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SOLVING A CLASS OF LINEAR SKOROKHOD STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. Solving stochastic differential equations with respect to the Skorokhod integral, i.e. without adaptedness assumptions, on Wiener space is usually rather difficult. On the isomorphic Fock space, the Skorokhod integral has an elementary form.
Using this fact we will determine the solutions of a class of linear stochastic differential equations and illustrate the general solution by a few examples. We show one example for an explosion: the solution leaves Wiener space, but is still a well-defined stochastic process. This calls for a more flexible definition of the Skorokhod integral on Wiener space.

1. Introduction

Let $(B_t)_{t\geq 0}$ be a Brownian motion, $f:\mathbb{R}_+\mapsto\mathbb{R}$ a square integrable deterministic function and $h=(h(t))_{t\geq 0}$ a square integrable stochastic process. Then the following stochastic processes are well-defined:

$$\begin{aligned}
Z_t^1 &= \mathsf{e}^{\int_0^t \mathsf{d}B_s\, h(s)-\frac{1}{2}\|h\|^2},\\
Z_t^2 &= \left(\int_0^\infty \mathsf{d}B_s\, f(s)-\int_0^t \mathsf{d}s\, f(s)h(s)\right)\mathsf{e}^{\int_0^t \mathsf{d}B_s\, h(s)-\frac{1}{2}\|h\|^2},\\
Z_t^3 &= \mathsf{e}^{\int_0^\infty \mathsf{d}B_s\, f(s)+\int_0^t \mathsf{d}B_s\, h(s)-\frac{1}{2}\|h\mathbf{1}_{[0,t)}+f\|^2}.
\end{aligned}$$

At first sight, these processes look completely different and it is not clear which properties these processes have in common. In this work we want to show that all three processes solve the same stochastic differential equation (SDE)

$$\mathsf{d}Z_t = h_t * Z_t\, \mathsf{d}B_t. \tag{1.1}$$

Here $h_t * Z_t$ denotes the Fock product on Wiener space which includes the usual product with a deterministic h (where all h_t are constant random variables). The different processes arise from different initial conditions, namely

$$\begin{aligned}
Z_0^1 &\equiv 1,\\
Z_0^2 &= \int_0^\infty \mathsf{d}B_s\, f(s),\\
Z_0^3 &= \mathsf{e}^{\int_0^\infty \mathsf{d}B_s\, f(s)-\frac{1}{2}\|h\|^2}.
\end{aligned}$$

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Since Z_0^1 is deterministic, the process $(Z_t^1)_{t\geq 0}$ is adapted to the Brownian filtration. The associated SDE can thus be interpreted and solved by applying the Itô theory of stochastic integration.

On the other hand, already the initial conditions of the processes $(Z_t^2)_{t\geq 0}$ and $(Z_t^3)_{t\geq 0}$ are not adapted to the Brownian filtration and we cannot use Itô integrals any more. Therefore, we need a generalisation of the concept of Itô integration. In this work we use the Skorokhod integral.

The definition and application of the Skorokhod integral on Wiener space is difficult (cf. [3]). But as shown in [8, 11, 2], the Skorokhod integral has an especially simple representation on Fock space. In [9] this representation was used to solve the SDE (1.1) for deterministic h and general initial conditions by elementary calculations.

The aim of this paper is to apply this method to the more general linear SDE

$$\mathrm{d}Z_t = h_t * Z_t\,\mathrm{d}B_t + m_t * Z_t\,\mathrm{d}t, \tag{1.2}$$

where m,h are certain stochastic processes. To use the same method, it is essential to use the Fock product instead of multiplication of random variables. We will translate this SDE into Fock space and present the solution of the generalised version (1.2) in a straightforward way. By some examples we will demonstrate the potential of this method. In particular, we can handle integrands which are not square integrable.

Note that for the linear SDE (1.2) existence and uniqueness was already established in [4], with ordinary product, under some boundedness assumptions on the coefficients. Unfortunately, it seems impossible to translate boundedness from Wiener space to Fock space directly.

In Section 1 we will introduce Wiener and Fock space, and two operators, the Malliavin derivative and the Skorokhod integral. Section 2 contains our translation of equation (1.2) into Fock space. Further, we introduce the Fock space solution of a generalised version of this equation. In Section 3 we compute some processes from this solution for different initial conditions. One example shows an explosion indicating that the Skorokhod integral should be extended beyond the scheme of square integrable processes. Finally, in Section 4, we will prove Theorem 3.3.

