

LARGE DEVIATIONS FOR STOCHASTIC TIDAL DYNAMICS EQUATION

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ABSTRACT. In this work, we study the large deviation principle of Wentzell-Freidlin type for the stochastic tidal dynamics equation with multiplicative noise in an open domain. The results are established by using a generalization of the Minty Browder method and also exploiting an inherent control theoretic structure of large deviation theory.

1. Introduction

Large deviation theory finds its application in many areas of applied probability theory and its importance has been well established ([15], [16], [24], [40], [43]). Several authors have studied large deviations for stochastic partial differential equations (SPDE). One of the initial works was carried out by Faris et al. [20] on heat equation with Gaussian type randomness. Sowers [38] studied large deviation property of reaction-diffusion equation with non-Gaussian perturbation. A large deviation principle (LDP) for the two dimensional Navier-Stokes equation with additive noise was established by Chang [9] whereas Cardon-Weber [7] considered a Burger's type SPDE. The LDP for stochastic reaction diffusion equations with non-Lipschitz reaction term was established by Cerrai and Rockner [8]. The Wentzell-Freidlin type large deviation principle for the two dimensional stochastic Navier-Stokes equations with multiplicative noise was studied by Sritharan and Sundar [39] and a Donsker-Varadhan type result was proved by Gourcy [23]. Budhiraja et al. considered infinite dimensional stochastic models and established the large deviation principle in [6]. The LDP for stochastic shell model of turbulence was established by Manna et al. [28] whereas an inviscid shell model was studied by Bessaih and Millet [2]. Swiech [42] studied large deviations in Hilbert spaces using Hamilton-Jacobi theory. Liu [26] has established LDP for a class of stochastic evolution equations. Rockner et al. [35] studied the large deviations for the stochastic tamed 3D Navier-Stokes equations, whilst Cheushov and Millet [13] have considered the 2D hydrodynamical type systems which includes 2D Navier-Stokes equation as a special case. An LDP for 2D Stochastic Navier-Stokes equation with free boundary was discussed by Bessaih and Millet [3]. The large deviations for a

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stochastic Burgers' equation was established by [37] using the weak convergence approach.

Ocean tides have been investigated by many authors starting from Isaac Newton ([25], [33]). We consider a stochastic analogue of a tidal dynamics model studied by Manna et al. [27] originally proposed in the deterministic context by Marchuk and Kagan [30]. The existence and uniqueness of pathwise strong solutions for the stochastic tidal dynamics equation with additive noise was established in [27] using Galerkin approximation and a generalization of the Minty-Browder technique [31]. In this paper, we will extend the stochastic theory to multiplicative noise and prove Wentzell-Freidlin type large deviation to this stochastic model of tidal dynamics.

2. Abstract Formulation

The tidal dynamics system developed by Marchuk and Kagan [30] for suitably normalized velocity $u \in \mathbb{R}^2$ and tide height $z \in \mathbb{R}$ is

$$\partial_t u + Au + B(u) + \nabla z = f(t) \text{ in } \mathcal{O} \times [0, T], \quad (2.1)$$

$$\partial_t z + \text{Div}(hu) = 0 \text{ in } \mathcal{O} \times [0, T]. \quad (2.2)$$

We consider the stochastic counterpart of (2.1)-(2.2) subjected to a random force with a multiplicative noise $\Gamma(t, x, u)$ as

$$\frac{\partial u}{\partial t} + Au + B(u) + \nabla z = f(t) + \Gamma(t, x, u), \quad (2.3)$$

$$\frac{\partial z}{\partial t} + \text{Div}(hu) = 0, \quad (2.4)$$

with the initial conditions $u(x, 0) = u_0(x), z(x, 0) = z_0(x)$ for $x \in \mathcal{O}$, an open domain in \mathbb{R}^2 with C^∞ boundary $\partial\mathcal{O}$, and the boundary condition $u(t, x) = 0$ for $x \in \partial\mathcal{O}$ and all $t \in [0, T]$. Also A and B are defined as

$$A = \begin{pmatrix} -\alpha\Delta & -\beta \\ \beta & -\alpha\Delta \end{pmatrix}; \quad (2.5)$$

$$B(u) = \gamma|u + w^0|(u + w^0). \quad (2.6)$$

Moreover $\alpha, \beta > 0$; and $w^0(x, t)$ is a known random function; $\gamma(x) = \frac{r}{h(x)}$ is a given smooth function, where $r > 0$ and $h(x)$ is continuously differentiable such that

$$\kappa = \min_{x \in \mathcal{O}} h(x), \quad \mu = \max_{x \in \mathcal{O}} h(x), \quad \text{and } L = \max_{x \in \mathcal{O}} |\nabla h(x)|, \quad (2.7)$$

with κ, μ and L being positive constants. Also $f(t)$ is a random forcing term and the noise term $\Gamma(t, x, u)$ is modeled abstractly as $\Gamma(t, x, u) = \sigma(t, u)dW(t)$ where $\{W_t\}$ is a Hilbert space-valued Wiener process and the multiplicative noise operator $\sigma(t, u)$ will be precisely defined below.

Let $\mathbb{H}^{-1}(\mathcal{O})$ denote the dual of the Sobolev space $\mathbb{H}_0^1(\mathcal{O})$ (see [1] for details on Sobolev spaces). Then we have the dense, continuous embedding

$$\mathbb{H}_0^1(\mathcal{O}) \subset \mathbb{L}^2(\mathcal{O}) \subset \mathbb{H}^{-1}(\mathcal{O}).$$

The inner product in the Hilbert space $\mathbb{L}^2(\mathcal{O})$ and the induced duality between the spaces $\mathbb{H}_0^1(\mathcal{O})$ and $\mathbb{H}^{-1}(\mathcal{O})$ are denoted by (\cdot, \cdot) and $\langle \cdot, \cdot \rangle$ respectively. Then for

$u \in \mathbb{L}^2(\mathcal{O})$ and $v \in \mathbb{H}_0^1(\mathcal{O})$, it follows that $\langle u, v \rangle = (u, v)$ (see [41]). Let us assume the random forcing term $f(t)$ to be $\mathbb{L}^2(\mathcal{O})$ -valued for all t .

Let us consider $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability space equipped with an increasing family $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ of sub-sigma fields of \mathcal{F} satisfying the usual conditions of right continuity and \mathbb{P} -completeness. Let Q be a positive, symmetric, trace class operator on $\mathbb{L}^2(\mathcal{O})$. Define the space $\mathcal{H}_0 = Q^{\frac{1}{2}}\mathbb{L}^2(\mathcal{O})$. Then \mathcal{H}_0 is a Hilbert space with the inner product

$$(u, v)_0 = (Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}v), \quad \forall u, v \in \mathcal{H}_0.$$

The norm in the \mathcal{H}_0 space will be $\|u\|_0^2 = (u, u)_0$. Let L_Q denote the space of linear operators S such that $SQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator from $\mathbb{L}^2(\mathcal{O})$ to $\mathbb{L}^2(\mathcal{O})$. That is for any orthonormal basis $\{e_k\}$ in $\mathbb{L}^2(\mathcal{O})$,

$$\sum_{k=1}^{\infty} \|SQ^{1/2}e_k\|_{\mathbb{L}^2}^2 < +\infty.$$

The norm of L_Q is obtained as follows:

$$\|S\|_{L_Q}^2 = \sum_{k=1}^{\infty} \|SQ^{1/2}e_k\|_{\mathbb{L}^2(\mathcal{O})}^2 = \sum_{k=1}^{\infty} (Q^{1/2}S^*SQ^{1/2}e_k, e_k)_{\mathbb{L}^2(\mathcal{O})} = \text{Tr}(SQS^*).$$

Let W be an $\{\mathcal{F}_t\}$ -adapted $\mathbb{L}^2(\mathcal{O})$ -valued Wiener process with covariance operator Q . We shall also impose the following assumptions on the multiplicative noise coefficient $\sigma : [0, T] \times \mathbb{H}_0^1(\mathcal{O}) \rightarrow L(\mathbb{L}^2(\mathcal{O}); \mathbb{L}^2(\mathcal{O}))$:

- (1) The function $\sigma \in \mathbb{C}([0, T] \times \mathbb{H}_0^1(\mathcal{O}); L(\mathbb{L}^2(\mathcal{O}); \mathbb{L}^2(\mathcal{O})))$.
- (2) There exists a positive constant \tilde{C}_1 such that

$$\|\sigma(t, u) - \sigma(t, v)\|_{L(\mathbb{L}^2(\mathcal{O}); \mathbb{L}^2(\mathcal{O}))} \leq \tilde{C}_1 \|u - v\|_{\mathbb{H}_0^1} \quad \text{for all } t \in [0, T], u, v \in \mathbb{H}_0^1(\mathcal{O}).$$

- (3) There exists a positive constant \tilde{C}_2 such that

$$\|\sigma(t, u)\|_{L(\mathbb{L}^2(\mathcal{O}); \mathbb{L}^2(\mathcal{O}))} \leq \tilde{C}_2(1 + \|u\|_{\mathbb{H}_0^1}) \quad \text{for all } t \in [0, T], u \in \mathbb{H}_0^1(\mathcal{O}).$$

These assumptions imply the following properties in terms of the L_Q norm of σ :

$$\|\sigma\|_{L_Q}^2 = \text{Tr}\{\sigma Q \sigma^*\},$$

stated as hypotheses (either directly or as consequences of properties (i)-(iii)) for $\sigma : [0, T] \times \mathbb{H}_0^1(\mathcal{O}) \rightarrow L(\mathbb{L}^2(\mathcal{O}); \mathbb{L}^2(\mathcal{O}))$ as (see [21]):

- (H1) The function $\sigma \in \mathbb{C}([0, T] \times \mathbb{H}_0^1(\mathcal{O}); L_Q)$.
- (H2) For all $t \in (0, T)$, there exists a positive constant C_1 such that for all $u, v \in \mathbb{H}_0^1$,

$$\|\sigma(t, u) - \sigma(t, v)\|_{L_Q} \leq C_1 \|u - v\|_{\mathbb{H}_0^1}. \tag{2.8}$$

- (H3) For all $t \in (0, T)$ and $u \in \mathbb{H}_0^1$, the following linear growth condition holds:

$$\|\sigma(t, u)\|_{L_Q}^2 \leq C_2(1 + \|u\|_{\mathbb{H}_0^1}^2), \tag{2.9}$$

where C_2 is a positive constant.

