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TWO-DIMENSIONAL 1-MEIXNER RANDOM VECTORS AND THEIR SEMI-QUANTUM OPERATORS

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Abstract. A definition of d-dimensional n-Meixner random vectors is given first. This definition involves the commutators of their semi-quantum operators. A characterization of all non-degenerate, two-dimensional 1-Meixner random vectors is presented next.

1. Introduction

In the one-dimensional case, the Szegö–Jacobi parameters of a random variable, having finite moments of all orders, appear naturally in the recursive formula relating the orthogonal polynomials generated by that random variable. In the multi-dimensional case, the quantum operators: annihilation, preservation, and creation provide a natural replacement of the Szegö–Jacobi parameters. The quantum operators can be used to describe qualitative properties of joint distributions of a finite set of random variables, like polynomially symmetry and factorisability, see [1] and [2], and square summability of the support, see [3]. The commutators of the quantum operators can also be used to recover the joint moments of a finite family of random variables, both in the commutative and non-commutative cases, see [15] and [18]. The q-commutators of the quantum operators can also be used to recover the joint moments of not necessary commuting random variables, see [9].

The class of Meixner random variables is particularly suited for recovering the moments from the commutators of the quantum operators, because of the Lie Algebra structure possessed by the vector space spanned by the identity and quantum operators of these random variables. In [17] the Meixner random variables of class $\mathcal{M}_L$, which contains all Meixner random variables except the symmetric two parameter hyperbolic random variables, were characterized in this way. The symmetric Meixner random variables can also be characterized using the double commutators of their annihilation and creation operators. Thus putting the two characterizations together, we can cover the whole Meixner class, in the same way that a sphere can be covered by two maps as a manifold. It must be mentioned that the class of Meixner random variables has been intensively studied by many authors, and many different approaches have been used in deriving it. Among

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these approaches, we would like to mention the powerful renormalization method used by Asai, Kubo, Kuo, and Namli in [4], [5], [6], [7], [10], [11].

In this paper we propose to split the preservation operators into two halves, and group one half with the annihilation and the other with the creation operators. In this way we create the semi-quantum operators, and using them we can give a complete characterization of the entire Meixner class (in one map). By using the semi-quantum instead of the quantum operators, we keep the number of commutators at minimum. We will see that to characterize the Meixner class we have to use double commutators. However, we can also consider simple, triple, quadruple, and so on commutators, and extend the notion of Meixner random variables. In this way we define the classes of 1-Meixner, 2-Meixner, 3-Meixner, and so on, random variables. The classic Meixner random variables become just the 2-Meixner random variables.

Moreover, since the quantum and semi-quantum operators can also be defined in the multi-dimensional case, for every natural number d, we can define the notion of d-dimensional 1-Meixner, 2-Meixner, 3-Meixner, and so on, random vectors.

The purpose of this paper is to come up with a natural definition of the d-dimensional n-Meixner random vectors and start a systematic study of them, by focusing on the 2-dimensional 1-Meixner random vectors. We hope that in subsequent works we will tackle the 2-dimensional 2-Meixner random vectors, 1-dimensional 3-Meixner random vectors, and finally attempt to attack a general finite dimension d. It must be mentioned that the work in dimensions $d \geq 2$ is much harder than that in the one dimensional case.

The paper is structured as follows. In section 2 we present a minimal background of the quantum and semi-quantum operators. In section 3 we characterize the classic Meixner random variables in terms of their semi-quantum operators, and use this characterization to define the notion of d-dimensional n-Meixner random vectors, for any natural numbers n and d. In section 4 we present a connection between the derivative of the logarithm of the Laplace transforms, of the Gaussian and gamma distributed random variables, and Möbius transformations that preserve the upper half-plane. Finally, in section 5 we characterize the non-degenerate 2-dimensional 1-Meixner random vectors.

2. Background

Let d be a fixed natural number. We consider d random variables: X_1, X_2, $\ldots$, X_d, defined on the same probability space and having finite moments of all orders. That means, if E denotes the expectation, then for all $i \in \{1, 2, \ldots, d\}$ and all $p > 0$, we have

$$E\left[|X_i|^p\right] \quad < \quad \infty. \tag{2.1}$$

For each non-negative integer n, we denote by F_n the space of all random variables of the form $f(X_1, X_2, \ldots, X_d)$, where f is a polynomial of d variables, with complex coefficients, of degree at most n. Since X_1, X_2, $\ldots$, X_d have finite moments of all orders, we have

$$\mathbb{C} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset L^2(\Omega, \mathcal{F}, P). \tag{2.2}$$

Because for each $n \geq 0$, F_n is a finite dimensional vector space, F_n is a closed subspace of $L^2(\Omega, \mathcal{F}, P)$. Thus, we can orthogonalize the spaces $\{F_n\}_{n\geq 0}$, defining for all $n \geq 0$,

$$G_n \quad := \quad F_n \ominus F_{n-1}, \tag{2.3}$$

where $F_n \ominus F_{n-1}$ denotes the orthogonal complement of F_{n-1} into F_n. Here for $n = 0$, $G_0 := F_0$, since we define $F_{-1} := \{0\}$. We also define $G_{-1} := \{0\}$.

For all $n \geq 0$, we call G_n the *space of all homogenous polynomial random variables of degree* n. We also define the space of all *polynomial random variables*:

$$F \quad = \quad \cup_{n=0}^{\infty} F_n. \tag{2.4}$$

We change now the way we view $X_1, X_2, \ldots, X_d$, regarding them not as random variables but as multiplication operators. Namely, for each $1 \leq i \leq d$, we define $X_i : F \to F$ as the linear operator mapping:

$$f(X_1, X_2, \ldots, X_d) \quad \mapsto \quad X_i f(X_1, X_2, \ldots, X_d), \tag{2.5}$$

for all polynomial random variables $f(X_1, X_2, \ldots, X_d)$.

The following lemma, see [1], can be easily verified using the symmetry of the multiplication operators $X_1, X_2, \ldots, X_d$.

Lemma 2.1. *For all $1 \leq i \leq d$ and all non-negative integers n, we have*

$$X_i G_n \quad \perp \quad G_k, \tag{2.6}$$

for all $k \neq n-1, n, n+1$, where "$\perp$" means "orthogonal to".

It follows from the above lemma, that for all $1 \leq i \leq d$ and all $n \geq 0$, we have

$$X_i G_n \quad \subset \quad G_{n-1} \oplus G_n \oplus G_{n+1}. \tag{2.7}$$

Thus if $f \in G_n$, there exist three unique homogenous polynomial random variables: $f_{n-1,i} \in G_{n-1}$, $f_{n,i} \in G_n$, and $f_{n+1,i} \in G_{n+1}$, such that:

$$X_i f \quad = \quad f_{n-1,i} + f_{n,i} + f_{n+1,i}. \tag{2.8}$$

We define three linear operators:

$$D_n^-(i) : G_n \quad \to \quad G_{n-1}, \tag{2.9}$$

$$D_n^-(i) f \quad := \quad f_{n-1,i}, \tag{2.10}$$

and call $D_n^-(i)$ an *annihilation operator* since it decreases the degree of a homogenous polynomial by 1 unit,

$$D_n^0(i) : G_n \quad \to \quad G_n, \tag{2.11}$$

$$D_n^0(i) f \quad := \quad f_{n,i}, \tag{2.12}$$

and call $D_n^0(i)$ a *preservation operator* since it preserves the degree of a homogenous polynomial, and

$$D_n^+(i) : G_n \quad \to \quad G_{n+1}, \tag{2.13}$$

$$D_n^+(i) f \quad := \quad f_{n+1,i}, \tag{2.14}$$

and call $D_n^+(i)$ a *creation operator* since it increases the degree of a homogenous polynomial by 1 unit.

Lemma 2.1 can be written now:

Lemma 2.2. *For all $1 \leq i \leq d$ and all $n \geq 0$, we have*

$$X_i|G_n = D_n^-(i) + D_n^0(i) + D_n^+(i), \tag{2.15}$$

where $X_i|G_n$ denotes the restriction of the multiplication operator X_i to the space G_n.

We extend now the definition of the annihilation, preservation, and creation operators to the space F of all polynomial random variables, in the following linear way. If $f \in F$, then there exist and are unique: $f_0 \in G_0$, $f_1 \in G_1$, $f_2 \in G_2$, ..., with only finitely many of them being non-zero, such that:

$$f = f_0 + f_1 + f_2 + \cdots . \tag{2.16}$$

We define the *annihilation operator* corresponding to X_i as:

$$a^-(i)f = D_0^-(i)f_0 + D_1^-(i)f_1 + D_2^-(i)f_2 + \cdots , \tag{2.17}$$

preservation operator corresponding to X_i as:

$$a^0(i)f = D_0^0(i)f_0 + D_1^0(i)f_1 + D_2^0(i)f_2 + \cdots , \tag{2.18}$$

and *creation operator* corresponding to X_i as:

$$a^+(i)f = D_0^+(i)f_0 + D_1^+(i)f_1 + D_2^+(i)f_2 + \cdots . \tag{2.19}$$

Lemma 2.1 becomes now:

Lemma 2.3. *For all $1 \leq i \leq d$, we have*

$$X_i = a^-(i) + a^0(i) + a^+(i). \tag{2.20}$$

In this equality, the domain of X_i, $a^-(i)$, $a^0(i)$, and $a^+(i)$ is understood to be the space F of all polynomial random variables.