2. Basic Notions

We first introduce the relevant spaces and operators. The formulations are adopted from [9]. Throughout the paper all spaces and functions are real valued. We denote by $\mathbb{N}$ the nonnegative integers and by $\mathbb{R}$ the real numbers.

2.1. The Wiener space. Let $\mathbb{R}_+$ be the nonnegative real numbers, $\mathfrak{B}$ the σ-algebra of Borel sets, and ℓ Lebesgue measure over $\mathbb{R}_+$. Moreover let $[\Omega,\mathscr{F},P]$ be a probability space and $(B_t)_{t\geq 0}$ a *Brownian motion* on Ω with reference measure ℓ. With this, we construct the *stochastic integral* $W = \big(W(f)\big)_{f\in L^2(\ell)}$ of deterministic functions by $W(f) = \int_0^\infty \mathrm{d}B_s\, f(s)$.

We write $\mathscr{F}_W$ for the σ-algebra generated by W over Ω and P_W for the restriction $P|\mathscr{F}_W$. The elements of the Wiener space $L^2(P_W) = L^2(\Omega,\mathscr{F}_W,P_W)$ are called square integrable functionals of $(B_t)_{t\geq 0}$.

Now we follow [13] and define two operators: the *Malliavin derivative* D and the *Skorokhod integral* δ. The linear unbounded operators $\mathrm{D} : Dom(\mathrm{D}) \to L^2(\ell\otimes P_W)$ and δ :

$Dom(\delta) \to L^2(P_W)$ both have dense domains $Dom(\mathrm{D}) \subset L^2(P_W)$ and $Dom(\delta) \subset L^2(\ell \otimes P_W)$. The Skorokhod integral δ is the adjoint operator to the Malliavin derivative D (one could take this as a definition of δ instead of the explicit one as in [13]). Since it is not needed, we do not repeat the definition here. Some useful characterisations of D and δ are presented below. For $Z \in Dom(\delta)$ we write $\delta(Z) = \int \mathsf{d}B_x Z_x$ interpreting δ as a stochastic integral w.r.t. Brownian motion. For a finite interval $[0,t)$ we will write $\delta(f_{t)}) = \int_0^t \mathsf{d}B_s f_s$ using the notation $f_{t)} = f\mathbf{1}_{[0,t)}$.

2.2. The symmetric Fock space. We define now the space $\mathscr{M}$ as an L^2-space over the set of finite point configurations. By the Wiener chaos decomposition, $\mathscr{M}$ is naturally isomorphic to the Wiener space $L^2(P_W)$. Details of this approach can be found e.g. in [8, 6].

We define the set

$$M = \left\{\varphi :\ \varphi \text{ is a measure on } [\mathbb{R}_+, \mathfrak{B}] \text{ s.t. } \varphi(A) \in \mathbb{N} \text{ for every } A \in \mathfrak{B}\right\}$$

of all finite counting measures on $[\mathbb{R}_+, \mathfrak{B}]$. Note that we can write every $\varphi \in M$ in the form $\varphi = \sum_{j=1}^n \delta_{x_j}$ [12].

We denote by define $\mathfrak{M}$ the smallest σ-algebra over M such that for each $A \in \mathfrak{B}$ the map $\varphi \mapsto \varphi(A)$ is measurable. Now we introduce the exponential measure F on $[M, \mathfrak{M}]$ (see also [5]) by setting for $B \in \mathfrak{M}$:

$$F(B) := \mathbf{1}_B(\mathfrak{o}) + \sum_{n=1}^{\infty} \frac{1}{n!} \int \ell^n(\mathsf{d}[x_1, \ldots, x_n])\, \mathbf{1}_B\left(\textstyle\sum_{j=1}^n \delta_{x_j}\right).$$

Here $\mathfrak{o} \in M$ denotes the empty point configuration, i.e. $\mathfrak{o}(A) = 0$ for every $A \in \mathfrak{B}$. We call $\mathscr{M} = L^2(F)$ the *Fock space* over M.

The following class of functions proves useful.

Definition 2.1. For every function $g : \mathbb{R}_+ \to \mathbb{R}$ we define the *coherent function* $\Psi_g : M \to \mathbb{R}$ by

$$\Psi_g(\mathfrak{o}) = 1, \quad \Psi_g(\textstyle\sum_{j=1}^n \delta_{x_j}) = \displaystyle\prod_{j=1}^n g(x_j), \tag{2.1}$$

where $x_1, \ldots, x_n \in \mathbb{R}_+$, $n \geq 1$.