Let us define the bilinear form $a(\cdot, \cdot) : \mathbb{H}_0^1(\mathcal{O}) \times \mathbb{H}_0^1(\mathcal{O}) \rightarrow \mathbb{R}$ as

$$a(u, v) = \alpha(\nabla u, \nabla v) + \beta[(u_1, v_2) - (u_2, v_1)], \quad (2.10)$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$, and formally,

$$a(u, v) = (Au, v). \quad (2.11)$$

We could observe that the bilinear form $a(\cdot, \cdot)$ is continuous and coercive in $\mathbb{H}_0^1(\mathcal{O})$ (for unbounded domain we have Gårding inequality as seen below):

$$|a(u, v)| \leq C_a \|u\|_{\mathbb{H}^1} \|v\|_{\mathbb{H}^1}, \quad \forall u, v \in \mathbb{H}_0^1(\mathcal{O}), \quad (2.12)$$

$$(Au, u) = a(u, u) = \alpha \|\nabla u\|_{\mathbb{L}^2}^2 = \alpha [\|u\|_{\mathbb{H}^1}^2 - \|u\|_{\mathbb{L}^2}^2], \quad \forall u \in \mathbb{H}_0^1(\mathcal{O}), \quad (2.13)$$

for some positive constant C_a . From (2.12) and (2.13), $A : \mathbb{H}_0^1(\mathcal{O}) \rightarrow \mathbb{H}^{-1}(\mathcal{O})$ is well defined by Lax Milgram lemma. We also need the following lemmas:

Lemma 2.1. *Let u and v be in $\mathbb{L}^4(\mathcal{O}, \mathbb{R}^2)$. Then the following estimate holds:*

$$\langle B(u) - B(v), u - v \rangle \geq 0. \quad (2.14)$$

Proof. Fix $x \in \mathcal{O}$ and take $U = u(x) + w^0(x)$; $V = v(x) + w^0(x)$ so that $U, V \in \mathbb{R}^2$. Consider the Euclidean product:

$$\begin{aligned} (U|U| - V|V|, U - V) &= |U|^3 + |V|^3 - (U, V)(|U| + |V|) \\ &\geq |U|^3 + |V|^3 - |U||V|(|U| + |V|) \\ &= (|U| - |V|)^2(|U| + |V|) > 0, \end{aligned}$$

where Cauchy-Schwarz inequality is used. Since $\gamma(x)$ is also positive, the integrand in (2.14) turns out to be positive and noting that $u - v = U - V$, we conclude that the inner product $\langle B(u) - B(v), u - v \rangle$ is positive. \square

It should be noted that the above inequality is true for arbitrary u and v in $\mathbb{L}^4(\mathcal{O}, \mathbb{R}^2)$. In particular, if v is chosen to be 0, then using the definition of $B(\cdot)$, we get

$$\begin{aligned} (B(u), u) &\geq (\gamma w^0 |w^0|, u) \geq -\frac{r}{\kappa} \|w^0 |w^0|\|_{\mathbb{L}^2} \|u\|_{\mathbb{L}^2} \\ &= -\frac{r}{\kappa} \|w^0\|_{\mathbb{L}^4}^2 \|u\|_{\mathbb{L}^2}, \\ (B(u), u) &\geq -\frac{r}{2\kappa} [\|w^0\|_{\mathbb{L}^4}^4 + \|u\|_{\mathbb{L}^2}^2]. \end{aligned} \quad (2.15)$$

This inequality will be very useful in proving the energy inequality to estimate the nonlinear operator. The following lemma will be helpful to form a bound for $B(\cdot)$.

Lemma 2.2. *For $u, v \in \mathbb{L}^4(\mathcal{O})$,*

$$\|B(v)\|_{\mathbb{L}^2} \leq C_b (\|v\|_{\mathbb{L}^4} + \|w^0\|_{\mathbb{L}^4})^2, \quad (2.16)$$

$$\|B(u) - B(v)\|_{\mathbb{L}^2} \leq C_b [\|u\|_{\mathbb{L}^4} + \|v\|_{\mathbb{L}^4} + \|w^0\|_{\mathbb{L}^4}] \|u - v\|_{\mathbb{L}^4}, \quad (2.17)$$

where C_b is an arbitrary positive constant.

Proof. From (2.6), we have

$$\begin{aligned}\|B(v)\|_{\mathbb{L}^2}^2 &= \int_{\mathcal{O}} |\gamma(x)|v + w^0|(v + w^0)|^2 dx \leq C_b \int_{\mathcal{O}} |v + w^0|^4 dx, \\ \|B(v)\|_{\mathbb{L}^2} &\leq C_b (\|v + w^0\|_{\mathbb{L}^4})^2 \leq C_b (\|v\|_{\mathbb{L}^4} + \|w^0\|_{\mathbb{L}^4})^2.\end{aligned}$$

In order to prove (2.17), consider

$$\begin{aligned}B(u) - B(v) &= \gamma|u + w^0|(u + w^0) - \gamma|v + w^0|(v + w^0) \\ &= \gamma[|u + w^0|(u - v) + (v + w^0)(|u + w^0| - |v + w^0|)] \\ &\leq C_b|u + w^0||u - v| + |v + w^0||u - v|,\end{aligned}$$

$$\|B(u) - B(v)\|_{\mathbb{L}^2} \leq C_b [\|u + w^0\|_{\mathbb{L}^4} + \|v + w^0\|_{\mathbb{L}^4}] \|u - v\|_{\mathbb{L}^4},$$

where Holder's and Minkowski's inequalities are used in the last step. \square

Lemma 2.3. *For a nonlinear operator F on $\mathbb{H}_0^1(\mathcal{O})$ defined by $F(u) := Au + B(u) - f$, and a small constant $0 < \epsilon < \frac{\alpha\kappa}{2C_1}$ with C_1 as in hypothesis (H2), the following monotonicity property holds: For $u, v \in \mathbb{H}_0^1(\mathcal{O})$,*

$$\langle F(u) - F(v), hu - hv \rangle - \epsilon \|\sigma(t, u) - \sigma(t, v)\|_{L^Q}^2 + \tilde{N} \|u - v\|_{\mathbb{L}^2}^2 \geq 0. \quad (2.18)$$

Proof. From the definition of the operator A given by (2.5) and the assumptions (2.7) on $h(x)$, by making use of Holder's and Young's inequalities, we get successively,

$$\begin{aligned}\langle Au, hu \rangle &= \alpha(\nabla u, h\nabla u) + \alpha(\nabla u, u\nabla h) \\ &\geq \alpha\kappa \|\nabla u\|_{\mathbb{L}^2}^2 - \alpha L \|u\|_{\mathbb{L}^2} \|\nabla u\|_{\mathbb{L}^2} \\ &\geq \alpha\kappa \|\nabla u\|_{\mathbb{L}^2}^2 - \frac{\alpha\kappa}{2} \|\nabla u\|_{\mathbb{L}^2}^2 - \frac{\alpha L^2}{2\kappa} \|u\|_{\mathbb{L}^2}^2 \\ &= \frac{\alpha\kappa}{2} \|u\|_{\mathbb{H}^1}^2 - \tilde{N} \|u\|_{\mathbb{L}^2}^2,\end{aligned}$$

where $\tilde{N} = \frac{\alpha(L^2 + \kappa^2)}{2\kappa}$. Since A is a linear operator, we have for $v \in \mathbb{H}_0^1(\mathcal{O})$,

$$\langle A(u - v), hu - hv \rangle \geq \frac{\alpha\kappa}{2} \|u - v\|_{\mathbb{H}^1}^2 - \tilde{N} \|u - v\|_{\mathbb{L}^2}^2. \quad (2.19)$$

Also if we proceed in similar way as in the proof of Lemma 2.1, we observe that

$$\langle B(u) - B(v), hu - hv \rangle \geq 0. \quad (2.20)$$

Then from (2.19) and (2.20), we have

$$\langle F(u) - F(v), hu - hv \rangle \geq \frac{\alpha\kappa}{2} \|u - v\|_{\mathbb{H}^1}^2 - \tilde{N} \|u - v\|_{\mathbb{L}^2}^2.$$

The required property (2.18) is now obtained by using hypothesis (H2) and choosing $\epsilon < \frac{\alpha\kappa}{2C_1}$. \square

We consider the stochastic system (2.3)-(2.4) with a small perturbation of the noise term and denote the corresponding solution by (u^ϵ, z^ϵ) . The perturbed version of the system (2.3)-(2.4) can be phrased in the variational form as

$$d(u^\epsilon, \phi) + ((Au^\epsilon, \phi) + (B(u^\epsilon), \phi) + (\nabla z^\epsilon, \phi))dt = (f, \phi)dt + \sqrt{\epsilon}(\sigma dW, \phi); \quad (2.21)$$

$$d(z^\epsilon, \zeta) + (\text{Div}(hu^\epsilon), \zeta)dt = 0, \quad (2.22)$$

for all $\phi \in \mathbb{H}_0^1(\mathcal{O}); \zeta \in \mathbb{L}^2(\mathcal{O})$ with $u^\epsilon(0) = u_0$ and $z^\epsilon(0) = z_0$. Next we prove the existence and uniqueness of the stochastic perturbed system (2.21)-(2.22).

3. Existence and Uniqueness

We first define a finite dimensional Galerkin approximation of the stochastic tidal dynamics system (2.21)-(2.22) as follows: Let $\{e_1, e_2, \dots\}$ be a complete orthonormal system in the Hilbert space $\mathbb{L}^2(\mathcal{O})$ belonging to the space $\mathbb{H}_0^1(\mathcal{O})$. Let $\mathbb{L}_n^2(\mathcal{O})$ denote the n -dimensional subspace of $\mathbb{L}^2(\mathcal{O})$ and $\mathbb{H}_0^1(\mathcal{O})$ of all linear combinations of the first n elements $\{e_1, e_2, \dots, e_n\}$. Let P_n denote the orthogonal projection of $\mathbb{L}^2(\mathcal{O})$ to $\mathbb{L}_n^2(\mathcal{O})$. Define $W_n = P_n W$ and $\sigma_n = P_n \sigma$. Let us also define $(u_n^\epsilon, z_n^\epsilon)$ as the solution to the projection of the system (2.21)-(2.22) in the finite dimensional space $\mathbb{L}_n^2(\mathcal{O})$ given by

$$d(u_n^\epsilon, \phi) + ((Au_n^\epsilon, \phi) + (Bu_n^\epsilon, \phi) + (\nabla z_n^\epsilon, \phi))dt = (f, \phi)dt + \sqrt{\epsilon}(\sigma_n dW_n, \phi); \quad (3.1)$$

$$d(z_n^\epsilon, \zeta) + (\text{Div}(hu_n^\epsilon), \zeta)dt = 0, \quad (3.2)$$

for all $\phi \in \mathbb{L}_n^2(\mathcal{O}); \zeta \in \mathbb{L}_n^2(\mathcal{O})$ with $u_n^\epsilon(0) = P_n u_0$ and $z_n^\epsilon(0) = P_n z_0$. We begin with the following energy estimates.

Theorem 3.1. *Let*

$$\begin{aligned} f &\in \mathbb{L}^4(\Omega; \mathbb{L}^4(0, T; \mathbb{L}^2(\mathcal{O}))); \quad w^0 \in \mathbb{L}^4(\Omega; \mathbb{L}^4(0, T; \mathbb{H}_0^1(\mathcal{O}))); \\ u_0 &\in \mathbb{L}^4(\Omega; \mathbb{L}^2(\mathcal{O})); \quad z_0 \in \mathbb{L}^4(\Omega; \mathbb{L}^2(\mathcal{O})). \end{aligned}$$

If $(u_n^\epsilon, z_n^\epsilon)$ denotes the unique strong solution of the system (3.1)-(3.2), then under the assumptions (H1)-(H3), the following energy estimates are satisfied:

(a) For all $\epsilon < \frac{1}{8C_2^2} \wedge \frac{\alpha}{2C_2} \wedge \frac{\alpha^2}{32C_2^2}$ and $0 \leq t \leq T$,

$$\begin{aligned} \mathbb{E} \{ \{ \|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 \} + \alpha \mathbb{E} \int_0^t \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \leq C \left\{ \mathbb{E} [\|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 \right. \\ \left. + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2] + \frac{r}{\kappa} \mathbb{E} \int_0^t \|w^0(s)\|_{\mathbb{L}^4}^4 ds + \mathbb{E} \int_0^t \|f(s)\|_{\mathbb{L}^2}^2 ds + \epsilon C_2 T \right\} e^{Kt} \quad (3.3) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E} \left\{ \sup_{0 \leq t \leq T} [\|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2] \right\} + \frac{\alpha}{2} \mathbb{E} \int_0^T \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\ \leq C \left\{ \mathbb{E} [\|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2] + \frac{r}{\kappa} \mathbb{E} \int_0^T \|w^0(s)\|_{\mathbb{L}^4}^4 ds \right. \\ \left. + \mathbb{E} \int_0^T \|f(s)\|_{\mathbb{L}^2}^2 ds + (1 + \epsilon) C_2 T \right\} e^{\tilde{K}T}, \quad (3.4) \end{aligned}$$

where C and \tilde{K} are positive constants.