The families of operators: $\{a^-(i)\}_{1 \leq i \leq d}$, $\{a^0(i)\}_{1 \leq i \leq d}$, and $\{a^+(i)\}_{1 \leq i \leq d}$ are called the *joint quantum operators of* X_1, X_2, ..., X_d. It must be mentioned that, for all $1 \leq i \leq d$, we have

$$\left(a^+(i)\right)^* = a^-(i) \tag{2.21}$$

and

$$\left(a^0(i)\right)^* = a^0(i), \tag{2.22}$$

in the sense that, for all f and g in F, we have

$$\langle a^+(i)f, g\rangle = \langle f, a^-(i)g\rangle \tag{2.23}$$

and

$$\langle a^0(i)f, g\rangle = \langle f, a^0(i)g\rangle, \tag{2.24}$$

where $\langle \cdot, \cdot\rangle$ denotes the inner product of the space $L^2(\Omega, \mathcal{F}, P)$.

Also, as it was proven in [1] and [2], the fact that for all $1 \leq i, j \leq d$, we have

$$X_iX_j = X_jX_i, \tag{2.25}$$

can be expressed in terms of the commutators of the joint quantum operators as:

$$\left[a^-(i), a^-(j)\right] = 0, \tag{2.26}$$

$$\left[a^-(i), a^0(j)\right] = \left[a^-(j), a^0(i)\right], \tag{2.27}$$

$$\left[a^0(i), a^0(j)\right] = \left[a^-(j), a^+(i)\right] - \left[a^-(i), a^+(j)\right], \tag{2.28}$$

$$\left[a^0(i), a^+(j)\right] = \left[a^0(j), a^+(i)\right], \tag{2.29}$$

and

$$\left[a^+(i), a^+(j)\right] = 0. \tag{2.30}$$

We will call the commutation relationships (2.26), (2.27), (2.28), (2.29), and (2.30), the *axioms of Commutative Probability.*

For all $1 \le i \le d$, we define the operators:

$$U_i, V_i : F \to F, \tag{2.31}$$

$$U_i = a^-(i) + \frac{1}{2}a^0(i) \tag{2.32}$$

and call it a *semi-annihilation operator*, and

$$V_i = a^+(i) + \frac{1}{2}a^0(i) \tag{2.33}$$

and call it a *semi-creation operator*.

For all $1 \le i \le d$, we have

$$X_i = U_i + V_i \tag{2.34}$$

and

$$V_i^* = U_i. \tag{2.35}$$

We call $\{U_i\}_{1\le i\le d}$ and $\{V_i\}_{1\le i\le d}$ the *joint semi-quantum operators generated by* X_1, X_2, ..., X_d.

Using the axioms of Commutative Probability, one can prove the following proposition, see [16]:

Proposition 2.4. *Let X_1, X_2, ..., X_d be d random variables defined on the same probability space and having finite moments of all orders. Let $\{U_i\}_{1\le i\le d}$ and $\{V_i\}_{1\le i\le d}$ be the joint semi-quantum operators generated by X_1, X_2, ..., X_d. Then, for all $1 \le i,\ j \le d$, we have*

(1)
$$[U_i, X_j] = [U_j, X_i]. \tag{2.36}$$

(2)
$$[X_i, V_j] = [X_j, V_i]. \tag{2.37}$$

(3) *The operators $[U_i,\ X_j]$ and $[X_i,\ V_j]$ are self-adjoint.*

To close this section, we must mention the connection with the classic theory of one-dimensional orthogonal polynomials.

If $d = 1$, then we have only one random variable, one creation, one preservation, one annihilation, one semi-creation, and one semi-annihilation operator, which we denote simply by X, a^+, a^0, a^-, U, and V, respectively.

Every homogenous chaos space G_n has dimension at most 1, since the co-dimension of the space F_{n-1} (spanned by 1, X, ..., X^{n-1}) into F_n (spanned by 1, X, ..., X^{n-1}, X^n) is at most 1. If the random variable X takes on only finitely many different values with positive probability, then if k is the number of such values, we have

$$\dim(G_n) = \begin{cases} 1 & \text{if} \quad n \leq k-1 \\ 0 & \text{if} \quad n \geq k \end{cases}, \tag{2.38}$$

where "dim" denotes the dimension. If the support of the probability distribution of X is an infinite set, then, for all $n \geq 0$, we have $\dim(G_n) = 1$.

If $\dim(G_n) = 1$, then there exists only one polynomial $f_n \in G_n$ having the leading coefficient equal to 1. Since $XG_n \subset G_{n+1} + G_n + G_{n-1}$, there exist two real numbers α_n and ω_n, such that:

$$Xf_n(X) = f_{n+1}(X) + \alpha_n f_n(X) + \omega_n f_{n-1}(X). \tag{2.39}$$

For $n = 0$, since $f_{-1} = 0$, we can choose ω_0 however we want. The numbers $\{\alpha_n\}_{n \geq 0}$ and $\{\omega_n\}_{n \geq 1}$ are called the *Szegö–Jacobi parameters* of X.

In this case, for all $n \geq 0$, we have

$$a^- f_n = \omega_n f_{n-1}, \tag{2.40}$$

$$a^0 f_n = \alpha_n f_n, \tag{2.41}$$

$$a^+ f_n = f_{n+1}, \tag{2.42}$$

$$Uf_n = \frac{\alpha_n}{2} f_n + \omega_n f_{n-1}, \tag{2.43}$$

and

$$Vf_n = f_{n+1} + \frac{\alpha_n}{2} f_n. \tag{2.44}$$

3. General Meixner Random Vectors

The classic Meixner random variables, introduced for the first time in [13], are the random variables X, having finite moments of all orders, whose Szegö–Jacobi parameters are of the form:

$$\alpha_n = \alpha n + \alpha_0 \tag{3.1}$$

and

$$\omega_n = \beta n^2 + (t - \beta)n, \tag{3.2}$$

for all $n \geq 1$, where α_0, α, β, and t are real numbers, with $t > 0$, and if $\beta < 0$, then $t \in -\mathbb{N}\beta$. Replacing eventually X by $-X$, we may assume that $\alpha \geq 0$. The class of Meixner random variables can be subdivided into six families, as follows:

- If $\alpha = \beta = 0$, then X is a *Gaussian* random variable, i.e., a continuous random variable given by the density function

$$f(x) = \frac{1}{\sqrt{2\pi t}} e^{-(x-\alpha_0)^2/(2t)}.$$

- If $\beta = 0$ and $\alpha \neq 0$, then X is a rescaled and shifted *Poisson* random variable, i.e.,

$$\mu_X = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \delta_{\alpha(k-\lambda)+\alpha_0},$$

where $\lambda := t/\alpha^2$.
- If $\beta > 0$ and $\alpha^2 > 4\beta$, then X is a shifted *Pascal (negative binomial)* random variable, i.e.,

$$\mu_X = \sum_{k=0}^{\infty} \frac{\Gamma(r+k)}{k!\Gamma(r)} p^r (1-p)^k \delta_{k-[2t/(\alpha+d)]+\alpha_0},$$

where $d := \sqrt{\alpha^2 - 4\beta}$, $p := 2d/(\alpha + d)$, $r := t/\beta$.
- If $\beta > 0$ and $\alpha^2 = 4\beta$, then X is a shifted and rescaled *gamma distributed* random variable with shift parameter $2t/\alpha$ and scaling parameter $\alpha/2$, i.e.,

$$f(x) = \frac{2^{2t/\alpha}}{\alpha^{2t/\alpha}\Gamma(2t/\alpha)} x^{(2t/\alpha)-1} e^{-2x/\alpha} 1_{(0,\infty)}.$$

- If $\beta > 0$ and $\alpha^2 < 4\beta$, then up to a translation, X is a *two parameter hyperbolic secant* random variable:

$$f(x) = ce^{2\theta x/\gamma} \left|\Gamma(k + ix\gamma)\right|^2,$$

where $\gamma := \sqrt{4\beta - \alpha^2}$ and $\gamma + i\alpha = re^{i\theta}$, with $-\pi/2 < \theta < \pi/2$, $k := 2t/(r\gamma)$.
- If $\beta < 0$, then $t \in -\mathbb{N}\beta$, and in this case, up to a rescaling and translation, X is a *binomial* random variable:

$$\mu_X = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \delta_k,$$

where $n := -t/\beta$, $p := (1/2) \pm (1/2)\sqrt{c/(4+c)}$, and $c := -\alpha^2/\beta \geq 0$.

It was shown in [16] that the shifted and rescaled gamma distributed random variables (Meixner with $\alpha^2 = 4\beta > 0$) and the Gaussian random variables (Meixner with $\alpha = \beta = 0$), which can be viewed as a limit of gamma distributed random variables as $\alpha \to 0$, are precisely those random variables X, having finite moments of all orders, for which the commutator between the semi-annihilation operator U and X is of the form:

$$[U, X] = bX + cI, \tag{3.3}$$

where b and c are real numbers. Taking the adjoint in both sides of (3.3), this condition is equivalent to:

$$[X, V] = bX + cI. \tag{3.4}$$

One of the reasons for which equality (3.3) is possible is the fact that, according to Proposition 2.4, $[U,\, X]$ is a self-adjoint operator. Without this fact, the equality (3.3) could never happen, since its right-hand side is the self-adjoint operator $bX + cI$.

In a future paper, we will see that the Meixner random variables (all six of them) are precisely the random variables X, having finite moments of all order, for which the double commutator $[[U, X], X]$ is of the form:

$$
\begin{aligned}
[[U, X], X] &= bV - bU & (3.5)\\
&= b(X - 2U), & (3.6)
\end{aligned}
$$

where b is a real number. In the above equality, the coefficients of U and V are opposite one to another, since due to the fact that both operators $[U, X]$ and X are self-adjoint, their commutator $[[U, X], X]$ is anti-self-adjoint.