Note that $\Psi_g \in \mathscr{M}$ if and only if $g \in L^2(\ell)$. In that case we can compute the norm of Ψ_g by

$$\|\Psi_g\|^2_{\mathscr{M}} = \mathsf{e}^{\|g\|^2}.$$

With the help of coherent functions one can characterise an isomorphism $\mathscr{U}$ from $\mathscr{M}$ onto $L^2(P_W)$:

Proposition 2.2 (Proposition 2.3.2, Theorem 2.4.1 in [8])**.**

(i) *The set* $\{\Psi_g : g \in L^2(\ell)\}$ *of coherent functions is total in* $\mathscr{M}$.
(ii) *There is exactly one unitary operator* $\mathscr{U} : \mathscr{M} \mapsto L^2(P_W)$ *fulfilling*

$$\mathscr{U}\Psi_g = \mathsf{e}^{W(g) - \frac{1}{2}\|g\|^2} \quad (g \in L^2(\ell)).$$

This characterisation of the isomorphism $\mathscr{U}$ is sufficient for this work. Details of the construction of the isomorphism $\mathscr{U}$ are given in [8].

2.3. Operators on $\mathscr{M}$. The Malliavin derivative D and the Skorokhod integral δ correspond to operators $\mathscr{D}$ and $\mathscr{S}$ via the isomorphism $\mathscr{U}$. Let $\mathrm{I} = \mathrm{I}_{\mathrm{L}^2(\ell)}$ denote the identity operator on $L^2(\ell)$. Then we set

$$\mathscr{D} = (\mathrm{I} \otimes \mathscr{U}^{-1})\mathrm{D}\mathscr{U}. \tag{2.2}$$

Since δ is adjoint to D we also define

$$\mathscr{S} = \mathscr{U}^{-1}\delta(\mathrm{I} \otimes \mathscr{U}). \tag{2.3}$$

Observe that both definitions include the domain of definition.

We cite from [8, 6] the simple definition of boths operators on Fock space.

Definition 2.3. The linear unbounded operators $\mathscr{D} : Dom(\mathscr{D}) \to L^2(\ell \otimes F)$ and $\mathscr{S} : Dom(\mathscr{S}) \to \mathscr{M}$ are given by

$$\begin{aligned} Dom(\mathscr{D}) =& \left\{ \Phi \in \mathscr{M} : \int F(\mathrm{d}\varphi)\varphi(\mathbb{R}_+)|\Phi(\varphi)|^2 < \infty \right\}, \\ \mathscr{D}\Phi(x,\varphi) :=& \Phi(\varphi + \delta_x) \quad \big(x \geq 0, \varphi \in M, \Phi \in Dom(\mathscr{D})\big), \end{aligned}$$

and

$$\begin{aligned} Dom(\mathscr{S}) =& \left\{ u \in L^2(\ell \otimes F) : \int F(\mathrm{d}\varphi) \Big| \int \varphi(\mathrm{d}x)\, u(x, \varphi - \delta_x) \Big|^2 < \infty \right\}, \\ \mathscr{S}U(\varphi) :=& \int \varphi(\mathrm{d}x)\, U(x, \varphi - \delta_x) \quad \big(\varphi \in M, U \in Dom(\mathscr{S})\big). \end{aligned}$$

These operators are densely and maximally defined, closed and mutually adjoint.

Using $\mathscr{U}$ again, we define the *Fock product* $*$ of two random variables X, Y in Wiener space by $X * Y = \mathscr{U}(\mathscr{U}^{-1}X\mathscr{U}^{-1}Y)$ provided $\mathscr{U}^{-1}X\mathscr{U}^{-1}Y \in \mathscr{M}$.

Remark 2.4. Since $cX = X * (\mathscr{U}c) = \mathscr{U}(c\mathscr{U}^{-1}X)$ for $c \in \mathbb{R}$, we can realise scalar multiplication with c as Fock multiplication with $\mathscr{U}c$. Strictly speaking, the constant function c is in $\mathscr{M}$ only if we consider a bounded interval $[0,T]$ instead of $\mathbb{R}_+$. When we solve the stochastic differential equation (1.1) we can achieve this by stopping integration at an arbitrary time $T > 0$. Then $\mathscr{U}c = c\mathrm{e}^{W(\mathbf{1}_{[0,T]}) - T/2}$ by Proposition 2.2.