(b) For all $0 \leq t \leq T$,

$$\begin{aligned} & \mathbb{E} \left(\|u_n^\epsilon(t)\|_{\mathbb{L}^2}^4 + \|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^4 \right) \\ & + \frac{3\alpha}{2} \mathbb{E} \int_0^t \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + \alpha \mathbb{E} \int_0^t \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\ & \leq C \left\{ \|u_n^\epsilon(0)\|_{\mathbb{L}^2}^4 + \|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^4 \right. \\ & \left. + \frac{r}{4\kappa} \mathbb{E} \int_0^t \|w^0(s)\|_{\mathbb{H}^1}^4 ds + 4\mathbb{E} \int_0^t \|f(s)\|_{\mathbb{L}^2}^4 ds + 7\epsilon C_2 T \right\} e^{\tilde{K}t}, \quad (3.5) \end{aligned}$$

where C and \tilde{K} are appropriate constants.

Proof. Using Itô formula ([32],[34]) for (3.1),

$$\begin{aligned} & d\|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + 2[\alpha\|\nabla u_n^\epsilon\|_{\mathbb{L}^2}^2 + (B(u_n^\epsilon(t)), u_n^\epsilon(t)) + (\nabla z_n^\epsilon(t), u_n^\epsilon(t))]dt \\ & = 2(f(t), u_n^\epsilon(t))dt + \epsilon \text{Tr}(\sigma_n(t), u_n^\epsilon(t))Q\sigma_n^*(t, u_n^\epsilon(t))dt \\ & \quad + 2\sqrt{\epsilon}(u_n^\epsilon(t), \sigma_n(t, u_n^\epsilon(t))dW_n(t)). \quad (3.6) \end{aligned}$$

Define a stopping time $\tau_N = \inf\{t : \|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \int_0^t \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \geq N\}$. Using the Cauchy-Schwarz inequality and then simplifying and integrating from 0 to $t \wedge \tau_N$,

$$\begin{aligned} & \|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + 2\alpha \int_0^{t \wedge \tau_N} \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \leq \|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 \\ & + \frac{r}{\kappa} \int_0^{t \wedge \tau_N} [\|w^0(s)\|_{\mathbb{L}^4}^4 + \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2] ds \\ & + \int_0^{t \wedge \tau_N} \left(\frac{4}{\alpha} \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 + \frac{\alpha}{4} \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 + \|f(s)\|_{\mathbb{L}^2}^2 + \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right) ds \\ & + \epsilon C_2 \int_0^{t \wedge \tau_N} (1 + \|u_n^\epsilon(s)\|_{\mathbb{H}^1}^2) ds + 2\sqrt{\epsilon} \int_0^{t \wedge \tau_N} (u_n^\epsilon(s), \sigma_n(s, u_n^\epsilon(s))dW_n(s)). \quad (3.7) \end{aligned}$$

Keeping this inequality as such and taking inner product of (3.2) with z_n^ϵ ,

$$d\|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + 2(\text{Div}(hu_n^\epsilon), z_n^\epsilon(t))dt = 0. \quad (3.8)$$

Now integrating and simplifying using Cauchy-Schwarz and Young's inequalities, we finally obtain

$$\begin{aligned} \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 & \leq \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \int_0^{t \wedge \tau_N} \left[\frac{\alpha}{4} \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right. \\ & \left. + L\|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 + \left(\frac{4\mu^2}{\alpha} + L \right) \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds. \quad (3.9) \end{aligned}$$

Adding the inequalities (3.7) and (3.9),

$$\begin{aligned}
& \left[\|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 \right] + \frac{3\alpha}{2} \int_0^{t \wedge \tau_N} \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\
& \leq \left[\|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2 \right] + \frac{r}{\kappa} \int_0^{t \wedge \tau_N} \|w^0(s)\|_{\mathbb{L}^4}^4 ds \\
& \quad + K \int_0^{t \wedge \tau_N} \left[\|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds + \int_0^{t \wedge \tau_N} \|f(s)\|_{\mathbb{L}^2}^2 ds \\
& \quad + \epsilon C_2 \int_0^{t \wedge \tau_N} (1 + \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2) ds \\
& \quad + 2\sqrt{\epsilon} \int_0^{t \wedge \tau_N} (u_n^\epsilon(s), \sigma_n(s, u_n^\epsilon(s)) dW_n(s)), \tag{3.10}
\end{aligned}$$

where $K = \max\left(\frac{r}{\kappa} + 1 + L + \epsilon C_2, \frac{4}{\alpha} + \frac{4\mu^2}{\alpha} + L\right)$. Now taking expectation,

$$\begin{aligned}
& \mathbb{E} \left\{ \|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 \right\} + \frac{3\alpha}{2} \mathbb{E} \int_0^{t \wedge \tau_N} \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\
& \leq \mathbb{E} \left[\|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2 \right] + \frac{r}{\kappa} \mathbb{E} \int_0^{t \wedge \tau_N} \|w^0(s)\|_{\mathbb{L}^4}^4 ds \\
& \quad + K \mathbb{E} \int_0^{t \wedge \tau_N} \left[\|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds + \mathbb{E} \int_0^{t \wedge \tau_N} \|f(s)\|_{\mathbb{L}^2}^2 ds \\
& \quad + \epsilon C_2 \mathbb{E} \int_0^{t \wedge \tau_N} (1 + \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2) ds. \tag{3.11}
\end{aligned}$$

If $\epsilon < \frac{\alpha}{2C_2}$, then

$$\begin{aligned}
& \mathbb{E} \left\{ \|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 \right\} + \alpha \mathbb{E} \int_0^{t \wedge \tau_N} \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\
& \leq \mathbb{E} \left[\|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2 \right] + \frac{r}{\kappa} \mathbb{E} \int_0^{t \wedge \tau_N} \|w^0(s)\|_{\mathbb{L}^4}^4 ds + K \mathbb{E} \int_0^{t \wedge \tau_N} \left[\|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right. \\
& \quad \left. + \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds + \mathbb{E} \int_0^{t \wedge \tau_N} \|f(s)\|_{\mathbb{L}^2}^2 ds + \epsilon C_2 T. \tag{3.12}
\end{aligned}$$

Applying Gronwall's inequality to $g_N(s) = \mathbb{E} \left[\mathbb{I}_{\tau_N > s} \left\{ \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right\} \right]$, where \mathbb{I} denotes the indicator function, we get

$$\begin{aligned}
& \mathbb{E} \left\{ \|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 \right\} + \alpha \mathbb{E} \int_0^{t \wedge \tau_N} \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\
& \leq C \left\{ \mathbb{E} \left[\|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2 \right] + \frac{r}{\kappa} \mathbb{E} \int_0^{t \wedge \tau_N} \|w^0(s)\|_{\mathbb{L}^4}^4 ds \right. \\
& \quad \left. + \mathbb{E} \int_0^{t \wedge \tau_N} \|f(s)\|_{\mathbb{L}^2}^2 ds + \epsilon C_2 T \right\} e^{Kt}. \tag{3.13}
\end{aligned}$$

On the other hand, taking supremum of (3.10) with respect to time from 0 to $T \wedge \tau_N$ and then taking expectation, we get

$$\begin{aligned}
& \mathbb{E} \left\{ \sup_{0 \leq t \leq T \wedge \tau_N} [\|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2] \right\} + \frac{3\alpha}{2} \mathbb{E} \int_0^{T \wedge \tau_N} \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\
& \leq \mathbb{E}[\|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2] + \frac{r}{\kappa} \mathbb{E} \int_0^{T \wedge \tau_N} \|w^0(s)\|_{\mathbb{L}^4}^4 ds \\
& \quad + K \mathbb{E} \int_0^{T \wedge \tau_N} [\|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2] ds + \mathbb{E} \int_0^{T \wedge \tau_N} \|f(s)\|_{\mathbb{L}^2}^2 ds \\
& \quad + \epsilon C_2 \mathbb{E} \int_0^{T \wedge \tau_N} (1 + \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2) ds \\
& \quad + 2\sqrt{\epsilon} \mathbb{E} \left\{ \sup_{0 \leq t \leq T \wedge \tau_N} \left| \int_0^t (u_n^\epsilon(s), \sigma_n(s, u_n^\epsilon(s)) dW_n(s)) \right| \right\}. \quad (3.14)
\end{aligned}$$

Now making use of the Burkholder-Davis-Gundy inequality [29],

$$\begin{aligned}
& \mathbb{E} \left\{ \sup_{0 \leq t \leq T \wedge \tau_N} \left| \int_0^t (u_n^\epsilon(s), \sigma_n(s, u_n^\epsilon(s)) dW_n(s)) \right| \right\} \\
& \leq \sqrt{2} C_2 \mathbb{E} \left\{ \left(\int_0^{T \wedge \tau_N} (1 + \|u_n^\epsilon(s)\|_{\mathbb{H}^1}^2) \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \right)^{1/2} \right\} \\
& \leq \sqrt{2} C_2 \left(\mathbb{E} \left\{ \sup_{0 \leq t \leq T \wedge \tau_N} \|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 \right\} + \mathbb{E} \int_0^{T \wedge \tau_N} \|u_n^\epsilon(s)\|_{\mathbb{H}^1}^2 ds + T \right).
\end{aligned}$$

Using this in (3.14) and assuming $\epsilon < \frac{1}{8C_2^2} \wedge \frac{\alpha}{2C_2} \wedge \frac{\alpha^2}{32C_2^2}$,

$$\begin{aligned}
& \mathbb{E} \left\{ \sup_{0 \leq t \leq T \wedge \tau_N} [\|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2] \right\} + \frac{\alpha}{2} \mathbb{E} \int_0^{T \wedge \tau_N} \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\
& \leq C \left\{ \mathbb{E}[\|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2] + \frac{r}{\kappa} \mathbb{E} \int_0^{T \wedge \tau_N} \|w^0(s)\|_{\mathbb{L}^4}^4 ds \right. \\
& \quad \left. + \tilde{K} \mathbb{E} \int_0^{T \wedge \tau_N} [\|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2] ds + \mathbb{E} \int_0^{T \wedge \tau_N} \|f(s)\|_{\mathbb{L}^2}^2 ds + (1 + \epsilon) C_2 T \right\},
\end{aligned}$$

where the positive constant $\tilde{K} = K + 1$. By applying Gronwall's inequality,

$$\begin{aligned}
& \mathbb{E} \left\{ \sup_{0 \leq t \leq T \wedge \tau_N} [\|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2] \right\} + \frac{\alpha}{2} \mathbb{E} \int_0^{T \wedge \tau_N} \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\
& \leq C \left\{ \mathbb{E}[\|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2] + \frac{r}{\kappa} \mathbb{E} \int_0^{T \wedge \tau_N} \|w^0(s)\|_{\mathbb{L}^4}^4 ds \right. \\
& \quad \left. + \mathbb{E} \int_0^{T \wedge \tau_N} \|f(s)\|_{\mathbb{L}^2}^2 ds + (1 + \epsilon) C_2 T \right\} e^{\tilde{K}T}. \quad (3.15)
\end{aligned}$$

Define

$$\Omega_N = \{\omega \in \Omega : \|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \frac{\alpha}{2} \int_0^t \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds < N\}.$$

Then we have

$$\begin{aligned} & \int_{\Omega_N} \left(\|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \frac{\alpha}{2} \int_0^t \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \right) \mathbb{P}(d\omega) \\ & + \int_{\Omega \setminus \Omega_N} \left(\|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \frac{\alpha}{2} \int_0^t \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \right) \mathbb{P}(d\omega) \leq C. \end{aligned}$$