With these facts in mind, and the observation that in what follows, when we have an odd number of commutators the left-hand side is self-adjoint, while when n is even, the left-hand side is an anti-self-adjoint operator, we are now ready to present the definition of the general d-dimensional n-Meixner random vectors:

Definition 3.1. Let $X_1, X_2, \ldots, X_d$ be d random variables defined on the same probability space, and having finite moments of all orders. Let n be a natural number. We say that $(X_1, X_2, \ldots, X_d)$ is an *n-Meixner random vector* if:

- If n is odd, then for all $i, i_1, i_2, \ldots, i_n$ in $\{1, 2, \ldots, d\}$, we have

$$
[\cdots[[U_i, X_{i_1}], X_{i_2}], \cdots X_{i_n}] = \sum_{j=1}^{d} b_{i,i_1 i_2 \ldots i_n, j} X_j + c_{i,i_1 i_2 \ldots i_n} I, \qquad (3.7)
$$

  for some real numbers $b_{i,i_1 i_2 \ldots i_n, j}$ and $c_{i,i_1 i_2 \ldots i_n}$, $1 \leq j \leq d$.
- If n is even, then for all $i, i_1, i_2, \ldots, i_n$ in $\{1, 2, \ldots, d\}$, we have

$$
\begin{aligned}
[\cdots[[U_i, X_{i_1}], X_{i_2}], \cdots X_{i_n}] &= \sum_{j=1}^{d} b_{i,i_1 i_2 \ldots i_n, j}(V_j - U_j) & (3.8)\\
&= \sum_{j=1}^{d} b_{i,i_1 i_2 \ldots i_n, j}(X_j - 2U_j), & (3.9)
\end{aligned}
$$

  for some real numbers $b_{i,i_1 i_2 \ldots i_n, j}$, $1 \leq j \leq d$.

We also say that the d-dimensional random vector $(X_1, X_2, \ldots, X_d)$ is *non-degenerate*, if the operators $I, X_1, X_2, \ldots, X_d$ are linearly independent. If the space of all polynomial functions of d variables, $P(x_1, x_2, \ldots, x_d)$, is dense in $L^2(\mathbb{R}^d, \mu)$, where μ denotes the joint probability distribution of the random variables $X_1, X_2, \ldots, X_d$, then the non-degeneracy condition is equivalent to the fact that the random variables: $1, X_1, X_2, \ldots, X_d$ are linearly independent, where 1 denotes the constant random variable equal to 1.

According to this definition the shifted and rescaled gamma distributed random variables (including the Gaussian random variables as a limiting case) form the one-dimensional 1-Meixner random vectors (variables). The classic Meixner random variables (all six types of them) form the one-dimensional 2-Meixner random vectors. It is very important to mention that, this definition (using the semi-quantum operators) allows us to both encompass all the six types of Meixner random vectors and keep the number of commutator conditions to a minimum (only one condition). Moreover, it allows us to study 3-Meixner, 4-Meixner, ... random vectors.

4. A Connection Between the Laplace Transform of Gamma Distributed Random Variables and Möbius Transformations

Let X be a gamma distributed random variable, i.e., a continuous random variable whose density function is:

$$f(x) = \frac{1}{\Gamma(t)} x^{t-1} \exp(-x) 1_{(0,\infty)}(x), \tag{4.1}$$

for some $t > 0$. Let us compute the Laplace transform of X. For all λ in a neighborhood of 0, we have

$$\begin{aligned} \varphi_X(\lambda) &:= E[\exp(\lambda x)] \\ &= \frac{1}{\Gamma(t)} \int_0^\infty x^{t-1} \exp(-x) \exp(\lambda x) dx \\ &= \frac{1}{\Gamma(t)} \int_0^\infty x^{t-1} \exp(-(1-\lambda)x)\, dx. \end{aligned}$$

Making the change of variable:

$$u := (1-\lambda)x, \tag{4.2}$$

we obtain:

$$\varphi_X(\lambda) = (1-\lambda)^{-t}. \tag{4.3}$$

Considering now a rescaling by a factor of p, where $p \neq 0$, and a shift of X by the number q, i.e.,

$$Y := pX + q, \tag{4.4}$$

we obtain:

$$\begin{aligned} \varphi_Y(\lambda) &:= E[\exp(\lambda Y)] \\ &= \exp(q\lambda)\varphi_X(p\lambda) \\ &= \exp(q\lambda)(1-p\lambda)^{-t}. \end{aligned}$$

Thus we have

$$\ln(\varphi_Y(\lambda)) = q\lambda - t\ln(1-p\lambda). \tag{4.5}$$

Taking the derivative with respect to λ, we obtain:

$$\begin{aligned} \psi_Y(\lambda) &:= \frac{d}{d\lambda} \ln(\varphi_Y(\lambda)) \\ &= q + \frac{pt}{1-p\lambda} \\ &= \frac{-pq\lambda + q + pt}{-p\lambda + 1}. \end{aligned} \tag{4.6}$$

Defining now the numbers:

$$a := -pq, \tag{4.7}$$

$$b := q + pt, \tag{4.8}$$

$$c := -p \tag{4.9}$$

$$\neq 0, \tag{4.10}$$

and

$$d := 1, \tag{4.11}$$

we obtain:

$$\psi_Y(\lambda) = \frac{a\lambda + b}{c\lambda + d}. \tag{4.12}$$

Moreover, since:

$$\begin{aligned} \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= ad - bc \\ &= -pq + pq + p^2 t \\ &= p^2 t \\ &> 0, \end{aligned}$$

$\psi_Y(\lambda)$ is the restriction of a Möbius transformation, with positive determinant, $c \neq 0$, and $d \neq 0$ (thus, by dividing all coefficients by d we can take $d = 1$), to a neighborhood of 0.

Conversely, given a Möbius transformation T with real coefficients, positive determinant, $c \neq 0$, and $d \neq 0$, we divide first all coefficients by d, and make $d = 1$. Then we define:

$$p := -c, \tag{4.13}$$

$$q := \frac{a}{c}, \tag{4.14}$$

$$t := \frac{a - bc}{c^2}. \tag{4.15}$$

We can see that $p \neq 0$ and $t > 0$. If X is t-Gamma distributed, then $pX + q$ has the Laplace transform equal to T, on a neighborhood of 0 .

If X is a Gaussian random variable with mean α_0 and variance t, then for all λ real (even complex), we have

$$\begin{aligned} \varphi(\lambda) &:= E\left[\exp(\lambda X)\right] \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \exp(\lambda x) \exp\Big(-(x - \alpha_0)^2/(2t)\Big)\, dx \\ &= \exp\Big(\lambda\alpha_0 + \lambda^2 t/2\Big). \end{aligned}$$

Thus, we have

$$\ln(\varphi(\lambda)) = \frac{\lambda^2 t}{2} + \lambda\alpha_0.$$

Differentiating with respect to λ, we obtain:

$$\psi(\lambda) := \frac{d}{d\lambda} \ln(\varphi(\lambda)) \tag{4.16}$$

$$= t\lambda + \alpha_0. \tag{4.17}$$

Thus, the derivative of the logarithm of the Laplace transform of X, is of the form:

$$\psi(\lambda) \quad := \quad a\lambda + b, \tag{4.18}$$

where $a > 0$, that means a Möbius transformation with positive determinant in which $c = 0$.

In both cases, gamma and Gaussian, it is easy to see that:

$$E[Y] \quad = \quad b. \tag{4.19}$$

Therefore, Y is centered, i.e., $E[Y] = 0$, if and only if $b = 0$, which means that the associated Möbius transformation maps 0 into 0.

Putting these facts together, we obtain the following characterization of the derivative of the logarithm of the Laplace transform of the one-dimensional 1-Meixner random vectors.

Lemma 4.1. *A random variable having finite moments of all orders is a non-degenerate* 1*-Meixner random variable if and only if the derivative of the logarithm of its Laplace transform:*

$$\psi(\lambda) \quad := \quad \frac{d}{d\lambda}\left\{\ln E\left[\exp(\lambda X)\right]\right\} \tag{4.20}$$

is equal to the restriction of a Möbius transformation with real coefficients, positive determinant, in which $d \neq 0$*, to a neighborhood of* 0*:*

$$\psi(\lambda) \quad := \quad \frac{a\lambda + b}{c\lambda + d}. \tag{4.21}$$

Geometrically, this can be restated as: the derivative of the logarithm of the Laplace transform, defined on a neighborhood of 0*, is equal to a Möbius transformation* T *that preserves the upper half plane* $H_+ := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$*, and* $T(0) \neq \infty$*. Moreover, the distinction between the two types of distribution within the* 1*-Meixner random variables is the following:*

- *If* $T(\infty) \neq \infty$*, then* X *is a shifted and rescaled gamma distributed random variable.*
- *If* $T(\infty) = \infty$*, then* X *is a Gaussian random variable.*

The 1*-Meixner random variable is centered if and only if* $T(0) = 0$*.*

5. Two-dimensional Gamma Random Vectors

In this section we consider a two-dimensional random vector, that means a pair of random variables $(X,\ Y)$, defined on the same probability space and having finite moments of all orders. Let

$$X \quad = \quad U_x + V_x \tag{5.1}$$

and

$$Y \quad = \quad U_y + V_y, \tag{5.2}$$

where U_x and U_y are their joint semi-annihilation operators, and V_x and V_y their joint semi-creation operators.

Before going further, let us first give a quick characterization of the 1-Meixner random variables, see [16].

Lemma 5.1. *Let X be a random variable having finite moments of all orders. Then X is a one-dimensional* 1*-Meixner random vector (variable) if and only if X is a classic Meixner random variable of one of the following two types: shifted and rescaled gamma distributed random variable or Gaussian random variable. If X is such a Meixner random variable, with parameters α, β, with $\alpha^2 = 4\beta$, t, and α_0, and U and V are its semi-quantum operators, then*

$$[U, X] = \frac{\alpha}{2}X + \left(t - \frac{\alpha\alpha_0}{2}\right) I. \tag{5.3}$$

Let us recall the definition of a two-dimensional 1-Meixner random vector.

Definition 5.2. Let (X, Y) be a two-dimensional random vector as described above. We say that (X, Y) is a *two-dimensional 1-Meixner random vector* if the real vector space W spanned by the identity operator I and the joint semi-quantum operators: U_x, U_y, V_x, and V_y, equipped with the bracket given by the commutator $[\cdot, \cdot]$ forms a Lie Algebra.