3. Formulation of Equation and Solution

We denote by $\Lambda_W(f_{t)}) = \int_0^t \mathrm{d}s\, f_s$ the Lebesgue integral of mappings $f \in L^2(\ell \otimes P_W)$. With this notation, the formal equation (1.2) is interpreted as

$$Z_t = Z_0 + \delta(h_{t)} * Z) + \Lambda_W(m_{t)} * Z) \quad (t \geq 0).$$

By application of (2.3) and using $\Lambda_F(g)(\varphi) = \int \mathrm{d}s\, g(s,\varphi)$, $g \in L^2(\ell \otimes F)$, $\varphi \in M$ we derive the isomorphic version on Fock space as

$$X_t = X_0 + \mathscr{S}(h_{t)} * X) + \Lambda_F(m_{t)} * X). \tag{3.1}$$

Since we are mainly interested in the solution on Fock space, we introduce a new class of solutions to (3.1). In the sequel, a process on M is a measurable mapping $X : \mathbb{R}_+ \times M \to \mathbb{R}$.

Sometimes we identify it with the family $(X_t)_{t\geq 0}$, $X_t : M \to \mathbb{R}$, $X_t(\varphi) = X(t,\varphi)$. A process $g : \mathbb{R}_+ \times M \to \mathbb{R}$ is called *locally integrable*, if

$$\int_0^t \mathrm{d}s |g(s,\varphi)| < \infty \quad (t \in \mathbb{R}_+, \varphi \in M).$$

Definition 3.1. Let $h,m : \mathbb{R}_+ \times M \to \mathbb{R}$ be processes on M. A process $X = (X_t)_{t\geq 0}$ is called *pointwise solution* of equation (3.1) if both hX and mX are locally integrable and for all $\varphi \in M$ the following identity holds for all $t \geq 0$

$$X_t(\varphi) = X_0(\varphi) + \int_0^t \varphi(\mathrm{d}s)\, h_s(\varphi - \delta_s) X_s(\varphi - \delta_s) + \int_0^t \mathrm{d}s\, m_s(\varphi) X_s(\varphi). \tag{3.2}$$

A pointwise solution $(X_t)_{t\geq 0}$ is called *unique* if $X_t(\varphi) = Y_t(\varphi)$ for all pointwise solutions $(Y_t)_{t\geq 0}$ of (3.1) and all $t \geq 0$, $\varphi \in M$.

Remark 3.2. We want to emphasize two important points. Firstly, for all $t \geq 0$ equation (3.1) yields equation (3.2) for F-a.a. $\varphi \in M$. One can derive then existence of modifications of X for which (3.2) is true for F-a.a. $\varphi \in M$ for all $t \geq 0$. See [10] for the solution of a similar problem. Since this step is rather technical, we prefer the above simpler notion for the sake of clarity.

Secondly, note that we do not need square integrability of the processes to define a solution of (3.1) on Fock space. Of course, we would need it for the transfer to Wiener space.

To describe the solution, we need some further notation. For $t \geq 0$ and $\varphi \in M$ we denote by $\varphi_{t)}$ the *restriction* of φ to $[0,t)$, i.e. $\varphi_{t)}(A) = \varphi([0,t) \cap A)$ for all $A \in \mathfrak{B}$. A measure $\widehat{\varphi} \in M$ is a *subconfiguration* of $\varphi \in M$ if $\widehat{\varphi}(A) \leq \varphi(A)$ for every $A \in \mathfrak{B}$. We express this fact by $\widehat{\varphi} \leq \varphi$.

For $t \geq 0$ we define a mapping $K_t : M \times M \to \mathbb{R}$ by

$$K_t(\varphi,\psi) = \begin{cases} \mathrm{e}^{\int_0^t \mathrm{d}s\, m(s,\psi+\varphi_{s)})} \prod_{\delta_x \geq \varphi} h_x(\psi + \varphi_{x)}), & x_\varphi \leq t \\ 0, & x_\varphi > t, \end{cases}$$

where $\psi, \varphi \in M$ and $x_\varphi := \max\{x : \delta_x \leq \varphi\}$. With these notations we can state the following

Theorem 3.3. *For measurable functions $h : \mathbb{R}_+ \times M \to \mathbb{R}$, $X_0 : M \to \mathbb{R}$ and locally integrable mappings $m : \mathbb{R}_+ \times M \to \mathbb{R}$ there exists a unique pointwise solution of (3.1). It is given by*

$$X_t(\varphi) = \sum_{\widehat{\varphi} \leq \varphi_{t)}} X_0(\varphi - \widehat{\varphi}) K_t(\widehat{\varphi}, \varphi - \widehat{\varphi}) \quad (t \geq 0, \varphi \in M). \tag{3.3}$$

4. Examples

We now discuss some examples with different coefficients to demonstrate the potential of our method. The following properties of coherent functions are straight forward:

Lemma 4.1 (Lemma 2.2 in [7]). *Let* $f, g : \mathbb{R}_+ \to \mathbb{R}$ *and* $\varphi, \varphi_1, \varphi_2 \in M$. *Then*

$$\Psi_f(\varphi_1 + \varphi_2) = \Psi_f(\varphi_1)\Psi_f(\varphi_2), \tag{4.1}$$

$$\Psi_{fg}(\varphi) = \Psi_f(\varphi)\Psi_g(\varphi), \tag{4.2}$$

$$\Psi_{f+g}(\varphi) = \sum_{\widehat{\varphi} \le \varphi} \Psi_f(\widehat{\varphi})\Psi_g(\varphi - \widehat{\varphi}). \tag{4.3}$$

It is easy to check that $\Psi_g \in Dom(\mathscr{D})$ for each $g \in L^2(\ell)$ and

$$\mathscr{D}\Psi_g = g \otimes \Psi_g \tag{4.4}$$

such that $\mathscr{D}$ is characterised by its restriction to coherent functions.

To compute the explicit form of some of the solutions on Wiener space, we need the following facts (cf. [13], Prop. 3.5). If $\xi \in Dom(\mathrm{D})$ and $f \in L^2(\ell)$ then $f \otimes \xi \in Dom(\delta)$ and it holds:

$$\delta(f \otimes \mathbf{1}) = W(f) \tag{4.5}$$

$$\delta(f \otimes \xi) = W(f)\xi - \langle f, \mathrm{D}\xi \rangle_{L^2(\ell)}. \tag{4.6}$$

Adapted processes. We consider deterministic coefficients. In particular, let $X_0 = c\,\mathbf{1}_{\{\mathfrak{o}\}}$ and $m, h \in L^2(\ell)$. Applying (3.3) gives us for $t \ge 0$ and $\varphi \in M$

$$X_t(\varphi) = c\,\mathrm{e}^{\int_0^t \mathrm{d}s\, m(s)} \sum_{\widehat{\varphi} \le \varphi_{t)}} \mathbf{1}_{\{\mathfrak{o}\}}(\varphi - \widehat{\varphi})\Psi_h(\widehat{\varphi}) = c\,\mathrm{e}^{\int_0^t \mathrm{d}s\, m(s)} \Psi_{h\mathbf{1}_{[0,t)}}(\varphi).$$

Using Lemma 2.2 we can transform X_t to Wiener space. Then $Z_t = \mathscr{U} X_t$ can be written as

$$Z_t = \mathscr{U} X_t = c\,\mathrm{e}^{\int_0^t \mathrm{d}s\, m(s)} \mathrm{e}^{\int_0^t \mathrm{d}B_s\, h(s) - \frac{1}{2}\int_0^t \mathrm{d}s\, |h(s)|^2} \quad (t \ge 0).$$

Since the Fock product realises the multiplication with deterministic random variables we see that $(Z_t)_{t \ge 0}$ is also the unique solution of the SDE

$$\mathrm{d}Z_t = h_t Z_t\, \mathrm{d}B_t + m_t Z_t\, \mathrm{d}t, \quad Z_0 = c \quad (t \ge 0).$$

The solution $(Z_t)_{t \ge 0}$ is even adapted. The well-known geometric Brownian motion is the special case where h, m are constants.

Nondeterministic initial condition. Now we treat an example that was discussed in [9] and motivated this work. Let $h \in L^2(\ell)$, $m \equiv 0$ and

$$X_0(\varphi) = \begin{cases} f(x), & \varphi = \delta_x \\ 0, & \text{otherwise} \end{cases} = \mathscr{S}(f \otimes \mathbf{1}_{\{\mathfrak{o}\}})(\varphi).$$

Then (3.3) yields the solution

$$X_t(\varphi) = \sum_{\delta_s \le \varphi_{t)}} f(s)\Psi_h(\varphi - \delta_s) = \mathscr{S}(f \otimes \Psi_{h\mathbf{1}_{[0,t)}})(\varphi) \quad (t \ge 0, \varphi \in M).$$

Applying Lemma 2.2, (4.6) and (4.4) we can provide a version of $(X_t)_{t \ge 0}$ on Wiener space by

$$Z_t = \mathscr{U} X_t = \Big(\int_0^\infty \mathrm{d}B_s\, f(s) - \int_0^t \mathrm{d}s\, f(s)h(s)\Big) \mathrm{e}^{\int_0^t \mathrm{d}B_s\, h(s) - \frac{1}{2}\int_0^t \mathrm{d}s |h(s)|^2} \quad (t \ge 0).$$

The process $(Z_t)_{t\geq 0}$, which is the process $(Z_t^2)_{t\geq 0}$ from the introduction, is the unique solution of the following SDE on Wiener space

$$\mathrm{d}Z_t = h_t * Z_t \,\mathrm{d}B_t, \quad Z_0 = W(f) \quad (t \geq 0)$$

interpreted with Skorokhod differential.

