v_{G,r}(\psi) = v_{F,h}(Kl\psi), \psi \in L_2(G),$$

*is an $r$-projected $G$-Brownian motion.*

As presented here, this statement was not given in [11], only its special cases have been formulated, which addressed particular statistical problems.

If in $v_{F,h}$ the function $h$, parallel to which the Brownian motion was projected, is chosen as $q$, then we would use the term $F$-Brownian bridge and drop $h$ from the notation. So, consider the $F$-Brownian bridge $v_F(\phi), \phi \in L_2(F)$. This is the limit in distribution of the (function-parametric) empirical process based on a sample of $n$ i.i.d. $(F)$ random variables when $n$ tends to infinity. Let $G$ be another distribution, such that $G$ and $F$ are mutually absolutely continuous, and consider $G$-Brownian bridge $v_G(\psi), \psi \in L_2(G)$. This is certainly the limit in distribution of the empirical process based on i.i.d.($G$) random variables. This empirical process is constructed for statistical inference concerning $G$, not $F$. The distributions $F$ and $G$ are only loosely related through being mutually absolutely continuous. Yet, we now know that there is a simple isometry between $v_F$ and $v_G$, which basically makes the two inference problems equivalent.

Let $K_1$ be the operator defined as

$$K_1\phi = \phi - \frac{2}{\|l-q\|_F^2}(l-q)\langle l-q, \phi\rangle_F.$$

This operator is unitary, self-adjoint and it maps $l$ into $q$ and $q$ into $l$. In other words, we choose here both $r$ and $h$ equal to $q$.

**Proposition 3.2.** [11]. *With functions $l$ and $q$ defined as above, if $v_F$ is $F$-Brownian bridge, then the process $v_G$, defined as*

$$v_G(\psi) = v_F(K_1 l\psi),\ \psi \in L_2(G),$$

*is a $G$-Brownian bridge.*

To map (usually absolutely continuous) distribution $\tilde{F}$ given on $\mathbb{R}^d$ into the distribution given on the unit cube $[0,1]^d$ is a well understood and often used step in statistical theory. For example, if $\tilde{F}_1, \ldots, \tilde{F}_d$ are marginal distribution functions of $\tilde{F}$, then

$$F(x_1, \ldots, x_d) = \tilde{F}(y_1, \ldots, y_d), \quad x_j = \tilde{F}_j(y_j),$$

is the distribution function on $[0,1]^d$ with uniform marginals, Such distributions, as we know, are called copula distributions or copulas (see, e.g., [6], [15]).

Now let $F$ be a distribution on $[0,1]^d$ – in particular a copula – and consider the $F$-Brownian bridge $v_F$. To standardize the problems of statistical inference it would be very useful to have a simple map of $v_F$ into the standard Brownian bridge $v$, which corresponds to the case of $F$ uniform on $[0,1]^d$.

The previous proposition creates this possibility. We present this particular case in the point-parametric version, as it may be more convenient for many readers.

Let $F$ be an absolutely continuous distribution on $[0,1]^d$ and denote its density by $f$. Denote also $Q(x)$ the uniform distribution function on $[0,1]^d$ and

$$R(x) = \int_{y\in[0,x]} \sqrt{f(y)}dy \quad \text{with } R(1) = \int_{y\in[0,1]^d} \sqrt{f(y)}dy.$$

Then we have the next proposition.

**Proposition 3.3.** *If $v_F$ is an $F$-Brownian bridge, then*

$$v(x) = \int_{y\in[0,x]} \frac{1}{\sqrt{f(y)}} v_F(dy) - \frac{Q(x)-R(x)}{1-R(1)} \int_{y\in[0,1]^d} \frac{1}{\sqrt{f(y)}} v_F(dy),$$

*is the standard Brownian bridge on $[0,1]^d$.*

Now let us consider only the bridges, which have the same $F$. We can assume that $F$ is the uniform distribution on $[0,1]^d$. Let, however, these bridges satisfy different constrains: for $v_h$ we have $v_h(h) = 0$ and for $v$ we have $v(q) = 0$. As we mentioned before, processes like $v_h$ occur as a limiting object in the survival models for point processes. If the survival model is parametric, and it very often is, and if the parameter is $m$-dimensional, then we will need to consider $m$-dimensional $h$. However, to illustrate the phenomenon, it is sufficient to consider one-dimensional $h$. Proposition 2.3 described how to map $v_h$ into Brownian motion, while the proposition here shows how to map it into the standard Brownian bridge.