Since the commutators $[U_x, X]$, $[U_x, Y] = [U_y, X]$ and $[U_y, V_y]$ are symmetric, the above condition is equivalent to the fact that, there exist a, b, c, d, e, f, α, β, and γ real numbers, such that:

$$[U_x, X] = aX + bY + \alpha I, \tag{5.4}$$

$$[U_x, Y] = cX + dY + \beta I, \tag{5.5}$$

$$[U_y, X] = cX + dY + \beta I, \tag{5.6}$$

$$[U_y, Y] = eX + fY + \gamma I. \tag{5.7}$$

In the above definition the real number a must not be confused with any of the quantum operators a^-, a^0, and a^+.

Let us also recall the non-degeneracy condition:

Definition 5.3. Let (X, Y) be a two-dimensional random vector. We say that (X, Y) is *non-degenerate*, if the operators X, Y, and I are linearly independent, where I denotes the identity operator.

It is not hard to see that we have the following proposition:

Proposition 5.4. *Let (X, Y) be a two-dimensional random vector and let U_x, U_y, V_x, and V_y be the joint semi-quantum operators of X and Y. The following statements are equivalent:*

(1) *(X, Y) is non-degenerate.*
(2) *U_x, U_y, and I are linearly independent.*
(3) *V_x, V_y, and I are linearly independent.*

Theorem 5.5. *Let X and Y be two independent non-degenerate* 1*-Meixner random variables defined on the same probability space, namely, X and Y are either shifted rescaled gamma distributed or Gaussian random variables. Then the random vector (X, Y) is a non-degenerate two-dimensional* 1*-Meixner random vector.*

Proof. Let $\{a_x^\epsilon\}_{\epsilon \in \{-,0,+\}}$ and $\{a_y^\epsilon\}_{\epsilon \in \{-,0,+\}}$ be the joint quantum operators of X and Y. Let $\{a_{i,x}^\epsilon\}_{\epsilon \in \{-,0,+\}}$ be the individual quantum operators of X, and $\{a_{i,y}^\epsilon\}_{\epsilon \in \{-,0,+\}}$ be the individual quantum operators of Y. That means, the domain of $a^{\epsilon}_{i,x}$ is the space F_X of all polynomial random variables of the form $P(X)$, the domain of $a^{\epsilon}_{i,y}$ is the space F_Y of all polynomial random variables of the form $P(Y)$, while the domain of a^{ϵ}_{x} and a^{ϵ}_{y} is the space F of all polynomial random variables of the form $P(X, Y)$, for all $\epsilon \in \{-, 0, +\}$. As it was shown in [1], since X and Y are independent, and $F = F_X \otimes F_Y$, we have

$$a^{\epsilon}_{x} \quad = \quad a^{\epsilon}_{i,x} \otimes I_Y \tag{5.8}$$

and

$$a^{\epsilon}_{y} \quad = \quad I_X \otimes a^{\epsilon}_{i,y}, \tag{5.9}$$

for all $\epsilon \in \{-, 0, +\}$, where I_X and I_Y denote the identity operator of F_X and F_Y, respectively.

Because of this fact, if U_x and U_y denote the joint semi-annihilation operators of X and Y, $U_{i,x}$ the individual semi-annihilation operator of X, and $U_{i,y}$ the individual semi-annihilation operator of Y, then we have

$$U_x \quad = \quad U_{i,x} \otimes I_Y \tag{5.10}$$

and

$$U_y \quad = \quad I_X \otimes U_{i,y}. \tag{5.11}$$

Since X and Y are 1-Meixner random variables, there exist p, q, r, and s, such that:

$$[U_{i,x}, X|F_X] \quad = \quad pX|F_X + qI_X \tag{5.12}$$

and

$$[U_{i,y}, Y|F_Y] \quad = \quad rY|F_Y + sI_Y. \tag{5.13}$$

Thus, we have

$$\begin{aligned}
[U_x, X] &= [U_{i,x} \otimes I_Y, X|F_X \otimes I_Y] \\
&= [U_{i,x}, X|F_X] \otimes I_Y \\
&= (pX|F_X + qI_X) \otimes I_Y \\
&= p\,(X|F_X \otimes I_Y) + q\,(I_X \otimes I_Y) \\
&= pX + qI \\
&= pX + 0Y + qI.
\end{aligned} \tag{5.14}$$

Similarly, we have

$$[U_x, Y] \quad = \quad 0X + 0Y + 0I, \tag{5.15}$$

$$[U_y, X] \quad = \quad 0X + 0Y + 0I, \tag{5.16}$$

and

$$[U_y, Y] \quad = \quad 0X + rY + sI. \tag{5.17}$$

Thus, we can see that (X, Y) is a two-dimensional 1-Meixner random vector.

It is obvious that since X and Y are non-degenerate independent 1-Meixner random variables, the random variables X, Y, and 1 are independent. Hence (X, Y) is a non-degenerate 1-Meixner random vector. □

We also have the following simple lemma:

Lemma 5.6. *If $(X,\, Y)$ is a non-degenerate two-dimensional 1-Meixner random vector, then for any invertible linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$, and any fixed vector $(w_1,\, w_2) \in \mathbb{R}^2$, the random vector:*

$$(S,T) \quad := \quad A(X,Y) + (w_1, w_2) \tag{5.18}$$

is also a non-degenerate two-dimensional 1-Meixner random vector.

Proof. If the matrix associated to the invertible linear map A, in the standard basis, is:

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix}, \tag{5.19}$$

then

$$S \quad = \quad pX + qY + w_1 \tag{5.20}$$

and

$$T \quad = \quad rX + sY + w_2. \tag{5.21}$$

It follows now from [17] that:

$$a_s^\epsilon \quad = \quad pa_x^\epsilon + qa_y^\epsilon \tag{5.22}$$

and

$$a_t^\epsilon \quad = \quad ra_x^\epsilon + sa_y^\epsilon, \tag{5.23}$$

for all $\epsilon \in \{-,\, +\}$, while

$$a_s^0 \quad = \quad pa_x^0 + qa_y^0 + w_1 I \tag{5.24}$$

and

$$a_t^0 \quad = \quad ra_x^0 + sa_y^0 + w_2 I, \tag{5.25}$$

where $\{a_s^\epsilon\}_{\epsilon\in\{-,0,+\}}$ and $\{a_t^\epsilon\}_{\epsilon\in\{-,0,+\}}$ are the joint quantum operators of S and T. Thus, if U_s and U_t denote the joint semi-quantum operators of S and T, then we have

$$U_s \quad = \quad pU_x + qU_y + \frac{1}{2}w_1 I \tag{5.26}$$

and

$$U_t \quad = \quad rU_x + sU_y + \frac{1}{2}w_2 I. \tag{5.27}$$

Since $(X,\, Y) = A^{-1}(S,\, T) - A^{-1}(w_1,\, w_2)$, and $(X,\, Y)$ is a two-dimensional 1-Meixner random vector, we have

$$\begin{aligned} [U_s, S] &= \left[pU_x + qU_y + \frac{1}{2}w_1 I, pX + qY + w_1 I\right] \\ &= p^2\,[U_x, X] + pq\,[U_x, Y] + qp\,[U_y, X] + q^2\,[U_y, Y] \\ &\in \mathbb{R}X + \mathbb{R}Y + \mathbb{R}I \\ &= \mathbb{R}S + \mathbb{R}T + \mathbb{R}I. \end{aligned} \tag{5.28}$$

Similarly,

$$\begin{aligned} [U_s, T] &= [U_t, S] \\ &\in \mathbb{R}S + \mathbb{R}T + \mathbb{R}I \end{aligned} \tag{5.29}$$

and

$$[U_t, T] \in \mathbb{R}S + \mathbb{R}T + \mathbb{R}I. \tag{5.30}$$

Therefore, $(S,\ T)$ is a two-dimensional Meixner random vector. Since $(X,\ Y)$ is non-degenerate and A is invertible, $(S,\ T)$ is also non-degenerate. □

It is easy to see that the quantum and semi-quantum operators, are based entirely on the joint moments of the random variables, due to the fact that the coefficients in the Gram-Schmidt orthogonalization procedure are determined by expectations of monomials in X and Y. Because of this fact, one can introduce the following equivalence relation, see also [17]:

Definition 5.7. Let X and Y be random variables defined on the probability space $(\Omega,\ \mathcal{F},\ P)$, and X' and Y' be random variables defined on the probability space $(\Omega',\ \mathcal{F}',\ P')$, having finite moments of all orders. We say that the random vectors $(X,\ Y)$ and $(X',\ Y')$ are *moment equal*, and denote it by:

$$(X, Y) \equiv (X', Y'), \tag{5.31}$$

if for all m and n non-negative integers, we have

$$E\left[X^m Y^n\right] = E'\left[X'^m Y'^n\right], \tag{5.32}$$

where E and E' denote the expectation with respect to P and P', respectively.

Now, we start the arduous path of characterizing all non-degenerate, two-dimensional 1-Meixner random vectors. Throughout the remaining part of the paper, we assume that $(X,\ Y)$ is a non-degenerate two-dimensional random vector, whose joint semi-annihilation operators U_x and U_Y satisfy the commutation relationships (5.4), (5.5), (5.6), and (5.7).

Replacing X by $X - E[X]$, and Y by $Y - E[Y]$, we can see that in formulas (5.4), (5.5), (5.6), and (5.7), the numbers denoted by the Roman letters: a, b, c, d, e, and f remain the same, while the numbers denoted by the Greek letters: α, β, and γ change into three new numbers, which for simplicity will still be denoted by α, β, and γ. With these shifts, we have $E[X] = E[Y] = 0$, and we say that X and Y are *centered*. Observe that if $(X,\ Y)$ is non-degenerate, after centering X and Y, $(X,\ Y)$ remains non-degenerate.