Nondeterministic drift. Now we regard the following coefficients: Let $X_0 = \Psi_f$, $f, h \in L^2(\ell)$ and

$$m_s(\varphi) = g(s)|\varphi|, \quad g \in L^2(\ell).$$

Application of (3.3), (4.2) and (4.3) leads to

$$X_t(\varphi) = \sum_{\widehat{\varphi}\leq\varphi_{t)}} \Psi_f(\varphi-\widehat{\varphi})\Psi_h(\widehat{\varphi})\Psi_{\rho_t}(\widehat{\varphi})\Psi_{\rho_t(0)}(\varphi-\widehat{\varphi}) = \Psi_{\rho_t(0)f+\rho_t h\mathbf{1}_{[0,t)}}(\varphi),$$

where $\rho_t(x) = \mathrm{e}^{\int_x^t \mathrm{d}s\, g(s)}$. With Lemma 2.2 we get the version of $(X_t)_{t\geq 0}$ on Wiener space by

$$Z_t = \mathscr{U}X_t = \mathrm{e}^{\rho_t(0)\int_{\mathbb{R}_+}\mathrm{d}B_s f(s) + \int_0^t \mathrm{d}B_s h(s)\rho_t(s) - \frac{1}{2}\|\rho_t h \mathbf{1}_{[0,t)} + \rho_t(0) f\|^2} \quad (t \geq 0).$$

This process $(Z_t)_{t\geq 0}$ is the unique solution of

$$\mathrm{d}Z_t = h_t Z_t \,\mathrm{d}B_t + \gamma g_t Z_t \,\mathrm{d}t, \quad Z_0 = \mathrm{e}^{\int_{\mathbb{R}_+}\mathrm{d}B_s f(s) - \frac{1}{2}\|f\|^2} \quad (t \geq 0)$$

where γ denotes the usual Malliavin operator as introduced in [13]. In [3] a slightly more general example was discussed on Wiener space. If m vanishes, we get the process $\left(Z_t^3\right)_{t\geq 0}$ from the introduction.

Nondeterministic noise strength. Finally, let us take a look at the following example: $X_0 = \mathbf{1}_{\{\mathfrak{o}\}}$, $m \equiv 0$ and

$$h_t(\varphi) = \frac{c(|\varphi|+1)}{c(|\varphi|)}, \quad (t \geq 0) \tag{4.7}$$

for some function $c : \mathbb{N} \to (0, \infty)$ with $c(0) = 1$. This means that the strength of the noise depends on the number of particles in the configuration $\varphi \in M$ only. Now we obtain for our solution from (3.3)

$$X_t(\varphi) = \sum_{\widehat{\varphi}\leq\varphi_{t)}} \mathbf{1}_{\{\mathfrak{o}\}}(\varphi-\widehat{\varphi})\frac{c(|\varphi|)}{c(|\varphi-\widehat{\varphi}|)} = c(|\varphi|)\Psi_{\mathbf{1}_{[0,t)}}(\varphi) \quad (t \geq 0, \varphi \in M). \tag{4.8}$$

It is not clear whether we are able to find a closed form of $(X_t)_{t\geq 0}$ on Wiener space. In the special case

$$c(n) = \frac{n!}{\lfloor \frac{n}{2} \rfloor !}(2\alpha)^{\frac{n}{2}} \quad (n \in \mathbb{N}) \tag{4.9}$$

we can do so.

Lemma 4.2. *Fix $\alpha > 0$, $t \geq 0$. If h and c are given by (4.7) and (4.9) respectively then the pointwise solution X_t from (4.8) fulfils $X_t \in \mathscr{M}$ if and only if $t < (4\alpha)^{-1}$.*

Proof. By the definition of F, we obtain

$$\int |X_t(\varphi)|^2 F(\mathrm{d}\varphi) = \sum_{n=0}^{\infty} \frac{(t\alpha)^n}{\lfloor \frac{n}{2} \rfloor !^2} 2^n n! < \infty.$$

Stirlings formula shows that the series converges if and only if $4\alpha t < 1$. □