Dropping the first integral and noting from the definition of Ω_N that the integrand

$$\left(\|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \frac{\alpha}{2} \int_0^t \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \right) \geq N$$

is in $\Omega \setminus \Omega_N$, we get

$$N \int_{\Omega \setminus \Omega_N} \mathbb{P}(d\omega) \leq C \text{ and so } \mathbb{P}(\Omega \setminus \Omega_N) \leq \frac{C}{N}.$$

Note also that

$$\mathbb{P}\{\omega \in \Omega : \tau_N < T\} = \mathbb{P}(\Omega \setminus \Omega_N) \leq \frac{C}{N}. \quad (3.16)$$

From this we observe that

$$\limsup_{N \rightarrow \infty} \mathbb{P}\{\omega \in \Omega : \tau_N < T\} = 0,$$

and hence $T \wedge \tau_N \rightarrow T$. Thus we arrive at the required estimate (3.4). The estimate (3.3) is obtained from (3.13) by making use of the same argument. In order to prove the estimate (3.5), we first use Itô product formula to the processes $\|u_n^\epsilon\|_{\mathbb{L}^2}^2$ and $\|z_n^\epsilon\|_{\mathbb{L}^2}^2$ in (3.6) and (3.8) respectively to obtain

$$\begin{aligned} & d(\|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2) + 2 \left[\alpha \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 (B(u_n^\epsilon(t)), u_n^\epsilon(t)) \right. \\ & \left. + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 (\nabla z_n^\epsilon(t), u_n^\epsilon(t)) \right] dt + 2 \|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 (\text{Div}(hu_n^\epsilon(t)), z_n^\epsilon(t)) dt \\ & = 2 \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 (f(t), u_n^\epsilon(t)) dt + \epsilon \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 \text{Tr}(\sigma_n(t, u_n^\epsilon(t)) Q \sigma_n^*(t, u_n^\epsilon(t))) dt \\ & \quad + 2\sqrt{\epsilon} \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 (u_n^\epsilon(t), \sigma_n(t, u_n^\epsilon(t)) dW_n(t)). \end{aligned} \quad (3.17)$$

Define the stopping time

$$\begin{aligned} \tau_N = \inf \{ t : & \|u_n^\epsilon(t)\|_{\mathbb{L}^2}^4 + \|u_n^\epsilon(t)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t)\|_{\mathbb{L}^2}^4 \\ & + \frac{3\alpha}{2} \int_0^t \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + \alpha \int_0^t \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds > N \}. \end{aligned}$$

Integrating with respect to t from 0 to $t \wedge \tau_N$,

$$\begin{aligned} & \|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + 2 \int_0^{t \wedge \tau_N} \left[\alpha \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right. \\ & \left. + \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (B(u_n^\epsilon(s)), u_n^\epsilon(s)) + \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (\nabla z_n^\epsilon(s), u_n^\epsilon(s)) \right] ds \\ & + 2 \int_0^{t \wedge \tau_N} \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (\text{Div}(hu_n^\epsilon(s)), z_n^\epsilon(s)) ds \\ & = \|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + 2 \int_0^{t \wedge \tau_N} \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (f(s), u_n^\epsilon(s)) ds \\ & + \epsilon \int_0^{t \wedge \tau_N} \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \text{Tr}(\sigma_n(s, u_n^\epsilon(s)) Q \sigma_n^*(s, u_n^\epsilon(s))) ds \\ & + 2\sqrt{\epsilon} \int_0^{t \wedge \tau_N} \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (u_n^\epsilon(s), \sigma_n(s, u_n^\epsilon(s)) dW_n(s)). \quad (3.18) \end{aligned}$$

If we apply Itô formula to the function $\|u_n^\epsilon\|_{\mathbb{L}^2}^4$ and then integrate from 0 to $t \wedge \tau_N$, we arrive at

$$\begin{aligned} & \|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^4 + 4 \int_0^{t \wedge \tau_N} \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \left[\alpha \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 + (B(u_n^\epsilon(s)), u_n^\epsilon(s)) \right. \\ & \left. + (\nabla z_n^\epsilon(s), u_n^\epsilon(s)) \right] ds = \|u_n^\epsilon(0)\|_{\mathbb{L}^2}^4 + \int_0^{t \wedge \tau_N} 4 \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (f(s), u_n^\epsilon(s)) ds \\ & + 6\epsilon \int_0^{t \wedge \tau_N} \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \text{Tr}(\sigma_n(s, u_n^\epsilon(s)) Q \sigma_n^*(s, u_n^\epsilon(s))) ds \\ & + 4\sqrt{\epsilon} \int_0^{t \wedge \tau_N} \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (u_n^\epsilon(s), \sigma_n(s, u_n^\epsilon(s)) dW_n(s)). \quad (3.19) \end{aligned}$$

From (3.8),

$$\|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^4 + 4 \int_0^{t \wedge \tau_N} \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (\text{Div}(hu_n^\epsilon(s)), z_n^\epsilon(s)) ds = \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^4. \quad (3.20)$$

Adding (3.18), (3.19) and (3.20), using Cauchy-Schwarz and Young's inequalities, and then simplifying, we finally obtain

$$\begin{aligned}
& (\|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^4 + \|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^4) \\
& + 2\alpha \int_0^{t \wedge \tau_N} \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + 4\alpha \int_0^{t \wedge \tau_N} \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\
& \leq \|u_n^\epsilon(0)\|_{\mathbb{L}^2}^4 + \|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^4 + \frac{r}{4\kappa} \int_0^{t \wedge \tau_N} \|w^0(s)\|_{\mathbb{H}^1}^4 ds \\
& + 4 \int_0^{t \wedge \tau_N} \|f(s)\|_{\mathbb{L}^2}^4 ds + K \int_0^{t \wedge \tau_N} (\|u_n^\epsilon(s)\|_{\mathbb{L}^2}^4 + \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \\
& + \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^4) ds + \epsilon C_2 \int_0^{t \wedge \tau_N} \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (1 + \|u_n^\epsilon(s)\|_{\mathbb{H}^1}^2) ds \\
& + 6\epsilon C_2 \int_0^{t \wedge \tau_N} \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (1 + \|u_n^\epsilon(s)\|_{\mathbb{H}^1}^2) ds \\
& + 4\sqrt{\epsilon} \int_0^{t \wedge \tau_N} \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (u_n^\epsilon(s), \sigma_n(s, u_n^\epsilon(s)) dW_n(s)), \tag{3.21}
\end{aligned}$$

where $K = \max \left\{ \frac{r}{2\kappa} + \frac{L}{4} + 1, \frac{r}{2\kappa} + \frac{2\mu^2+1}{\alpha} + \frac{L}{2} + \frac{1}{4}, \frac{\mu^2+2\alpha}{2\alpha} + \frac{3L}{4} \right\}$. Taking expectation and further simplifying,

$$\begin{aligned}
& \mathbb{E} (\|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^4 + \|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^4) \\
& + 2\alpha \mathbb{E} \int_0^{t \wedge \tau_N} \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + 4\alpha \mathbb{E} \int_0^{t \wedge \tau_N} \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\
& \leq \|u_n^\epsilon(0)\|_{\mathbb{L}^2}^4 + \|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^4 + \frac{r}{4\kappa} \mathbb{E} \int_0^{t \wedge \tau_N} \|w^0(s)\|_{\mathbb{H}^1}^4 ds \\
& + 4\mathbb{E} \int_0^{t \wedge \tau_N} \|f(s)\|_{\mathbb{L}^2}^4 ds + \tilde{K} \mathbb{E} \int_0^{t \wedge \tau_N} (\|u_n^\epsilon(s)\|_{\mathbb{L}^2}^4 + \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \\
& + \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^4) ds + \epsilon C_2 \mathbb{E} \int_0^{t \wedge \tau_N} \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (1 + \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2) ds \\
& + 6\epsilon C_2 \mathbb{E} \int_0^{t \wedge \tau_N} \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 (1 + \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2) ds,
\end{aligned}$$

with $\tilde{K} = K + 6\epsilon C_2$. If $\epsilon < \frac{\alpha}{2C_2}$, then applying Gronwall's inequality,

$$\begin{aligned}
& \mathbb{E} (\|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^4 + \|u_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^4) \\
& + \frac{3\alpha}{2} \mathbb{E} \int_0^{t \wedge \tau_N} \|z_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + \alpha \mathbb{E} \int_0^{t \wedge \tau_N} \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \|\nabla u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\
& \leq C \left\{ \|u_n^\epsilon(0)\|_{\mathbb{L}^2}^4 + \|u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2 + \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^4 \right. \\
& \left. + \frac{r}{4\kappa} \mathbb{E} \int_0^{t \wedge \tau_N} \|w^0(s)\|_{\mathbb{H}^1}^4 ds + 4\mathbb{E} \int_0^{t \wedge \tau_N} \|f(s)\|_{\mathbb{L}^2}^4 ds + 7\epsilon C_2 T \right\} e^{\tilde{K}T}, \tag{3.22}
\end{aligned}$$

from which we obtain the estimate (3.5) by showing that $t \wedge \tau_N \rightarrow t$ as was done while proving (3.4). \square

Having proved the required energy estimates, we now move on to the proof of existence.

Theorem 3.2. *Let f, w^0, u_0 and z_0 be such that*

$$\left. \begin{aligned} f &\in \mathbb{L}^4(\Omega; \mathbb{L}^4(0, T; \mathbb{L}^2(\mathcal{O}))); & w^0 &\in \mathbb{L}^4(\Omega; \mathbb{L}^4(0, T; \mathbb{H}_0^1(\mathcal{O}))); \\ u_0 &\in \mathbb{L}^4(\mathbb{L}^2(\mathcal{O})); & z_0 &\in \mathbb{L}^4(\mathbb{L}^2(\mathcal{O})). \end{aligned} \right\} \quad (3.23)$$

If $\epsilon > 0$ is small enough as in Theorem 3.1, then under the assumptions (H2) and (H3) on σ , there exists a pathwise unique strong solution $(u^\epsilon(t, x), z^\epsilon(t, x))$ to the system (2.21)-(2.22) with the regularity

$$\begin{aligned} u^\epsilon &\in \mathbb{L}^2(\Omega; \mathbb{C}([0, T]; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^2(0, T; \mathbb{H}_0^1(\mathcal{O}))); \\ z^\epsilon &\in \mathbb{L}^2(\Omega; \mathbb{C}([0, T]; \mathbb{L}^2(\mathcal{O}))). \end{aligned}$$

Proof. Let $\Omega_T = \Omega \times [0, T]$. Using the energy estimates that have been derived so far, along with the Banach-Alaoglu theorem, we can extract a subsequence $\{u_{n_k}^\epsilon\}$ of $\{u_n^\epsilon\}$ which converges to the following limits. For simplicity, if we still denote the index n_k by n ,

$$\begin{aligned} &: u_n^\epsilon \rightarrow u^\epsilon \text{ weak star in } \mathbb{L}^2(\Omega; \mathbb{L}^\infty(0, T; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^2(\Omega_T; \mathbb{H}_0^1(\mathcal{O}))); \\ &: u_n^\epsilon(T) \rightarrow \eta^\epsilon \text{ weakly in } \mathbb{L}^2(\Omega; \mathbb{L}^2(\mathcal{O})); \\ &: z_n^\epsilon \rightarrow z^\epsilon \text{ weakly in } \mathbb{L}^2(\Omega_T; \mathbb{L}^2(\mathcal{O})); \\ &: F(u_n^\epsilon) \rightarrow F_0^\epsilon \text{ weakly in } \mathbb{L}^2(\Omega_T; \mathbb{H}^{-1}(\mathcal{O})), \end{aligned}$$

where $F(u_n^\epsilon) = Au_n^\epsilon + B(u_n^\epsilon) - f$. The boundedness of $F(u_n^\epsilon)$ follows by virtue of (2.13), (2.16) and the estimate (3.5). Also by the linear growth property of σ and the estimate of u_n^ϵ in $\mathbb{L}^2(0, T; \mathbb{H}_0^1(\mathcal{O}))$,

$$: \sigma_n(\cdot, u_n^\epsilon) \rightarrow S^\epsilon \text{ weakly in } \mathbb{L}^2(\Omega_T; L_Q(\mathcal{H}_0; \mathbb{L}^2(\mathcal{O}))).$$

Let us now extend the coupled equation (3.1)-(3.2) to an open interval $(-\delta, T + \delta)$, simply by setting the terms outside the interval $[0, T]$ to be zero.