Proposition 5.8. *If $(X,\ Y)$ is a centered, non-degenerate, 1-Meixner random vector, then*

$$\alpha = E\left[X^2\right], \tag{5.33}$$

$$\beta = E\left[XY\right], \tag{5.34}$$

$$\gamma = E\left[Y^2\right]. \tag{5.35}$$

Proof. Let $\phi := 1$ (the constant polynomial random variable equal to 1). We have

$$\begin{aligned} E\left[X^2\right] &= \langle X^2\phi, \phi\rangle \\ &= \langle (U_x + V_x)X\phi, \phi\rangle \\ &= \langle U_x X\phi, \phi\rangle + \langle V_x X\phi, \phi\rangle \\ &= \langle XU_x\phi, \phi\rangle + \langle [U_x, X]\phi, \phi\rangle + \langle X\phi, U_x\phi\rangle \\ &= \langle [U_x, X]\phi, \phi\rangle, \end{aligned}$$

since:

$$\begin{aligned} U_x\phi &= a_x^-\phi + \frac{1}{2}a_x^0\phi \\ &= 0 + \frac{1}{2}E[X]\phi \\ &= 0. \end{aligned}$$

Thus, we have

$$\begin{aligned} E\left[X^2\right] &= \langle [U_x, X]\phi, \phi\rangle \\ &= \langle (aX + bY + \alpha I)\phi, \phi\rangle \\ &= aE[X] + bE[Y] + \alpha \\ &= \alpha. \end{aligned}$$

Similarly, we have $E[XY] = \beta$ and $E[Y^2] = \gamma$. □

Applying the Gram-Schmidt orthogonalization procedure to X and Y, we obtain a new centered, non-degenerate, two-dimensional 1-Meixner random vector (X', Y'), such that:

$$E\left[X'^2\right] = 1, \tag{5.36}$$
$$E\left[X'Y'\right] = 0, \tag{5.37}$$
$$E\left[Y'^2\right] = 1. \tag{5.38}$$

This means that for the random vector (X', Y'), we have

$$\alpha' = 1, \tag{5.39}$$
$$\beta' = 0, \tag{5.40}$$
$$\gamma' = 1. \tag{5.41}$$

Therefore, from now on we will assume that (X, Y) is a centered, non-degenerate, two-dimensional 1-Meixner random vector, with $\alpha = \gamma = 1$ and $\beta = 0$. We have the following recursive relations:

Lemma 5.9. *Suppose (X, Y) is a centered, non-degenerate, two-dimensional 1-Meixner random vector satisfying the commutations relationships (5.4), (5.5), (5.6), and (5.7), with $\alpha = \gamma = 1$ and $\beta = 0$. Then, for all non-negative integers m and n, we have*

$$\begin{aligned} & E\left[X^{m+1}Y^n\right] \\ = \; & cnE\left[X^{m+1}Y^{n-1}\right] + (am + dn)E\left[X^mY^n\right] \\ & + bmE\left[X^{m-1}Y^{n+1}\right] + mE\left[X^{m-1}Y^n\right] \end{aligned} \tag{5.42}$$

and

$$\begin{aligned} & E\left[X^m Y^{n+1}\right] \\ = \quad & dmE\left[X^{m-1}Y^{n+1}\right] + (fn+cm)E\left[X^m Y^n\right] \\ & +neE\left[X^{m+1}Y^{n-1}\right] + nE\left[X^m Y^{n-1}\right]. \end{aligned} \tag{5.43}$$

Proof. Let m and n be non-negative integers. Using Leibniz rule for commutators, we have

$$\begin{aligned} E\left[X^{m+1}Y^n\right] &= \langle (U_x+V_x)X^mY^n\phi,\phi\rangle \\ &= \langle U_xX^mY^n\phi,\phi\rangle + \langle X^mY^n\phi, U_x\phi\rangle \\ &= \sum_{i=0}^{m-1}\langle X^i\left[U_x,X\right]X^{m-1-i}Y^n\phi,\phi\rangle \\ &\quad + \sum_{j=0}^{n-1}\langle X^mY^j\left[U_x,Y\right]Y^{n-1-j}\phi,\phi\rangle + \langle X^mY^nU_x\phi,\phi\rangle \\ &= \sum_{i=0}^{m-1}\langle X^i\left[U_x,X\right]X^{m-1-i}Y^n\phi,\phi\rangle \\ &\quad + \sum_{j=0}^{n-1}\langle X^mY^j\left[U_x,Y\right]Y^{n-1-j}\phi,\phi\rangle, \end{aligned}$$

since $U_x\phi = 0$, due to the assumption that $E[X] = 0$.

Using now the commutator relationships (5.4) and (5.5), we obtain:

$$\begin{aligned} E\left[X^{m+1}Y^n\right] &= \sum_{i=0}^{m-1}\langle X^i\left(aX+bY+I\right)X^{m-1-i}Y^n\phi,\phi\rangle \\ &\quad + \sum_{j=0}^{n-1}\langle X^mY^j\left(cX+dY\right)Y^{n-1-j}\phi,\phi\rangle \\ &= \sum_{i=0}^{m-1}a\langle X^mY^n\phi,\phi\rangle + \sum_{i=0}^{m-1}b\langle X^{m-1}Y^{n+1}\phi,\phi\rangle \\ &\quad + \sum_{i=0}^{m-1}\langle X^{m-1}Y^n\phi,\phi\rangle + \sum_{j=0}^{n-1}c\langle X^{m+1}Y^{n-1}\phi,\phi\rangle \\ &\quad + \sum_{j=0}^{n-1}d\langle X^mY^n\phi,\phi\rangle \\ &= amE\left[X^mY^n\right] + bmE\left[X^{m-1}Y^{n+1}\right] + mE\left[X^{m-1}Y^n\right] \\ &\quad + cnE\left[X^{m+1}Y^{n-1}\right] + dnE\left[X^mY^n\right]. \end{aligned}$$

Formula (5.43) can be proven similarly by writing:

$$E\left[Y^{n+1}X^m\right] = \langle (U_y+V_y)Y^nX^m\phi,\phi\rangle.$$

□

We can prove now the following lemma:

Lemma 5.10. *With the assumptions of the previous lemma, if we define:*

$$K \quad := \quad \max\{|a| + |b| + |c| + |d| + 1, |f| + |e| + |d| + |c| + 1\}, \tag{5.44}$$

then, for all m and n non-negative integers, we have

$$|E[X^m Y^n]| \quad \leq \quad K^{m+n} m! n!. \tag{5.45}$$

Proof. We use induction on $s := m + n$. For $s = 0$, we must have $m = n = 0$, and (5.45) is evident.

Let us assume that the inequality (5.45) is true for all m and n non-negative integers, with $m + n$ equal to a fixed non-negative integer s, and prove that it continues to hold for all m and n non-negative integers for which $m + n = s + 1$.

Let $m + n = s + 1$.

Case 1: If $m \geq n$, then $m \geq 1$, and so $m = m' + 1$, where $m' := m - 1$ is a non-negative integer. Using the recursive relation (5.42), we obtain:

$$\begin{aligned}
& |E[X^m Y^n]| \\
\leq\ & |a| m' \left| E\left[X^{m'} Y^n\right]\right| + |b| m' \left| E\left[X^{m'-1} Y^{n+1}\right]\right| + m' \left| E\left[X^{m'-1} Y^n\right]\right| \\
& + |c| n \left| E\left[X^{m'+1} Y^{n-1}\right]\right| + |d| n \left| E\left[X^{m'} Y^n\right]\right| \\
\leq\ & |a|(m'+1) K^{m'+n} m'! n! + |b| m' K^{m'-1+n+1} (m'-1)! n! \\
& + m' K^{m'-1+n} (m'-1)! n! + \ |c| n K^{m'+1+n-1} (m'+1)! (n-1)! \\
& + |d| m K^{m'+n} m'! n! \\
\leq\ & K^{m'+n} (|a| + |b| + 1 + |c| + |d|) m! n! \\
\leq\ & K^{m+n} m! n!.
\end{aligned}$$

Case 2: If $n \geq m$, then we can use the recursive formula (5.43) and do a similar proof. □

The growth inequality (5.45) allows us to define the *joint Laplace transform*

$$\varphi(s, t) \quad := \quad E[\exp(sX + tY)] \tag{5.46}$$

of X and Y, and differentiate it, by differentiating its Taylor series term by term, for $(s,\ t)$ in a neighborhood of $(0,\ 0)$.

We have the following lemma:

Lemma 5.11. *With the assumptions from the previous lemma, for $(s,\ t)$ in a neighborhood of $(0,\ 0)$, the joint Laplace transform φ of X and Y satisfies the following two partial differential equations:*

$$(1 - as - ct)\frac{\partial \varphi}{\partial s} - (bs + dt)\frac{\partial \varphi}{\partial t} \quad = \quad s\varphi \tag{5.47}$$

and

$$-(cs + et)\frac{\partial \varphi}{\partial s} + (1 - ds - ft)\frac{\partial \varphi}{\partial t} \quad = \quad t\varphi. \tag{5.48}$$

Proof. Let us multiply the recursive formula (5.42) by $s^m t^n/(m!n!)$ and sum from $m = 0$ to ∞, and from $n = 0$ to ∞. Thus, we obtain:

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{s^m t^n}{m!n!}E\left[XX^mY^n\right] &= as\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{s^{m-1}t^n}{(m-1)!n!}E\left[XX^{m-1}Y^n\right]\\ &\quad + bs\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{s^{m-1}t^n}{(m-1)!n!}E\left[X^{m-1}Y^nY\right]\\ &\quad + s\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{s^{m-1}t^n}{(m-1)!n!}E\left[X^{m-1}Y^n\right]\\ &\quad + ct\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{s^m t^{n-1}}{m!(n-1)!}E\left[XX^mY^{n-1}\right]\\ &\quad + dt\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{s^m t^{n-1}}{m!(n-1)!}E\left[X^mY^{n-1}Y\right].\end{aligned}$$

This is equivalent to:

$$\frac{\partial\varphi}{\partial s}(s,t) = as\frac{\partial\varphi}{\partial s}(s,t) + bs\frac{\partial\varphi}{\partial t}(s,t) + s\varphi(s,t) + ct\frac{\partial\varphi}{\partial s}(s,t) + dt\frac{\partial\varphi}{\partial t}(s,t),$$

which in turn is equivalent to (5.47).