If $X_t \in \mathscr{M}$ we can apply the isomorphism $\mathscr{U}$ to (4.8). Using Proposition 2.2(ii) in the computation of $\mathscr{U}\Psi_{\tau\frac{1}{\sqrt{t}}\mathbf{1}_{[0,t]}}$ for all $\tau \in \mathbb{R}$ we arrive by equating coefficients of power series at

$$\mathscr{U}\left(\mathbf{1}_{M_n}\Psi_{\mathbf{1}_{[0,t]}}\right) = t^{n/2} n!\, H^{(n)}\left(\tfrac{B_t}{\sqrt{t}}\right)$$

where $M_n = \{\varphi \in M : \varphi(\mathbb{R}_+) = n\}$ denote the measurable set of all configurations with precisely n points and $H^{(n)}$ denotes the Hermite polynomial of degree n. This yields the following identity in Wiener space

$$Z_t = \mathscr{U} X_t = \sum_{n=0}^{\infty} \frac{(t\alpha)^{\frac{n}{2}}}{\lfloor \frac{n}{2} \rfloor !} 2^{\frac{n}{2}}\, n!\, H^{(n)}\left(\tfrac{B_t}{\sqrt{t}}\right).$$

By the properties of the Wiener chaos decomposition, this series convergences in L^2 and thus in probability provided that Z_t is square integrable. Now we use the following identity for Hermite polynomials (cf. [8] and p.340 in [1])

$$\sum_{n=0}^{\infty} \frac{n! 2^{\frac{n}{2}} H^{(n)}(x)}{\lfloor \frac{n}{2} \rfloor !} r^n = \frac{1 + 2xr + 4r^2}{(1+4r^2)^{\frac{3}{2}}}\, \mathrm{e}^{\frac{4x^2 r^2}{1+4r^2}} \tag{4.10}$$

to compute

$$Z_t = \left((1+4\alpha t)^{-\frac{1}{2}} + (1+4\alpha t)^{-\frac{3}{2}} B_t\right) \mathrm{e}^{\frac{4\alpha}{1+4\alpha t} B_t^2}.$$

Since (4.10) holds pointwise, we can consider the process $(Z_t)_{t \geq 0}$ as solution of the Skorokhod SDE (3.1). But, in order to give equation (3.1) a rigorous sense the usual definition of the Skorokhod integral has to be extended beyond square integrability first. Still, we are not aware of any attempts in this direction.

5. Proof of Theorem 3.3

We need the following result.

Lemma 5.1. *Let $\varphi \in M$. For $f : M \times M \to \mathbb{R}$ we have*

$$\int \varphi(\mathrm{d}s) \sum_{\widehat{\varphi} \leq \varphi - \delta_s} f(\widehat{\varphi}, \varphi - \delta_s - \widehat{\varphi}) = \sum_{\widehat{\varphi} \leq \varphi} \int \widehat{\varphi}(\mathrm{d}s)\, f(\widehat{\varphi} - \delta_s, \varphi - \widehat{\varphi}). \tag{5.1}$$

Proof. Let $n \in \mathbb{N}$ be arbitrary. For $\varphi \in M_n$ equation (5.1) attains the following form (for a index set A we write δ_A instead of $\sum_{j \in A} \delta_{x_j}$)

$$\sum_{j=1}^{n} \sum_{A \subseteq \{1,\dots,n\} \setminus \{j\}} f(\delta_A, \delta_{A^c \setminus \{j\}}) = \sum_{\widetilde{A} \subseteq \{1,\dots,n\}} \sum_{j \in \widetilde{A}} f(\delta_{\widetilde{A} \setminus \{j\}}, \delta_{\widetilde{A}^c},) \tag{5.2}$$

where $\widetilde{A} := A \cup \{j\}$. But with this the RHS of (5.1) and the RHS of (5.2) are equal for all $\varphi \in M_n$ and we have proved Lemma 5.1. □

Proof of Theorem 3.3: For existence of a solution of (3.1) it is sufficient to verify for the process X from (3.3) the following:

$$X_t(\varphi) = X_0(\varphi) + \int_0^t \varphi(\mathrm{d}s)\, h(s,\varphi-\delta_s) \sum_{\widehat{\varphi}\le\varphi-\delta_s} X_0(\varphi-\delta_s-\widehat{\varphi}) K_t(\widehat{\varphi},\varphi-\delta_s-\widehat{\varphi}) + \int_0^t \mathrm{d}s\, m(s,\varphi) \sum_{\widehat{\varphi}\le\varphi} X_0(\varphi-\widehat{\varphi}) K_s(\widehat{\varphi},\varphi-\widehat{\varphi}) \quad (t\ge 0, \varphi\in M). \tag{5.3}$$