Denote, similarly to the case of the previous proposition,

$$H(x) = \int_{y\in[0,x]} h(y)dy \quad \text{with } H(1) = \int_{y\in[0,1]^2} h(y)dy.$$

**Proposition 3.4.** *Let $v_h$ be the $h$-projected standard Brownian motion, defined in* (2.4) *with $m = 1$. Then*

$$v(x) = v_h(x) - \frac{Q(x) - H(x)}{1 - H(1)} v_h(1)$$

*is the standard Brownian bridge on* $[0,1]^d$.

Let us now show one other choice of the transformation of the present type for Kiefer processes. Let, again, $v$ denote the process defined in (2.5). As $v(1,s) = 0$ for all $s \in [0,1]$, the Kiefer process is "sharply" different from Brownian motion on $[0,1]^2$, but nevertheless the transformation below will map it into "almost" Brownian motion with a relatively "small" perturbation.

Let us choose $A$, a Borel subset of $[0,1]$, and consider the strip $A\times[0,1]$. Choose also a function $\gamma(t)$ so that

$$\int_A \gamma^2(t)dt = 1, \quad \text{and } \gamma(t) = 0,\ t \notin A.$$

It certainly is possible to choose $A$ dependent on $s$ and also make $\gamma(t)$ dependent on $s$. However, we will stay within a simple framework and choose $A$ as the interval $[1-\Delta, 1]$ and $\gamma^2$ as the uniform density on this interval. Now use in the operator $K_1$ the function $\gamma$ in place of $l$ and consider the resulting process:

$$v_A(\phi, ds) = v(\phi, ds) - \frac{2}{\|\gamma - q\|^2} v(\gamma, ds)\langle\gamma - q, \phi\rangle. \tag{3.1}$$

One can see that here

$$\frac{2}{\|\gamma - q\|^2} = \frac{1}{1 - \int_A \gamma(x)dx} = \frac{1}{1-\sqrt{\Delta}}$$

and

$$v(\gamma, ds) = -v(1-\Delta, ds)/\sqrt{\Delta}.$$

Using an indicator function $\phi(t') = \mathbf{1}(t' \le t)$, we obtain the following expression for $v_A$ in its point-parametric version:

$$v_A(t,s) = v(t,s) + \frac{v(1-\Delta, s)}{\sqrt{\Delta}(1-\sqrt{\Delta})}(\Gamma_A(t) - t), \tag{3.2}$$

where

$$\Gamma_A(t) = \int_0^t \gamma(t')dt'.$$

**Proposition 3.5.** *The process* $v_A(t,s)$*, defined in* (3.2)*, is the standard Brownian motion on* $[0,1-\Delta]\times[0,1]$*; on the remaining set* $[1-\Delta,1]\times[0,1]$ *the process* $v_A(t,s)-v_A(1-\Delta,s)$ *is the Kiefer process.*

We believe that equation (3.1) explains the structure of the process. The formal proof of the proposition only requires calculation of the variance of $v_A(t,s)$. For $t<1-\Delta$, we have $\Gamma_A(t)=0$ and

$$v_A(t,s)=v(t,s)-\frac{v(1-\Delta,s)}{\sqrt{\Delta}(1-\sqrt{\Delta})}t.$$

Therefore

$$\begin{aligned}Ev_A^2(t,s)&=st-st^2-2s\frac{t-t(1-\Delta)}{\sqrt{\Delta}(1-\sqrt{\Delta})}t+st^2\frac{\Delta(1-\Delta)}{\Delta(1-\sqrt{\Delta})^2}\\&=st-st^2\left[1+2\frac{\sqrt{\Delta}}{1-\sqrt{\Delta}}-\frac{1+\sqrt{\Delta}}{1-\sqrt{\Delta}}\right]\\&=st.\end{aligned}$$

For $t>1-\Delta$, after some rearrangements we easily see that

$$\begin{aligned}&v_A(t,s)-v_A(1-\Delta,s)\\&=v(t,s)-v(1-\Delta,s)+v(1-\Delta,s)\frac{t-(1-\Delta)}{\Delta}\end{aligned}$$

and the second claim becomes obvious. □

We conclude with the remark that the unitary operators applied to the function-parametric Brownian motion can produce unexpected and somewhat bizarre expressions. For example,

**Proposition 3.6.** *If* $w(t), t\in[0,1]$*, is a standard Brownian motion on* $[0,1]$*, then the processes*

$$w(t)-2tw(1)$$

*and*

$$w(t)-\frac{\sqrt{3}t^2-t}{2-\sqrt{3}}\left[\sqrt{3}\int_0^1 sw(ds)-w(1)\right]$$

*are also standard Brownian motions on* $[0,1]$*. The relationships between all three processes are one-to-one.*

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