Equation (5.48) is proven similarly, using the recursive relation (5.43). □

Since $\varphi(0,\, 0) = 1 > 0$, and φ is continuous, there exists a neighborhood $\mathcal{V}$ of $(0,\, 0)$, such that, for all $(s,\, t) \in \mathcal{V}$, $\varphi(s,\, t) > 0$. Thus, for all $(s,\, t) \in \mathcal{V}$, we may divide both sides of equations (5.47) and (5.48) by $\varphi(s,\, t)$, and obtain:

$$(1 - as - ct)\frac{1}{\varphi}\frac{\partial\varphi}{\partial s} - (bs + dt)\frac{1}{\varphi}\frac{\partial\varphi}{\partial t} = s \tag{5.49}$$

and

$$-(cs + et)\frac{1}{\varphi}\frac{\partial\varphi}{\partial s} + (1 - ds - ft)\frac{1}{\varphi}\frac{\partial\varphi}{\partial t} = t. \tag{5.50}$$

If we define the function $\psi(s,\, t) = \ln(\varphi(s,\, t))$, for all $(s,\, t) \in \mathcal{V}$, then (5.49) and (5.50) become:

$$(1 - as - ct)\frac{\partial\psi}{\partial s} - (bs + dt)\frac{\partial\psi}{\partial t} = s \tag{5.51}$$

and

$$-(cs + et)\frac{\partial\psi}{\partial s} + (1 - ds - ft)\frac{\partial\psi}{\partial t} = t. \tag{5.52}$$

Together the last two equations form a system with two unknowns, $\partial\psi/\partial s$ and $\partial\psi/\partial t$, whose determinant is:

$$\Delta(s,t) := (1 - as - ct)(1 - ds - ft) - (bs + dt)(cs + et). \tag{5.53}$$

Since $\Delta(0,\, 0) = 1 > 0$, and Δ is continuous, by eventually reducing the neighborhood $\mathcal{V}$ of $(0,\, 0)$, we may assume that for all $(s,\, t) \in \mathcal{V}$, $\Delta(s,\, t) > 0$. Thus, for all

$(s,\,t) \in \mathcal{V}$, $\Delta(s,\,t) \neq 0$, and so we can solve this system uniquely, obtaining:

$$\frac{\partial\psi}{\partial s} = \frac{s(1-ds-ft)+t(bs+dt)}{(1-as-bt)(1-ds-ft)-(bs+dt)(cs+et)} \tag{5.54}$$

and

$$\frac{\partial\psi}{\partial t} = \frac{t(1-as-ct)+s(cs+et)}{(1-as-bt)(1-ds-ft)-(bs+dt)(cs+et)}. \tag{5.55}$$

Let us define:

$$N := (1-as-bt)(1-ds-ft)-(bs+dt)(cs+et), \tag{5.56}$$

$$N_s := \frac{\partial N}{\partial s}, \tag{5.57}$$

and

$$N_t := \frac{\partial N}{\partial t}. \tag{5.58}$$

Differentiating both sides of (5.54) with respect to t, and both sides of (5.55) with respect to s, and using the fact that:

$$\frac{\partial^2\psi}{\partial t\partial s} = \frac{\partial^2\psi}{\partial s\partial t}, \tag{5.59}$$

we obtain:

$$\begin{aligned} &[(b-f)s+2dt]\,N - N_t\,[s(1-ds-ft)+t(bs+dt)] \\ =\ &[(e-a)t+2cs]\,N - N_s\,[t(1-as-ct)+s(cs+et)]\,, \end{aligned} \tag{5.60}$$

for all $(s,\,t) \in \mathcal{V}$.

Writing the condition that the coefficients of s from both sides of (5.60) are equal, we obtain:

$$(b-f)-(-b-f) = 2c,$$

which is equivalent to:

$$b = c. \tag{5.61}$$

Similarly, equating the coefficients of t, from both sides of (5.60), we conclude that:

$$d = e. \tag{5.62}$$

Let us introduce the notations:

$$D_1 := \begin{vmatrix} a & b \\ b & d \end{vmatrix} \tag{5.63}$$

$$= ad - b^2, \tag{5.64}$$

$$D_2 := \begin{vmatrix} b & d \\ d & f \end{vmatrix} \tag{5.65}$$

$$= bf - d^2, \tag{5.66}$$

and

$$D := \begin{vmatrix} a & b \\ d & f \end{vmatrix} \tag{5.67}$$

$$= af - bd. \tag{5.68}$$

Equating the coefficients of s^3, from both sides of (5.60), we conclude that:

$$b\,(D_1 + D_2) = 0. \tag{5.69}$$

Equating the coefficients of t^3, from both sides of (5.60), we conclude that:

$$d\,(D_1 + D_2) = 0. \tag{5.70}$$

Conditions (5.69) and (5.70) imply:

$$D_1 + D_2 = 0, \tag{5.71}$$

since if we assume that $D_1 + D_2 \neq 0$, then $b = d = 0$, and this implies: $D_1 = ad - b^2 = 0$ and $D_2 = bf - d^2 = 0$; so $D_1 + D_2 = 0$, which is a contradiction.

It can be seen now that if $b = c$, $d = e$, and $D_1 + D_2 = 0$, then (5.60) is indeed an identity and so there are now more extra conditions.

We would also like to mention that (5.71) is equivalent to the fact that, the following determinant of the commutator matrix coefficients:

$$\begin{vmatrix} a & b & 1 \\ b & d & 0 \\ d & f & 1 \end{vmatrix} = D_1 + D_2 \tag{5.72}$$

vanishes.

Let us define the following two matrices:

$$A := \begin{pmatrix} a & b \\ b & d \end{pmatrix} \tag{5.73}$$

and

$$F := \begin{pmatrix} b & d \\ d & f \end{pmatrix}. \tag{5.74}$$

We also define the *signum function*, sign : $\mathbb{R} \to \mathbb{R}$,

$$\text{sign(u)} = \begin{cases} 1 & \text{if} \quad u > 0, \\ 0 & \text{if} \quad u = 0, \\ -1 & \text{if} \quad u < 0 \end{cases}. \tag{5.75}$$

Proposition 5.12. *With the previous notations, let λ_1 and λ_2 be the eigenvalues of the matrix $-A$, indexed in the following way:*

$$\lambda_i = \frac{-(a+d) + (-1)^{i-1}\sigma\sqrt{(a-d)^2 + 4b^2}}{2}, \tag{5.76}$$

where:

$$\sigma := \text{sign}\left[(a-d)(b-f) + 4bd\right], \tag{5.77}$$

for all $i \in \{1, 2\}$.

Let μ_1 and μ_2 be the eigenvalues of the matrix $-F$, indexed in the following way:

$$\mu_i = \frac{-(b+f) + (-1)^{i-1}\sqrt{(b-f)^2 + 4d^2}}{2}, \tag{5.78}$$

for all $i \in \{1, 2\}$.

Then we have

$$N := (\lambda_1 s + \mu_1 t + 1)(\lambda_2 s + \mu_2 t + 1). \tag{5.79}$$

Proof. Since λ_1 and λ_2 are the roots of the quadratic equation:

$$\lambda^2 + (a+d)\lambda + (ad - b^2) = 0, \tag{5.80}$$

using Vieta relations, we have

$$\lambda_1 + \lambda_2 = -(a+d) \tag{5.81}$$

and

$$\lambda_1 \lambda_2 = ad - b^2. \tag{5.82}$$

Similarly, since μ_1 and μ_2 are the roots of the quadratic equation:

$$\mu^2 + (b+f)\mu + (bf - d^2) = 0, \tag{5.83}$$

using Vieta relations, we have

$$\mu_1 + \mu_2 = -(b+f) \tag{5.84}$$

and

$$\mu_1 \mu_2 = bf - d^2. \tag{5.85}$$

Let us compute now the expression $\lambda_1\mu_2 + \lambda_2\mu_1$. Here, the indexing of the roots is very important. We have

$$\begin{aligned}
&\lambda_1\mu_2 + \lambda_2\mu_1 \\
&= \frac{-(a+d) + \sigma\sqrt{(a-d)^2 + 4b^2}}{2} \cdot \frac{-(b+f) - \sqrt{(b-f)^2 + 4d^2}}{2} \\
&\quad + \frac{-(a+d) - \sigma\sqrt{(a-d)^2 + 4b^2}}{2} \cdot \frac{-(b+f) + \sqrt{(b-f)^2 + 4d^2}}{2} \\
&= \frac{(a+d)(b+f) - \sigma\sqrt{[(a-d)^2 + (2b)^2]\,[(b-f)^2 + (2d)^2]}}{2}.
\end{aligned}$$

Using now Lagrange identity, we obtain:

$$\begin{aligned}
&\lambda_1\mu_2+\lambda_2\mu_1\\
&= \frac{(a+d)(b+f)-\sigma\sqrt{[(a-d)^2+(2b)^2]\,[(b-f)^2+(2d)^2]}}{2}\\
&= \frac{(a+d)(b+f)}{2}\\
&\quad-\frac{\sigma\sqrt{[(a-d)(b-f)+(2b)(2d)]^2+[(a-d)\cdot 2d-2b\cdot(b-f)]^2}}{2}\\
&= \frac{(a+d)(b+f)-\sigma\sqrt{[(a-d)(b-f)+4bd]^2+4\left(ad-b^2+bf-d^2\right)^2}}{2}\\
&= \frac{(a+d)(b+f)-\sigma\sqrt{[(a-d)(b-f)+4bd]^2+4\left(D_1+D_2\right)^2}}{2}.
\end{aligned}$$