We denote the RHS of (5.3) by $G_t(\varphi)$. Applying Lemma 5.1 and using linearity of the Lebesgue integral we can write for $G_t(\varphi)$:

$$G_t(\varphi) = X_0(\varphi) + \sum_{\widehat{\varphi}\le\varphi} X_0(\varphi-\widehat{\varphi}) \Bigg[\int_0^t \widehat{\varphi}(\mathrm{d}s)\, h(s,\varphi-\delta_s) K_s(\widehat{\varphi}-\delta_s,\varphi-\widehat{\varphi}) + \int_0^t \mathrm{d}s\, m(s,\varphi) K_s(\widehat{\varphi},\varphi-\widehat{\varphi}) \Bigg] \quad (t\ge 0, \varphi\in M).$$

Thus it is enough to show that, abbreviating $\widetilde{\varphi} = \varphi - \widehat{\varphi}$,

$$\int_0^t \widehat{\varphi}(\mathrm{d}s)\, h(s,\varphi-\delta_s) K_s(\widehat{\varphi}-\delta_s,\widetilde{\varphi}) + \int_0^t \mathrm{d}s\, m(s,\varphi) K_s(\widehat{\varphi},\widetilde{\varphi}) = K_t(\widehat{\varphi},\widetilde{\varphi}) - \mathbf{1}_{\{\mathfrak{o}\}}(\widehat{\varphi}) \quad (t\ge 0, \varphi\in M). \tag{5.4}$$

Since $K_s(\varphi,\psi) = 0$ for $s < x_\varphi$ we get for the first integral on the LHS in (5.4)

$$\int_0^t \widehat{\varphi}(\mathrm{d}s) h(s,\varphi-\delta_s) K_s(\widehat{\varphi}-\delta_s,\varphi-\widehat{\varphi}) = \mathbf{1}_{M\setminus\{\mathfrak{o}\}}(\widehat{\varphi}) \prod_{\delta_s\le\widehat{\varphi}} h(s,\widetilde{\varphi}+\widehat{\varphi}_{s)}) \mathrm{e}^{\int_0^{x_{\widehat{\varphi}}} \mathrm{d}s\, m(s,\widetilde{\varphi}+\widehat{\varphi}_{s)})}. \tag{5.5}$$

The second integral on the LHS of (5.4) computes by the change-of-variable formula to

$$\int_0^t \mathrm{d}s\, m(s,\varphi) K_s(\widehat{\varphi},\widetilde{\varphi}) = \prod_{\delta_s\le\widehat{\varphi}} h(s,\widetilde{\varphi}+\widehat{\varphi}_{s)}) \left[\mathrm{e}^{\int_0^t \mathrm{d}s\, m(s,\widetilde{\varphi}+\widehat{\varphi}_{s)})} - \mathrm{e}^{\int_0^{x_{\widehat{\varphi}}} \mathrm{d}s\, m(s,\widetilde{\varphi}+\widehat{\varphi}_{s)})} \right]. \tag{5.6}$$

Adding the RHS's of (5.5) and (5.6) we obtain (5.4). Therefore we have shown that (3.3) is indeed a solution of (3.1).

For proving uniqueness we assume that there exists another pointwise solution $(Y_t)_{t\ge0}$ of (3.1) apart from $(G_t)_{t\ge0}$. Then the difference $Z_t = G_t - Y_t$ fulfils $Z_0 = 0$ and for all $\varphi\in M$ the equation

$$Z_t(\varphi) = \int_0^t \varphi(\mathrm{d}s)\, h(s,\varphi-\delta_s) Z_s(\varphi-\delta_s) + \int_0^t \mathrm{d}s\, m(s,\varphi) Z_s(\varphi) \quad (t\ge 0). \tag{5.7}$$

We assume further $Z \neq 0$. Then there exists a minimal $n \geq 0$ such that $Z_t(\varphi) \neq 0$ for some $\varphi \in M_n$ and for some $t > 0$. By minimality of n, the first integral on the RHS of (5.7) vanishes. Then (5.7) leads to

$$Z_t(\varphi) = \int_0^t \mathrm{d}s\, m(s,\varphi) Z_s(\varphi) \quad (t \geq 0, \varphi \in M_n).$$

Gronwalls lemma readily implies from $Z_0 = 0$ that $Z_t(\varphi) = 0$ for all $t \geq 0$ and $\varphi \in M_n$ contrary to our assumption. Therefore we have only one solution and the proof is complete. □

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