As in Chow [10] (see also Sritharan et al. [39] and Manna et al. [28]), consider a function $\xi(t)$ in $\mathbb{H}^1(-\delta, T + \delta)$ with $\xi(0) = 1$. For a fixed orthonormal sequence $\{e_j\}, j \in \mathbb{N}$ in $\mathbb{H}_0^1(\mathcal{O})$, define $e_j(t) = \xi(t)e_j$. Applying Itô formula to the function $(u_n^\epsilon(t), e_j(t))$,

$$\begin{aligned} (u_n^\epsilon(T), e_j(T)) - \int_0^T \left(u_n^\epsilon(s), \frac{de_j(s)}{ds} \right) ds + \int_0^T (F(u_n^\epsilon(s)) + \nabla z_n^\epsilon(s), e_j(s)) ds \\ = (u_n(0), e_j) + \sqrt{\epsilon} \int_0^T (e_j(s), \sigma_n(s, u_n^\epsilon(s)) dW_n(s)). \end{aligned} \quad (3.24)$$

For the present, let us fix the integer j and consider the stochastic integral on the right hand side of (3.24). Let \mathcal{P}_T denote the class of predictable processes with values in $\mathbb{L}^2(\Omega_T; L_Q(\mathcal{H}_0; \mathbb{L}^2(\mathcal{O})))$. Also define $J : \mathcal{P}_T \rightarrow \mathbb{L}^2(\Omega_T)$ by $J(G) = \int_0^t (e_j(s), G(s) dW(s))$. Then J is clearly linear and continuous. The weak convergence of $\sigma_n(\cdot, u_n^\epsilon) \rightarrow S^\epsilon$ implies that for any $V \in \mathcal{P}_T$, and $\Pi_n V = V_n \in \mathcal{P}_T$ where Π_n is the finite dimensional projection using the above basis,

$$(\sigma_n(s, u_n^\epsilon(s)) dW_n(s), \Gamma)_{\mathcal{P}_T} \rightarrow (S^\epsilon(s) dW(s), \Gamma)_{\mathcal{P}_T} \text{ for all } \Gamma \in \mathcal{P}_T,$$

where the inner product

$$(\sigma, \Gamma)_{\mathcal{P}_T} = \mathbb{E} \int_0^T \text{Tr}(\sigma(s)Q\Gamma^*(s))ds.$$

From this, we could conclude (as in Chow [12], Chapter 6.7, Proof of Theorem 7.5) that

$$J(\sigma_n(\cdot, u_n^\epsilon)) \rightarrow \int_0^t (e_j(s), S^\epsilon(s)dW(s)).$$

Thus, passing to the limits in equation (3.24),

$$\begin{aligned} (\eta^\epsilon, e_j)\xi(T) - \int_0^T \left(u^\epsilon(s), \frac{d\xi(s)}{ds}e_j \right) ds + \int_0^T (F_0^\epsilon(s) + \nabla z^\epsilon(s), \xi(s)e_j)ds \\ = (u(0), e_j) + \sqrt{\epsilon} \int_0^T (\xi(s)e_j, S^\epsilon(s)dW(s)). \end{aligned}$$

Choose a subsequence of functions $\{\xi_k\} \in \mathbb{H}^1(-\delta, T+\delta)$ such that $\xi_k(0) = 1, k \in \mathbb{N}$, and as $k \rightarrow \infty, \xi_k(s)$ converges to the Heaviside function $H(t-s)$ which equals 1 for $s \leq t$ and 0 otherwise. Replacing ξ_k instead of ξ and letting $k \rightarrow \infty$, we end up with

$$(u^\epsilon(t), e_j) + \int_0^t (F_0^\epsilon(s) + \nabla z^\epsilon(s), e_j)ds = (u^\epsilon(0), e_j) + \sqrt{\epsilon} \int_0^t (e_j, S^\epsilon(s)dW(s))$$

with $(u^\epsilon(T), e_j) = (\eta^\epsilon, e_j)$, for each $j = 1, 2, \dots$. This implies that

$$u^\epsilon(t) + \int_0^t (F_0^\epsilon(s) + \nabla z^\epsilon(s))ds = u^\epsilon(0) + \sqrt{\epsilon} \int_0^t S^\epsilon(s)dW(s)$$

with $u^\epsilon(T) = \eta^\epsilon$. Hence u^ϵ has the Itô differential

$$du^\epsilon(t) + (F_0^\epsilon(t) + \nabla z^\epsilon(t))dt = \sqrt{\epsilon}S^\epsilon(t)dW(t). \tag{3.25}$$

We now target to prove that $F_0^\epsilon = F(u^\epsilon)$ and $S^\epsilon = \sigma(\cdot, u^\epsilon)$. For this purpose, we first apply Itô formula to the function $e^{-\tilde{N}t}\|\sqrt{h}u^\epsilon\|_{\mathbb{L}^2}^2$ to obtain

$$\begin{aligned} d(e^{-\tilde{N}t}\|\sqrt{h}u^\epsilon(t)\|_{\mathbb{L}^2}^2) + e^{-\tilde{N}t}(2[(F_0^\epsilon, hu^\epsilon(t)) + (\nabla z^\epsilon(t), hu^\epsilon(t))] \\ + \tilde{N}\|\sqrt{h}u^\epsilon(t)\|_{\mathbb{L}^2}^2)dt = e^{-\tilde{N}t}(\epsilon\|\sqrt{h}S^\epsilon(t)\|_{L^Q}^2 dt + 2\sqrt{\epsilon}(hu^\epsilon(t), S^\epsilon(t)dW(t))). \end{aligned}$$

Writing similar expression for $e^{-\tilde{N}t}\|z^\epsilon(t)\|_{\mathbb{L}^2}^2$ and then integrating the sum of both equations from 0 to T and then taking expectation,

$$\begin{aligned} \mathbb{E}(e^{-\tilde{N}T}\|\sqrt{h}u^\epsilon(T)\|_{\mathbb{L}^2}^2 + e^{-\tilde{N}T}\|z^\epsilon(T)\|_{\mathbb{L}^2}^2 - \|\sqrt{h}u^\epsilon(0)\|_{\mathbb{L}^2}^2 - \|z^\epsilon(0)\|_{\mathbb{L}^2}^2) \\ + \mathbb{E} \int_0^T e^{-\tilde{N}s} \left[2(F_0^\epsilon(s), hu^\epsilon(s)) + \tilde{N}\|\sqrt{h}u^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds = \epsilon \mathbb{E} \int_0^T e^{-\tilde{N}s}\|\sqrt{h}S^\epsilon(s)\|_{L^Q}^2 ds. \end{aligned}$$

In a similar manner we get

$$\begin{aligned} & \mathbb{E}(e^{-\tilde{N}T} \|\sqrt{h}u_n^\epsilon(T)\|_{\mathbb{L}^2}^2 + e^{-\tilde{N}T} \|z_n^\epsilon(T)\|_{\mathbb{L}^2}^2 - \|\sqrt{h}u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 - \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2) \\ & \quad + \mathbb{E} \int_0^T e^{-\tilde{N}s} \left[2(F(u_n^\epsilon(s)), hu_n^\epsilon(s)) + \tilde{N} \|\sqrt{h}u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds \\ & = \epsilon \mathbb{E} \int_0^T e^{-\tilde{N}s} \|\sqrt{h}\sigma_n(s, u_n^\epsilon(s))\|_{L_Q}^2 ds. \end{aligned} \quad (3.26)$$

Making use of the fact that the initial conditions $u_n^\epsilon(0)$ and $z_n^\epsilon(0)$ converge to $u^\epsilon(0)$ and $z^\epsilon(0)$ respectively in \mathbb{L}^2 , and the lower semi-continuity of the \mathbb{L}^2 -norm to pass on to an infimum limit (see [31],[32]),

$$\begin{aligned} & \liminf_n \mathbb{E} \int_0^T e^{-\tilde{N}s} \left[\epsilon \|\sqrt{h}\sigma_n(s, u_n^\epsilon(s))\|_{L_Q}^2 - 2(F(u_n^\epsilon(s)), hu_n^\epsilon(s)) - \tilde{N} \|\sqrt{h}u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds \\ & = \liminf_n \mathbb{E}(e^{-\tilde{N}T} \|\sqrt{h}u_n^\epsilon(T)\|_{\mathbb{L}^2}^2 + e^{-\tilde{N}T} \|z_n^\epsilon(T)\|_{\mathbb{L}^2}^2 - \|\sqrt{h}u_n^\epsilon(0)\|_{\mathbb{L}^2}^2 - \|z_n^\epsilon(0)\|_{\mathbb{L}^2}^2) \\ & \geq \mathbb{E}(e^{-\tilde{N}T} \|\sqrt{h}u^\epsilon(T)\|_{\mathbb{L}^2}^2 + e^{-\tilde{N}T} \|z^\epsilon(T)\|_{\mathbb{L}^2}^2 - \|\sqrt{h}u^\epsilon(0)\|_{\mathbb{L}^2}^2 - \|z^\epsilon(0)\|_{\mathbb{L}^2}^2) \\ & = \mathbb{E} \int_0^T e^{-\tilde{N}s} \left[\epsilon \|\sqrt{h}S^\epsilon(s)\|_{L_Q}^2 - 2(F_0^\epsilon(s), hu^\epsilon(s)) - \tilde{N} \|\sqrt{h}u^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds. \end{aligned} \quad (3.27)$$

Now turning to the monotonicity estimate (2.18), integrating and then taking expectation,

$$\begin{aligned} & \mathbb{E} \int_0^T e^{-\tilde{N}s} \left[(F(u_n^\epsilon(s)) - F(v^\epsilon(s)), hu_n^\epsilon(s) - hv^\epsilon(s)) \right. \\ & \quad \left. - \epsilon \|\sigma_n(s, u_n^\epsilon(s)) - \sigma_n(s, v^\epsilon(s))\|_{L_Q}^2 + \tilde{N} \|u_n^\epsilon(s) - v^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds \geq 0, \end{aligned}$$

with $v^\epsilon \in \mathbb{L}^\infty(0, T; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^2(0, T; \mathbb{H}_0^1(\mathcal{O}))$. Splitting the inner products termwise and rearranging,

$$\begin{aligned} & \mathbb{E} \int_0^T e^{-\tilde{N}s} \left[(F(v^\epsilon(s)), hv^\epsilon(s) - hu_n^\epsilon(s)) + \epsilon [2(\sigma_n(s, u_n^\epsilon(s)), \sigma_n(s, v^\epsilon(s)))_{L_Q} \right. \\ & \quad \left. - \|\sigma_n(s, v^\epsilon(s))\|_{L_Q}^2] + \tilde{N} [\|v^\epsilon(s)\|_{\mathbb{L}^2}^2 - (u_n^\epsilon(s), v^\epsilon(s))] \right] ds \\ & \geq \mathbb{E} \int_0^T e^{-\tilde{N}s} \left[(F(u_n^\epsilon(s)), hv^\epsilon(s)) - (F(u_n^\epsilon(s)), hu_n^\epsilon(s)) \right. \\ & \quad \left. + \epsilon \|\sigma_n(s, u_n^\epsilon(s))\|_{L_Q}^2 - \tilde{N} \|u_n^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds. \end{aligned}$$