Since $D_1+D_2=0$, we obtain:

$$\begin{aligned}
\lambda_1\mu_2+\lambda_2\mu_1 &= \frac{(a+d)(b+f)-\sigma\sqrt{[(a-d)(b-f)+4bd]^2}}{2}\\
&= \frac{(a+d)(b+f)}{2}\\
&\quad-\frac{\operatorname{sign}[(a-d)(b-f)+4bd]\,|(a-d)(b-f)+4bd|}{2} \qquad (5.86)\\
&= \frac{(a+d)(b+f)-[(a-d)(b-f)+4bd]}{2}\\
&= af-bd. \qquad (5.87)
\end{aligned}$$

Therefore, using formulas (5.81), (5.83), (5.84), (5.85), and (5.87), we obtain:

$$\begin{aligned}
N &= (1-as-bt)(1-ds-ft)-(bs+dt)^2\\
&= 1-(a+d)s-(b+f)t+\left(ad-b^2\right)s^2+\left(bf-d^2\right)t^2+(af-bd)\,st\\
&= 1+(\lambda_1+\lambda_2)s+(\mu_1+\mu_2)t+\lambda_1\lambda_2 s^2+\mu_1\mu_2 t^2+(\lambda_1\mu_2+\lambda_2\mu_1)\,st\\
&= (1+\lambda_1 s+\mu_1 t)\,(1+\lambda_2 s+\mu_2 t)\,.
\end{aligned}$$

□

Observation 5.13. (1) If $(a-d)(b-f)\neq 0$ and $bd\neq 0$, then

$$\begin{aligned}
\operatorname{sign}[(a-d)(b-f)] &= \operatorname{sign}(bd) \qquad (5.88)\\
&= \sigma. \qquad (5.89)
\end{aligned}$$

(2) If $(a-d)(b-f)=0$, then

$$\operatorname{sign}(bd) = \sigma. \qquad (5.90)$$

(3) If $bd=0$, then

$$\operatorname{sign}[(a-d)(b-f)] = \sigma. \qquad (5.91)$$

Proof. We have

$$\begin{aligned}[(a-d)(b-f)]\cdot(bd) &= [d(a-d)]\cdot[b(b-f)]\\ &= \left(ad-d^2\right)\cdot\left(b^2-bf\right)\\ &= \left(ad-d^2\right)^2\\ &\geq 0,\end{aligned}$$

since:

$$ad-d^2 = b^2-bf \tag{5.92}$$

which follows from condition (5.71). Therefore, if $(a-d)(b-f)bd \neq 0$, the numbers $(a-d)(b-f)$ and bd have the same sign, and since σ is the signum function of the sum $(a-d)(b-f)+4bd$, we conclude that σ is the signum of each of them.

The other two statements of the observation are clear from the definition of σ. □

We have now the following proposition:

Proposition 5.14. *The vectors $\xi := (\lambda_1,\ \mu_1)$ and $\eta := (\lambda_2,\ \mu_2)$ are orthogonal. Moreover, we have*

$$D_\xi\psi(s,t) = \frac{\xi\cdot(s,t)}{1+\xi\cdot(s,t)} \tag{5.93}$$

and

$$D_\eta\psi(s,t) = \frac{\eta\cdot(s,t)}{1+\eta\cdot(s,t)}, \tag{5.94}$$

where D_ξ and D_η denote the Gâteaux derivatives after the directions ξ and η, and "$\cdot$" the dot product.

Proof. We have

$$\begin{aligned}\xi\cdot\eta &= (\lambda_1,\mu_1)\cdot(\lambda_2,\mu_2)\\ &= \lambda_1\lambda_2+\mu_1\mu_2\\ &= D_1+D_2\\ &= 0.\end{aligned}$$

Thus, ξ and η are orthogonal.

Using the differential equations (5.54) and (5.55) we obtain:

$$\begin{aligned}D_\xi\psi(s,t) &= \lambda_1\frac{\partial\psi}{\partial_s}(s,t)+\mu_1\frac{\partial\psi}{\partial_t}(s,t)\\ &= \lambda_1\frac{s(1-ds-ft)+t(bs+dt)}{N}+\mu_1\frac{t(1-as-bt)+s(bs+dt)}{N}\\ &= \frac{\lambda_1 s+\mu_1 t+(-d\lambda_1+b\mu_1)s^2+[(b-f)\lambda_1+(d-a)\mu_1]st}{N}\\ &\quad+\frac{(d\lambda_1-b\mu_1)t^2}{N}.\end{aligned} \tag{5.95}$$

Let us observe that the coefficient of s^2, which is the opposite of the coefficient of t^2, in the numerator of the last expression, is:

$$\begin{aligned} & -d\lambda_1 + b\mu_1 \\ = \quad & -d \cdot \frac{-(a+d) + \sigma\sqrt{(a-d)^2 + 4b^2}}{2} + b \cdot \frac{-(b+f) + \sqrt{(b-f)^2 + 4d^2}}{2} \\ = \quad & \frac{d(a+d) - b(b+f)}{2} + \frac{b\sqrt{(b-f)^2 + 4d^2} - \sigma d\sqrt{(a-d)^2 + 4b^2}}{2} \\ = \quad & \frac{d(a+d) - b(b+f)}{2}, \end{aligned}$$

since we have the following claim:
Claim 1: The following equality holds:

$$b\sqrt{(b-f)^2 + 4d^2} \quad = \quad \sigma d\sqrt{(a-d)^2 + 4b^2}. \tag{5.96}$$

To check (5.96), let us verify first that the squares of the two numbers (excluding σ from the second number) are the same. Indeed, this fact is equivalent to:

$$b^2\left[(b-f)^2 + 4d^2\right] \quad = \quad d^2\left[(a-d)^2 + 4b^2\right], \tag{5.97}$$

which, after cancelling $4b^2d^2$ from both sides, reduces to proving that:

$$b^2(b-f)^2 \quad = \quad d^2(a-d)^2. \tag{5.98}$$

The last equality follows from the fact that:

$$b(b-f) \quad = \quad d(a-d),$$

which is equivalent to:

$$D_1 + D_2 \quad = \quad 0.$$

Since the squares of the numbers $b\sqrt{(b-f)^2 + 4d^2}$ and $d\sqrt{(a-d)^2 + 4b^2}$ are the same, if one of them is zero, then the other must also be zero, so they are equal to zero even after putting back σ in the second number. If both of them are different from zero, then since $\sigma = \text{sign}(bd)$ (from Observation 5.13), we conclude that the two numbers have the same sign and square, and so they are equal.

Thus, we have

$$\begin{aligned} -d\lambda_1 + b\mu_1 \quad & = \quad \frac{d(a+d) - b(b+f)}{2} \\ & = \quad \frac{D_1 - D_2}{2} \\ & = \quad D_1 && (5.99) \\ & = \quad -D_2. && (5.100) \end{aligned}$$

The coefficient of st in the numerator of formula (5.95) is:

$$\begin{aligned}
&(b-f)\lambda_1 + (d-a)\mu_1 \\
&= (b-f)\frac{-(a+d)+\sigma\sqrt{(a-d)^2+4b^2}}{2} + (d-a)\frac{-(b+f)+\sqrt{(b-f)^2+4d^2}}{2} \\
&= \frac{-(b-f)(a+d)-(d-a)(b+f)}{2} \\
&\quad + \frac{(b-f)\sigma\sqrt{(a-d)^2+4b^2}-(a-d)\sqrt{(b-f)^2+4d^2}}{2} \\
&= \frac{-(b-f)(a+d)-(d-a)(b+f)}{2},
\end{aligned}$$

since

$$(b-f)\sigma\sqrt{(a-d)^2+4b^2} = (a-d)\sqrt{(b-f)^2+4d^2}, \tag{5.101}$$

due to the fact that their squares (excluding σ) are equal, since as before:

$$b(b-f) = d(a-d), \tag{5.102}$$

and if $(b-f)(a-d) \neq 0$, then $\operatorname{sign}[(b-f)(a-d)] = \sigma$ as proven in Observation 5.13.

Thus we obtain:

$$\begin{aligned}
(b-f)\lambda_1 + (d-a)\mu_1 &= \frac{-(b-f)(a+d)-(d-a)(b+f)}{2} \\
&= af - bd \\
&= D.
\end{aligned} \tag{5.103}$$

Therefore, we have

$$\begin{aligned}
&D_\xi \psi(s,t) \\
&= \frac{\lambda_1 s + \mu_1 t + (-d\lambda_1 + b\mu_1)s^2 + [(b-f)\lambda_1 + (d-a)\mu_1]st + (d\lambda_1 - b\mu_1)t^2}{N} \\
&= \frac{\lambda_1 s + \mu_1 t + D_1 s^2 + Dst + D_2 t^2}{(1+\lambda_1 s + \mu_1 t)(1+\lambda_2 s + \mu_2 t)} \\
&= \frac{\lambda_1 s + \mu_1 t + \lambda_1\lambda_2 s^2 + (\lambda_1\mu_2 + \lambda_2\mu_1)st + \mu_1\mu_2 t^2}{(1+\lambda_1 s + \mu_1 t)(1+\lambda_2 s + \mu_2 t)} \\
&= \frac{(\lambda_1 s + \mu_1 t)(1+\lambda_2 s + \mu_2 t)}{(1+\lambda_1 s + \mu_1 t)(1+\lambda_2 s + \mu_2 t)} \\
&= \frac{\lambda_1 s + \mu_1 t}{1+\lambda_1 s + \mu_1 t} \\
&= \frac{\xi \cdot (s,t)}{1+\xi \cdot (s,t)}.
\end{aligned}$$