Taking the limit and making use of (3.27), we get

$$\begin{aligned} & \mathbb{E} \int_0^T e^{-\tilde{N}s} \left[(F(v^\epsilon(s)), hv^\epsilon(s) - hu^\epsilon(s)) + \epsilon \left\{ 2(S^\epsilon(s), \sigma(s, v^\epsilon(s)))_{L_Q} \right. \right. \\ & \quad \left. \left. - \|\sigma(s, v^\epsilon(s))\|_{L_Q}^2 \right\} + \tilde{N} [\|v^\epsilon(s)\|_{\mathbb{L}^2}^2 - (u^\epsilon(s), v^\epsilon(s))] \right] ds \\ & \geq \mathbb{E} \int_0^T e^{-\tilde{N}s} \left[(F_0^\epsilon(s), hv^\epsilon(s) - hu^\epsilon(s)) + \epsilon \|S^\epsilon(s)\|_{L_Q}^2 - \tilde{N} \|u^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds. \end{aligned}$$

Merging back the inner products by simplification,

$$\begin{aligned} & \mathbb{E} \int_0^T e^{-\tilde{N}s} \left[(F_0^\epsilon(s) - F(v^\epsilon(s)), hv^\epsilon(s) - hu^\epsilon(s)) \right. \\ & \quad \left. + \epsilon \|S^\epsilon(s) - \sigma(s, v^\epsilon(s))\|_{L_Q}^2 - \tilde{N} \|u^\epsilon(s) - v^\epsilon(s)\|_{\mathbb{L}^2}^2 \right] ds \leq 0. \end{aligned} \quad (3.28)$$

Choosing $v^\epsilon = u^\epsilon$ readily results with $S^\epsilon(s) = \sigma(s, u^\epsilon(s))$. If we now choose $v^\epsilon = u^\epsilon + \lambda w^\epsilon$ with $\lambda > 0$ and $w^\epsilon \in \mathbb{L}^\infty(0, T; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^2(0, T; \mathbb{H}_0^1(\mathcal{O}))$, then dividing the resultant by λ and making $\lambda \rightarrow 0$, we end up with

$$\mathbb{E} \int_0^T e^{-\tilde{N}s} (F_0^\epsilon(s) - F(u^\epsilon(s)), hw^\epsilon(s)) ds \leq 0,$$

thus ending the proof of our existence result by showing $F_0^\epsilon = F(u^\epsilon)$. In order to prove the uniqueness of solution for the coupled system, assume that (u^ϵ, z^ϵ) and $(\tilde{u}^\epsilon, \tilde{z}^\epsilon)$ are two solutions of the system (2.21)–(2.22). Let $w^\epsilon = u^\epsilon - \tilde{u}^\epsilon$; $\zeta^\epsilon = z^\epsilon - \tilde{z}^\epsilon$. Then $(w^\epsilon, \zeta^\epsilon)$ satisfies

$$\begin{aligned} & dw^\epsilon(t) + [Aw^\epsilon(t) + B(u^\epsilon(t)) - B(\tilde{u}^\epsilon(t)) + \nabla \zeta^\epsilon(t)] dt \\ & \quad = \sqrt{\epsilon} (\sigma(t, u^\epsilon(t)) - \sigma(t, \tilde{u}^\epsilon(t))) dW(t); \\ & d\zeta^\epsilon(t) + \text{Div}(hw^\epsilon(t)) dt = 0. \end{aligned}$$

Recapitulating the procedure used to obtain the energy estimates,

$$\begin{aligned} & \|w^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + 2\alpha \int_0^{t \wedge \tau_N} \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \leq \frac{4}{\alpha} \int_0^{t \wedge \tau_N} \|\zeta^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\ & \quad + \frac{\alpha}{4} \int_0^{t \wedge \tau_N} \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + 2\sqrt{\epsilon} \int_0^{t \wedge \tau_N} (w^\epsilon(s), (\sigma(s, u^\epsilon(s)) - \sigma(s, \tilde{u}^\epsilon(s)))) dW(s) \\ & \quad + \epsilon \int_0^{t \wedge \tau_N} \|\sigma(s, u^\epsilon(s)) - \sigma(s, \tilde{u}^\epsilon(s))\|_{L_Q}^2 ds; \\ & \|\zeta^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 \leq \frac{\alpha}{4} \int_0^{t \wedge \tau_N} \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + \frac{4\mu^2}{\alpha} \int_0^{t \wedge \tau_N} \|\zeta^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\ & \quad + L \int_0^{t \wedge \tau_N} [\|w^\epsilon(s)\|_{\mathbb{L}^2}^2 + \|\zeta(s)\|_{\mathbb{L}^2}^2] ds. \end{aligned}$$

Adding the above two and taking expectation, we get

$$\begin{aligned} & \mathbb{E}[\|w^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + \|\zeta^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2] + \frac{3\alpha}{2} \mathbb{E} \int_0^{t \wedge \tau_N} \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\ & \quad \leq K \mathbb{E} \int_0^{t \wedge \tau_N} [\|w^\epsilon(s)\|_{\mathbb{L}^2}^2 + \|\zeta^\epsilon(s)\|_{\mathbb{L}^2}^2] ds + \epsilon \mathbb{E} \int_0^{t \wedge \tau_N} C_2 \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds, \end{aligned}$$

where $K = \frac{\mu^2 + 4}{\alpha} + L + \epsilon C_2$. If $\epsilon > 0$ is chosen so that $\epsilon < \frac{\alpha}{2C_2}$, then an application of Gronwall's inequality at once yields,

$$\mathbb{E}[\|w^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2 + \|\zeta^\epsilon(t \wedge \tau_N)\|_{\mathbb{L}^2}^2] + \alpha \mathbb{E} \int_0^{t \wedge \tau_N} \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \leq 0. \quad (3.29)$$

As $N \rightarrow \infty, t \wedge \tau_N \rightarrow t$, and hence the uniqueness of the solution for the given system can be confirmed. \square

4. Large Deviation Principle

The stochastic control approach for large deviations was highlighted in the works of Fleming [22] and it was combined with weak convergence methods by Dupuis and Ellis [19]. We use the theory developed by Boue and Dupuis [4] (a simpler proof of the theory is established in [41]) and as generalized to infinite dimensional processes by Budhiraja and Dupuis [5] for proving the large deviation principle for stochastic partial differential equations.

Let \mathcal{A} denote the class of \mathcal{H}_0 -valued $\{\mathcal{F}_t\}$ -predictable processes Φ such that $\int_0^T \|\Phi(s)\|_0^2 ds < \infty$ a.s. Let $S_M = \{v \in \mathbb{L}^2(0, T; \mathcal{H}_0) : \int_0^T \|v(s)\|_0^2 ds \leq M\}$. Then the set S_M endowed with the weak topology on $\mathbb{L}^2(0, T; \mathcal{H}_0)$ is a Polish space [18]. Define $\mathcal{A}_M = \{\Phi \in \mathcal{A} : \Phi(\omega) \in S_M, \text{ a.s.}\}$. Let \mathcal{Z} be a Polish space (will be the solution space in our case). Let $\mathcal{G}^\epsilon : \mathbb{C}([0, T]; \mathcal{H}_0) \rightarrow \mathcal{Z}$ be a measurable map. Define $X^\epsilon = \mathcal{G}^\epsilon(W(\cdot))$. According to [5, Theorem 4.4], the large deviation principle holds if the following proposition is satisfied:

Proposition 4.1. *Suppose that there exists a measurable map $\mathcal{G}^0 : \mathbb{C}([0, T]; \mathcal{H}_0) \rightarrow \mathcal{Z}$ such that the following hold:*

- (1) *Let $\{v^\epsilon : \epsilon > 0\} \subset \mathcal{A}_M$ for some $M < \infty$. Let v^ϵ converge in distribution as S_M -valued random elements to v . Then $\mathcal{G}^\epsilon \left(W(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^\cdot v^\epsilon(s) ds \right)$ converges in distribution to $\mathcal{G}^0 \left(\int_0^\cdot v(s) ds \right)$.*
- (2) *For every $M < \infty$, the set*

$$K_M = \left\{ \mathcal{G}^0 \left(\int_0^\cdot v(s) ds \right) : v \in S_M \right\} \tag{4.1}$$

is a compact subset of \mathcal{Z} .

For each $g \in \mathcal{Z}$, let

$$I(g) = \inf_{\left\{ v \in \mathbb{L}^2(0, T; \mathcal{H}_0) : g = \mathcal{G}^0 \left(\int_0^\cdot v(s) ds \right) \right\}} \left\{ \frac{1}{2} \int_0^T \|v(s)\|_0^2 ds \right\}, \tag{4.2}$$

where the infimum over an empty set is taken as ∞ . Then the family $\{X^\epsilon : \epsilon > 0\}$ satisfies the Laplace principle in \mathcal{Z} with the rate function I given by (4.2).

Let \mathcal{Z} denote the solution space $\mathbb{C}([0, T]; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^2(0, T; \mathbb{H}_0^1(\mathcal{O})) \times \mathbb{C}([0, T]; \mathbb{L}^2(\mathcal{O}))$. We note here that even though the topology of \mathcal{Z} used in the proof of the existence theorem 3.2 via weak convergence is of Lusin type [36], the weak convergence of the laws of the solutions proved in Theorem 4.6 below is for measures defined on the Borel subsets of \mathcal{Z} in the strong topology. Hence \mathcal{Z} is a separable Banach space in the rest of the paper and hence also Polish, and thus, the Laplace principle is equivalent to the large deviation principle (see Theorems 1.2.1 and 1.2.3 in [19]). We also would like to note that the separability of $\mathbb{C}([0, T]; \mathbb{L}^2(\mathcal{O}))$ following from the vector-valued generalization of the Stone-Weierstrass Theorem

[45] and that of $\mathbb{L}^2(0, T : \mathbb{H}_0^1(\mathcal{O}))$ from the separability of Lebesgue spaces with vector measures [17].

Let (u^ϵ, z^ϵ) denote the solution of the perturbed stochastic equation (2.21)-(2.22) with appropriate assumptions. Then there exists a Borel-measurable function $\mathcal{G}^\epsilon : \mathbb{C}([0, T] : \mathcal{H}_0) \rightarrow \mathcal{Z}$ such that $(u^\epsilon(\cdot), z^\epsilon(\cdot)) = \mathcal{G}^\epsilon(W(\cdot))$ a.s. (This is a consequence of the unique solvability of the strong pathwise solutions. Similar measurability theorems are well-known in stochastic Navier-Stokes equations, see for example Vishik and Fursikov [44], Chapter X, Corollary 4.2). We will prove the large deviation principle for this family $\{(u^\epsilon, z^\epsilon)\}$. The main theorem of this paper is the following:

Theorem 4.2. *Let $\{(u^\epsilon(\cdot), z^\epsilon(\cdot))\}$ denote the strong solution of the stochastic system (2.21)-(2.22). Then with f, w^0, u_0, z_0 as in (3.23), and the assumptions (H2) and (H3) on σ , the family $\{(u^\epsilon, z^\epsilon)\}$ satisfies the large deviation principle in $\mathcal{Z} = \mathbb{C}([0, T] : \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^2(0, T : \mathbb{H}_0^1(\mathcal{O})) \times \mathbb{C}([0, T] : \mathbb{L}^2(\mathcal{O}))$ with a good rate function*

$$I(g) = \inf_{\{v \in \mathbb{L}^2(0, T : \mathcal{H}_0) : g = \mathcal{G}^0(\int_0^\cdot v(s) ds)\}} \left\{ \frac{1}{2} \int_0^T \|v(s)\|_0^2 ds \right\}, \tag{4.3}$$

where the infimum over an empty set is taken as ∞ and $\mathcal{G}^0(\int_0^\cdot v(s) ds)$ denotes the solution (u_v, z_v) of the system

$$\begin{aligned} du_v + (Au_v + B(u_v) + \nabla z_v)dt &= fdt + \sigma(t, u_v)vdt; \\ dz_v + Div(hu_v)dt &= 0, \end{aligned}$$

with $(u_v(0), z_v(0)) = (u_0, z_0)$ and $v \in \mathcal{A}_M$.