Similarly, we can prove formula (5.94). □

We distinguish now between three cases:

Case 1. If $\xi \neq 0$ and $\eta \neq 0$, then since they are orthogonal, ξ and η are linearly independent. We can normalize them, defining:

$$\vec{u} = \frac{1}{\parallel \xi \parallel}\xi \tag{5.104}$$

and

$$\vec{v} = \frac{1}{\parallel \eta \parallel}\eta, \tag{5.105}$$

where $\parallel \cdot \parallel$ denotes the Euclidean norm. Formulas (5.93) and (5.94) become now:

$$D_{\vec{u}}\psi(s,t) = \frac{\vec{u}\cdot(s,t)}{1+ \parallel \xi \parallel \vec{u}\cdot(s,t)} \tag{5.106}$$

and

$$D_{\vec{v}}\psi(s,t) = \frac{\vec{v}\cdot(s,t)}{1+ \parallel \eta \parallel \vec{v}\cdot(s,t)}. \tag{5.107}$$

The last two equations can be easily integrated, and if we denote by $P_{\vec{u}}$ and $P_{\vec{v}}$, the orthogonal projections of $\mathbb{R}^2$ onto $\mathbb{R}\vec{u}$ and $\mathbb{R}\vec{v}$, we have

$$\psi(s,t) = \psi_1(P_{\vec{u}}(s,t)) + \psi_2(P_{\vec{v}}(s,t)). \tag{5.108}$$

Since the joint Laplace transform of X and Y satisfies $\varphi(s,\, t) = \exp(\psi(s,\, t))$, we have

$$\varphi(s,t) = \varphi_1(P_{\vec{u}}(s,t))\varphi_2(P_{\vec{v}}(s,t)), \tag{5.109}$$

for all $(s,\, t)$ in a neighborhood $\mathcal{V}$ of $(0,\, 0)$. This shows that up to the orthogonal transformation that maps $(1,\, 0)$ to $\vec{u}$ and $(0,\, 1)$ to $\vec{v}$, the joint Laplace transform of X and Y can be factorized as a product of a function of s and a function of t. This implies that up to an invertible transformation, the random vector $(X,\, Y)$ is moment equivalent to a random vector $(X',\, Y')$, where X' and Y', are independent random variables. Moreover, since $\parallel \xi \parallel \neq 0$ and $\parallel \eta \parallel \neq 0$, we see from (5.106) and (5.107) that the first derivatives of the logarithm of the Laplace transforms of X' and Y' are Möbius transformations, that map 0 into 0. Therefore, X' and Y' are centered and rescaled gamma (non-Gaussian) distributed random variables.

Case 2. If $\xi = (0,\, 0)$ and $\eta \neq (0,\, 0)$, then $\lambda_1 = \mu_1 = 0$. This implies two things:

First, we have

$$\begin{aligned} D_1 &= ad - b^2 \\ &= \lambda_1\lambda_2 \\ &= 0 \end{aligned} \tag{5.110}$$

and

$$\begin{aligned} D_2 &= bf - d^2 \\ &= \mu_1\mu_2 \\ &= 0. \end{aligned}$$

Therefore, we obtain:

$$ad = b^2 \tag{5.111}$$

and

$$bf = d^2. \tag{5.112}$$

Second, we have

$$\begin{aligned} \lambda_2 &= \lambda_1 + \lambda_2 \\ &= -(a+d) \end{aligned} \tag{5.113}$$

and

$$\begin{aligned} \mu_2 &= \mu_1 + \mu_2 \\ &= -(b+f). \end{aligned} \tag{5.114}$$

We divide this case into two subcases:
Subcase 1. If $b \neq 0$, then sice $ad = b^2$, we must have $a \neq 0$ and $d \neq 0$. Since $bf = d^2$, we have $f \neq 0$, too. Multiplying the two equations $ad = b^2$ and $bf = d^2$, since $bd \neq 0$, we obtain:

$$af = bd. \tag{5.115}$$

Since $\xi = (0,\, 0)$, we have

$$\begin{aligned} N &= [1 + \xi \cdot (s,t)]\,[1 + \eta \cdot (s,t)] \\ &= 1 + \eta \cdot (s,t). \end{aligned}$$

Let us define the non-zero vector:

$$\tau = (-b, a). \tag{5.116}$$

Let us observe first that the vectors τ and η are orthogonal, since:

$$\begin{aligned} \tau \cdot \eta &= (-b, a) \cdot (\lambda_2, \mu_2) \\ &= -b\lambda_2 + a\mu_2 \\ &= b(a+d) - a(b+f) \\ &= bd - af \\ &= 0. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} D_\tau \psi(s,t) &= -b\frac{\partial \psi}{\partial s}(s,t) + a\frac{\partial \psi}{\partial t}(s,t) \\ &= -b\frac{s(1 - ds - ft) + t(bs + dt)}{N} + a\frac{t(1 - as - bt) + s(bs + dt)}{N} \\ &= \frac{-bs + at + b(a+d)s^2 + (bf - a^2 + ad - b^2)st - b(a+d)t^2}{1 + \lambda_2 s + \mu_2 t}. \end{aligned}$$

Since $ad - b^2 = 0$, we can replace $ad - b^2$ by $b^2 - ad$ in the coefficient of st, and since $bd = af$, we can replace $b(a+d)$ by $a(b+f)$ as the coefficient of t^2 in the numerator of $D_\tau\psi(s,\, t)$. Thus, we obtain:

$$\begin{aligned} D_\tau\psi(s,t) &= \frac{-bs+at+b(a+d)s^2+(bf-a^2+b^2-ad)st-a(b+f)t^2}{1+\lambda_2 s+\mu_2 t} \\ &= \frac{-bs+at-b\lambda_2 s^2+(-b\mu_2+a\lambda_2)st+a\mu_2 t^2}{1+\lambda_2 s+\mu_2 t} \\ &= \frac{(-bs+at)(1+\lambda_2 s+\mu_2 t)}{1+\lambda_2 s+\mu_2 t} \\ &= -bs+at \\ &= \tau\cdot(s,t). \end{aligned}$$

Therefore, for the two non-zero orthogonal directions τ and η, we have

$$D_\tau\psi(s,t) = \tau\cdot(s,t) \tag{5.117}$$

and

$$D_\eta\psi(s,t) = \frac{\eta\cdot(s,t)}{1+\eta\cdot(s,t)}. \tag{5.118}$$

As before, we can see now that by using the orthogonal transformation mapping (1, 0) to $(1/\parallel\tau\parallel)\tau$ and (0, 1) to $(1/\parallel\xi\parallel)\xi$, that up to an invertible linear transformation the random vector $(X,\, Y)$ is moment equivalent to a pair of independent random variables $(X',\, Y')$, in which X' is Gaussian and Y' is gamma (non-Gaussian) distributed random variable.

Subcase 2. If $b = 0$, then since $bf = d^2$, we have $d = 0$, too. In this case the differential equations (5.54) and (5.55) become:

$$\frac{\partial\psi}{\partial s} = \frac{s}{1-as} \tag{5.119}$$

and

$$\frac{\partial\psi}{\partial t} = \frac{t}{1-ft}. \tag{5.120}$$

It is now clear that $(X,\, Y)$ is moment equivalent to a pair of two independent gamma (possibly Gaussian) distributed random variables $(X',\, Y')$. No application of an invertible linear transformation is needed.

The case when $\eta = (0,\, 0)$ and $\xi \neq (0,\, 0)$ can be done similarly.

Case 3. If $\xi = \eta = (0,\, 0)$, then we have $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$. This implies:

$$a+d = 0, \tag{5.121}$$

$$ad-b^2 = 0, \tag{5.122}$$

$$b+f = 0, \tag{5.123}$$

and

$$bf-d^2 = 0. \tag{5.124}$$

Multiplying relation (5.122) by -4 and adding it to equation (5.121) squared, we obtain:

$$(a-d)^2+4b^2 = 0. \tag{5.125}$$

Therefore, we have $b = 0$ and $a = d$.

Similarly, multiplying relation (5.124) by -4 and adding it to equation (5.123) squared, we get: $d = 0$ and $f = b$.

Thus, we have $a = b = d = f = 0$.

Therefore, in this case the partial differential equations (5.54) and (5.55) become:

$$\frac{\partial \psi}{\partial s}(s, t) = s \tag{5.126}$$

and

$$\frac{\partial \psi}{\partial t}(s, t) = t. \tag{5.127}$$

Since $\psi(0, 0) = 0$, integrating the last two partial differential equations, we obtain:

$$\psi(s, t) = \frac{1}{2}s^2 + \frac{1}{2}t^2. \tag{5.128}$$

Thus, we have

$$\begin{aligned} \varphi(s, t) &= \exp\left(\psi(s, t)\right) \\ &= \exp\left(s^2/2\right) \cdot \exp\left(t^2/2\right), \end{aligned} \tag{5.129}$$

and so (X, Y) is equivalent to a pair of two independent Gaussian random variables.

To summarize our discussion, we obtain the following characterization of the non-degenerate, two-dimensional 1-Meixner random vectors:

Theorem 5.15. *A two dimensional random vector (X, Y) is a non-degenerate,* 1-*Meixner random vector if and only if there exist an invertible linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ and a vector $(w_1, w_2) \in \mathbb{R}^2$, such that the random vector:*

$$(X', Y') = A(X, Y) + (w_1, w_2) \tag{5.130}$$

is moment equivalent to a pair of two independent rescaled and shifted gamma (including Gaussian) distributed random variables.

Equivalently, up to the moment equal equivalence relation, every non-degenerate, two-dimensional 1-*Meixner random vector is obtained through an invertible affine transformation applied to a random vector of two independent, non-degenerate, gamma (including Gaussian) distributed random variables.*

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