In order to prove the above theorem, the hypotheses of Proposition 4.1 will be shown to hold. First let us discuss the following theorem on existence of solutions to the system (2.21) - (2.22) with an additional control term. The proof follows easily by making use of the Girsanov argument as was done in [39].

Theorem 4.3. *Let $v \in \mathcal{A}_M, 0 < M < \infty$ and let $(u_v^\epsilon, z_v^\epsilon)$ denote the process $\mathcal{G}^\epsilon(W(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^\cdot v(s)ds)$. Then $(u_v^\epsilon, z_v^\epsilon)$ is the unique strong solution of*

$$du_v^\epsilon + [Au_v^\epsilon + B(u_v^\epsilon) + \nabla z_v^\epsilon]dt = fdt + \sigma(t, u_v^\epsilon)vdt + \sqrt{\epsilon}\sigma(t, u_v^\epsilon)dW; \tag{4.4}$$

$$dz_v^\epsilon + Div(hu_v^\epsilon)dt = 0 \tag{4.5}$$

with the initial conditions $(u_v^\epsilon(0), z_v^\epsilon(0)) = (u_0, z_0) \in \mathbb{L}^4(\Omega; \mathbb{L}^2(\mathcal{O})) \times \mathbb{L}^4(\Omega; \mathbb{L}^2(\mathcal{O}))$.

Proof. Since $v \in \mathcal{A}_M, 0 < M < \infty$, by Girsanov's theorem (see [14]), $\tilde{W}(\cdot) = W(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^\cdot v(s)ds$ is also a Wiener process with covariance form Q under the probability measure

$$d\tilde{\mathbb{P}}_v^\epsilon := \exp \left\{ -\frac{1}{\sqrt{\epsilon}} \int_0^T v(s)dW(s) - \frac{1}{2\epsilon} \int_0^T \|v(s)\|_0^2 ds \right\} d\mathbb{P}$$

and so there exists a solution to (2.21) - (2.22) with \tilde{W} in place of W . This in turn implies the existence of solutions to the stochastic controlled system (4.4)-(4.5) under the probability measure $d\tilde{\mathbb{P}}$. Likewise, the uniqueness of solution to (4.4)-(4.5) also follows by making use of the same Girsanov argument. \square

We now proceed to the deterministic case, where the equation is represented as

$$du_v + [Au_v + B(u_v) + \nabla z_v]dt = f(t)dt + \sigma(t, u_v)v(t)dt; \quad (4.6)$$

$$dz_v + \text{Div}(hu_v)dt = 0 \quad (4.7)$$

with the initial conditions $u_v(0) = u_0, z_v(0) = z_0$.

Theorem 4.4. *Let f, u_0 and z_0 be such that*

$$f \in \mathbb{L}^4(0, T; \mathbb{L}^2(\mathcal{O})); u_0 \in \mathbb{L}^2(\mathcal{O}); z_0 \in \mathbb{L}^2(\mathcal{O})$$

and σ satisfy the hypotheses (H2) and (H3). Then there exists a unique solution $(u_v(t, x), z_v(t, x))$ of the equation (4.6)-(4.7) with the regularity $(u_v, z_v) \in \mathcal{Z}$.

Proof. If we take inner product of equation (4.6) with u_v and (4.7) with z_v ,

$$\begin{aligned} d\|u_v(t)\|_{\mathbb{L}^2}^2 + 2[\alpha\|\nabla u_v(t)\|_{\mathbb{L}^2}^2 dt + (B(u_v(t)), u_v(t)) + (\nabla z_v(t), u_v(t))]dt \\ = 2(f(t), u_v(t))dt + 2(\sigma(t, u_v(t))v(t), u_v(t)); \end{aligned} \quad (4.8)$$

$$d\|z_v(t)\|_{\mathbb{L}^2}^2 + 2(\text{Div}(hu_v(t)), z_v(t))dt = 0. \quad (4.9)$$

Adding the above two equations, then integrating, using Cauchy-Schwarz and Young's inequalities and simplifying as was done while proving the energy estimates in Theorem 3.1, we finally obtain

$$\begin{aligned} \|u_v(t)\|_{\mathbb{L}^2}^2 + \|z_v(t)\|_{\mathbb{L}^2}^2 + \alpha \int_0^t \|\nabla u_v(s)\|_{\mathbb{L}^2}^2 ds \leq \|u_v(0)\|_{\mathbb{L}^2}^2 + \|z_v(0)\|_{\mathbb{L}^2}^2 \\ + \frac{r}{\kappa} \int_0^t \|w^0(s)\|_{\mathbb{L}^4}^4 ds + K \int_0^t [\|u_v(s)\|_{\mathbb{L}^2}^2 + \|z_v(s)\|_{\mathbb{L}^2}^2] ds \\ + \int_0^t \|f(s)\|_{\mathbb{L}^2}^2 ds + \frac{4C_2}{\alpha} \int_0^t \|v(s)\|_0^2 \|u_v(s)\|_{\mathbb{L}^2}^2 ds + \frac{\alpha T}{4}, \end{aligned}$$

where $K = \max\left\{\frac{r}{\kappa} + L + 1, \frac{2(\mu^2+2)}{\alpha} + L\right\}$. Taking supremum and applying Gronwall's inequality,

$$\begin{aligned} \sup_{0 \leq t \leq T} [\|u_v(t)\|_{\mathbb{L}^2}^2 + \|z_v(t)\|_{\mathbb{L}^2}^2] + \alpha \int_0^T \|\nabla u_v(s)\|_{\mathbb{L}^2}^2 ds \leq C \left\{ \|u_v(0)\|_{\mathbb{L}^2}^2 + \|z_v(0)\|_{\mathbb{L}^2}^2 \right. \\ \left. + \frac{r}{\kappa} \int_0^T \|w^0(s)\|_{\mathbb{L}^4}^4 ds + \int_0^T \|f(s)\|_{\mathbb{L}^2}^2 ds + \frac{\alpha T}{4} \right\} e^{(KT + \frac{4C_2}{\alpha} \int_0^T \|v(s)\|_0^2 ds)}. \end{aligned} \quad (4.10)$$

In addition, if we multiply each of the equations (4.8) and (4.9) by $\|u_v\|_{\mathbb{L}^2}^2$ and $\|z_v\|_{\mathbb{L}^2}^2$, add all the four equations obtained and then make similar manipulations as above, we could get

$$\begin{aligned} \sup_{0 \leq t \leq T} (\|u_v(t)\|_{\mathbb{L}^2}^4 + \|u_v(t)\|_{\mathbb{L}^2}^2 \|z_v(t)\|_{\mathbb{L}^2}^2 + \|z_v(t)\|_{\mathbb{L}^2}^4) + \frac{\alpha}{8} \int_0^T [\|u_v(s)\|_{\mathbb{L}^2}^2 \|\nabla u_v(s)\|_{\mathbb{L}^2}^2 \\ + 2\|z_v(s)\|_{\mathbb{L}^2}^2 \|\nabla u_v(s)\|_{\mathbb{L}^2}^2] ds \leq \left\{ \|u_v(0)\|_{\mathbb{L}^2}^4 + \|u_v(0)\|_{\mathbb{L}^2}^2 \|z_v(0)\|_{\mathbb{L}^2}^2 + \|z_v(0)\|_{\mathbb{L}^2}^4 \right. \\ \left. + \frac{r}{16\kappa} \int_0^T \|w^0(s)\|_{\mathbb{L}^4}^4 ds + \frac{1}{2} \int_0^T \|f(s)\|_{\mathbb{L}^2}^4 ds + \alpha T \right\} e^{(KT + \frac{8C_2}{\alpha} \int_0^T \|v(s)\|_0^2 ds)}. \end{aligned} \quad (4.11)$$

With these estimates (4.10) and (4.11), the proof of existence and uniqueness follows similar to that in Theorem 3.2. \square

Now we need to prove the following theorems in order to satisfy the hypothesis of Proposition 4.1.

Theorem 4.5 (Compactness). *Let M be any finite fixed positive number. Let $K_M := \{(u_v, z_v) \in \mathcal{Z} : v \in S_M\}$, where (u_v, z_v) is the unique solution in \mathcal{Z} of the controlled equation*

$$du_v(t) + [Au_v(t) + B(u_v(t)) + \nabla z_v(t)]dt = f(t)dt + \sigma(t, u_v(t))v(t)dt; \tag{4.12}$$

$$dz_v(t) + \text{Div}(hu_v(t))dt = 0 \tag{4.13}$$

with $(u_v(0), z_v(0)) = (u_0, z_0) \in \mathbb{L}^2(\mathcal{O}) \times \mathbb{L}^2(\mathcal{O})$. Then K_M is compact in \mathcal{Z} .

Proof. Let $\{(u_n, z_n)\} \in K_M$ denote the solution of the above equations (4.12)-(4.13) with the control v replaced by $v_n \in S_M$. Since S_M is weakly compact, there exists a subsequence of $\{v_n\}$ (still denoted by $\{v_n\}$) which converges to a limit v weakly in $\mathbb{L}^2(0, T; \mathcal{H}_0)$. Let (u, z) denote the solution of the controlled system

$$du(t) + [Au(t) + B(u(t)) + \nabla z(t)]dt = f(t)dt + \sigma(t, u(t))v(t)dt; \tag{4.14}$$

$$dz(t) + \text{Div}(hu(t))dt = 0. \tag{4.15}$$

Take $w_n = u_n - u; \zeta_n = z_n - z$. Then

$$dw_n(t) + [Aw_n(t) + (B(u_n(t)) - B(u(t))) + \nabla \zeta_n(t)]dt = [\sigma(t, u_n(t))v_n(t) - \sigma(t, u(t))v(t)]dt; \tag{4.16}$$

$$d\zeta_n(t) + \text{Div}(hw_n(t))dt = 0. \tag{4.17}$$

Taking inner product of the first equation with w_n ,

$$\begin{aligned} \frac{1}{2}d\|w_n(t)\|_{\mathbb{L}^2}^2 + [(Aw_n(t), w_n(t)) + (B(u_n(t)) - B(u(t)), u_n(t) - u(t)) \\ + (\nabla \zeta_n(t), w_n(t))]dt = (\sigma(t, u_n(t))v_n(t) - \sigma(t, u(t))v(t), w_n(t))dt. \end{aligned}$$

Making use of (2.13) and (2.14), the Cauchy-Schwarz and Young's inequalities, and then integrating

$$\begin{aligned} \|w_n(t)\|_{\mathbb{L}^2}^2 + 2\alpha \int_0^t \|\nabla w_n(s)\|_{\mathbb{L}^2}^2 ds \leq \int_0^t \left[\frac{4}{\alpha} \|\zeta_n(s)\|_{\mathbb{L}^2}^2 + \frac{\alpha}{4} \|\nabla w_n(s)\|_{\mathbb{L}^2}^2 \right] ds \\ + 2 \int_0^t \|\sigma(s, u_n(s))v_n(s) - \sigma(s, u(s))v(s)\|_{\mathbb{L}^2} \|w_n(s)\|_{\mathbb{L}^2} ds. \end{aligned} \tag{4.18}$$

Next we only have to estimate the last term in the right hand side of the above estimate. For this consider

$$\begin{aligned} &\|\sigma(s, u_n(s))v_n(s) - \sigma(s, u(s))v(s)\|_{\mathbb{L}^2} \\ &= \|(\sigma(s, u_n(s)) - \sigma(s, u(s)))v_n(s) + \sigma(s, u(s))(v_n(s) - v(s))\|_{\mathbb{L}^2} \\ &\leq \|\sigma(s, u_n(s)) - \sigma(s, u(s))\|_{L^Q} \|v_n(s)\|_0 + \|\sigma(s, u(s))(v_n(s) - v(s))\|_{\mathbb{L}^2}. \end{aligned}$$

Hence (4.18) becomes

$$\begin{aligned} \|w_n(t)\|_{\mathbb{L}^2}^2 + 2\alpha \int_0^t \|\nabla w_n(s)\|_{\mathbb{L}^2}^2 ds &\leq \int_0^t \left[\frac{4}{\alpha} \|\zeta_n(s)\|_{\mathbb{L}^2}^2 + \frac{\alpha}{4} \|\nabla w_n(s)\|_{\mathbb{L}^2}^2 \right] ds \\ &+ \frac{\alpha}{4C_1} \int_0^t \|\sigma(s, u_n(s)) - \sigma(s, u(s))\|_{L^Q}^2 ds + \frac{4C_1}{\alpha} \int_0^t \|v_n(s)\|_0^2 \|w_n(s)\|_{\mathbb{L}^2}^2 ds \\ &+ \int_0^t \|\sigma(s, u(s))(v_n(s) - v(s))\|_{\mathbb{L}^2}^2 ds + \int_0^t \|w_n(s)\|_{\mathbb{L}^2}^2 ds. \end{aligned} \quad (4.19)$$

Adding this with that of the equation obtained after taking inner product of (4.17) with ζ^n yields

$$\begin{aligned} \|w_n(t)\|_{\mathbb{L}^2}^2 + \|\zeta_n(t)\|_{\mathbb{L}^2}^2 + \alpha \int_0^t \|\nabla w_n(s)\|_{\mathbb{L}^2}^2 ds &\leq K \int_0^t [\|w_n(s)\|_{\mathbb{L}^2}^2 + \|\zeta_n(s)\|_{\mathbb{L}^2}^2] ds \\ &+ \frac{4C_1}{\alpha} \int_0^t \|v_n(s)\|_0^2 \|w_n(s)\|_{\mathbb{L}^2}^2 ds + \int_0^t \|\sigma(s, u(s))(v_n(s) - v(s))\|_{\mathbb{L}^2}^2 ds \end{aligned} \quad (4.20)$$

with $K = \max \left\{ \frac{\alpha}{4} + L + 1, \frac{2(\mu^2+2)}{\alpha} + L \right\}$. Now making use of Gronwall's inequality

$$\begin{aligned} &\sup_{0 \leq t \leq T} [\|w_n(t)\|_{\mathbb{L}^2}^2 + \|\zeta_n(t)\|_{\mathbb{L}^2}^2] + \alpha \int_0^T \|\nabla w_n(s)\|_{\mathbb{L}^2}^2 ds \\ &\leq C \left\{ \int_0^T \|\sigma(s, u(s))(v_n(s) - v(s))\|_{\mathbb{L}^2}^2 ds \right\} \exp \left(KT + \frac{4C_1}{\alpha} \int_0^T \|v_n(s)\|_0^2 ds \right) \end{aligned} \quad (4.21)$$

for some $C > 0$. Since v_n converges weakly to v in $\mathbb{L}^2(0, T; \mathcal{H}_0)$ and σ is a Hilbert-Schmidt operator and so compact, we get that $w_n \rightarrow 0$ in $\mathbb{C}([0, T]; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^2(0, T; \mathbb{H}_0^1(\mathcal{O}))$ and $\zeta_n \rightarrow 0$ in $\mathbb{C}([0, T]; \mathbb{L}^2(\mathcal{O}))$, thereby proving the compactness. \square

Theorem 4.6 (Weak Convergence). *Let $\{v^\epsilon : \epsilon > 0\} \subset \mathcal{A}_M$ converges in distribution to v with respect to the weak topology on $\mathbb{L}^2(0, T; \mathcal{H}_0)$. Then*

$\mathcal{G}^\epsilon \left(W(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^\cdot v^\epsilon(s) ds \right)$ converges in distribution to $\mathcal{G}^0 \left(\int_0^\cdot v(s) ds \right)$ in \mathcal{Z} as $\epsilon \rightarrow 0$.

Proof. Let v^ϵ converge to v in distribution as random elements taking values in S_M where S_M is equipped with the weak topology. Let $\{u^\epsilon, z^\epsilon\}$ denote the solution of the stochastic control equation

$$\begin{aligned} du^\epsilon(t) + [Au^\epsilon(t) + B(u^\epsilon(t)) + \nabla z^\epsilon(t)]dt &= f(t)dt + \sigma(t, u^\epsilon(t))v^\epsilon(t)dt \\ &+ \sqrt{\epsilon}\sigma(t, u^\epsilon(t))dW(t); \end{aligned} \quad (4.22)$$

$$dz^\epsilon(t) + \text{Div}(hu^\epsilon(t)) = 0 \quad (4.23)$$

with $(u^\epsilon(0), z^\epsilon(0)) = (u_0, z_0)$. Then Girsanov's theorem can be invoked to show that (u^ϵ, z^ϵ) can be represented as $(u^\epsilon, z^\epsilon) = \mathcal{G}^\epsilon \left(W(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^\cdot v^\epsilon(s) ds \right)$. Define $\mathcal{G}^0 : \mathbb{C}([0, T] : \mathcal{H}_0) \rightarrow \mathcal{Z}$ by

$$\mathcal{G}^0(h) = \begin{cases} (u, z), & \text{if } h = \int_0^\cdot v(s) ds \text{ for some } v \in \mathbb{L}^2(0, T; \mathcal{H}_0), \\ 0, & \text{otherwise,} \end{cases} \quad (4.24)$$

where (u, z) denotes the solution of the controlled equation

$$du(t) + [Au(t) + B(u(t)) + \nabla z(t)]dt = f(t)dt + \sigma(t, u(t))v(t)dt; \quad (4.25)$$

$$dz(t) + \text{Div}(hu(t)) = 0 \quad (4.26)$$

with $(u(0), z(0)) = (u_0, z_0)$. Since S_M is Polish, the Skorokhod representation theorem can be introduced to construct processes $(\tilde{v}^\epsilon, \tilde{v}, \tilde{W})$ such that the joint distribution of $(\tilde{v}^\epsilon, \tilde{W})$ is the same as that of (v^ϵ, W) , and the distribution of \tilde{v} coincides with that of v , and $\tilde{v}^\epsilon \rightarrow \tilde{v}$ a.s. in the weak topology of S_M . Let $w^\epsilon = u^\epsilon - u$ and $\zeta^\epsilon = z^\epsilon - z$. Then

$$\begin{aligned} dw^\epsilon(t) + [Aw^\epsilon(t) + (B(u^\epsilon(t)) - B(u(t))) + \nabla \zeta^\epsilon(t)]dt \\ = [\sigma(t, u^\epsilon(t))v^\epsilon(t) - \sigma(t, u(t))v(t)]dt + \sqrt{\epsilon}\sigma(t, u^\epsilon(t))dW(t); \end{aligned} \quad (4.27)$$

$$d\zeta^\epsilon(t) + \text{Div}(hw^\epsilon(t)) = 0. \quad (4.28)$$

By Itô formula,

$$\begin{aligned} d\|w^\epsilon(t)\|_{\mathbb{L}^2}^2 + 2[(Aw^\epsilon(t), w^\epsilon(t)) + (B(u^\epsilon(t)) - B(u(t)), u^\epsilon(t) - u(t)) \\ + (\nabla \zeta^\epsilon(t), w^\epsilon(t))]dt = 2(\sigma(t, u^\epsilon(t))v^\epsilon(t) - \sigma(t, u(t))v(t), w^\epsilon(t))dt \\ + \epsilon \text{Tr}(\sigma(t, u^\epsilon(t))Q\sigma^*(t, u^\epsilon(t)))dt + 2\sqrt{\epsilon}(w^\epsilon(t), \sigma(t, u^\epsilon(t))dW(t)). \end{aligned}$$

Using (2.13) and (2.14), then applying Cauchy-Schwarz and Young's inequalities, and then integration yields,

$$\begin{aligned} \|w^\epsilon(t)\|_{\mathbb{L}^2}^2 + 2\alpha \int_0^t \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \leq \frac{4}{\alpha} \int_0^t \|\zeta^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + \frac{\alpha}{4} \int_0^t \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\ + 2 \int_0^t (\sigma(s, u^\epsilon(s))v^\epsilon(s) - \sigma(s, u(s))v(s), w^\epsilon(s))ds \\ + \epsilon \int_0^t \text{Tr}(\sigma(s, u^\epsilon(s))Q\sigma^*(s, u^\epsilon(s)))ds + 2\sqrt{\epsilon} \int_0^t (w^\epsilon(s), \sigma(s, u^\epsilon(s))dW(s)). \end{aligned} \quad (4.29)$$

Define the stopping time argument,

$$\begin{aligned} \tau_{N,\epsilon} = \inf \left\{ t : \sup_{0 \leq s \leq t} [\|u(s)\|_{\mathbb{L}^2}^2 + \|u^\epsilon(s)\|_{\mathbb{L}^2}^2] > N \right. \\ \left. \text{or } \int_0^t (\|\nabla u^\epsilon(s)\|_{\mathbb{L}^2}^2 + \|\nabla u(s)\|_{\mathbb{L}^2}^2)ds > N \right\}. \end{aligned}$$

Then estimating the expectation of the stochastic integral term in (4.29) by means of the Burkholder-Davis-Gundy inequality as done earlier,

$$\begin{aligned} \mathbb{E} \left\{ \sup_{0 \leq t \leq T \wedge \tau_{N,\epsilon}} \left| \int_0^t (w^\epsilon(s), \sigma(s, u^\epsilon(s))dW(s)) \right| \right\} \\ \leq \sqrt{2}C_2 \left(\mathbb{E} \left[\sup_{0 \leq t \leq T \wedge \tau_{N,\epsilon}} \|w^\epsilon(t)\|_{\mathbb{L}^2}^2 \right] + \mathbb{E} \int_0^{T \wedge \tau_{N,\epsilon}} \|u^\epsilon(s)\|_{\mathbb{H}^1}^2 ds + T \wedge \tau_{N,\epsilon} \right). \end{aligned} \quad (4.30)$$

Also consider the term

$$\begin{aligned} & (\sigma(s, u^\epsilon(s))v^\epsilon(s) - \sigma(s, u(s))v(s), w^\epsilon(s)) \\ &= ((\sigma(s, u^\epsilon(s)) - \sigma(s, u(s)))v^\epsilon(s) + \sigma(s, u(s))(v^\epsilon(s) - v(s)), w^\epsilon(s)) \\ &\leq C_1 \|u^\epsilon(s) - u(s)\|_{\mathbb{H}^1} \|v^\epsilon(s)\|_0 \|w^\epsilon(s)\|_{\mathbb{L}^2} + \|\sigma(s, u(s))(v^\epsilon(s) - v(s))\|_{\mathbb{L}^2} \|w^\epsilon(s)\|_{\mathbb{L}^2}. \end{aligned}$$

With this estimate, (4.29) becomes, after applying Cauchy-Schwarz inequality and our assumption (H3) on σ ,

$$\begin{aligned} & \|w^\epsilon(t)\|_{\mathbb{L}^2}^2 + \frac{3\alpha}{2} \int_0^t \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \leq \frac{4}{\alpha} \int_0^t \|\zeta^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + \frac{\alpha}{4} \int_0^t \|w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\ &+ \frac{4C_1}{\alpha} \int_0^t \|v^\epsilon(s)\|_0^2 \|w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + \int_0^t \|\sigma(s, u(s))(v^\epsilon(s) - v(s))\|_{\mathbb{L}^2}^2 ds \\ &+ \int_0^t \|w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + \epsilon C_2 \int_0^t (1 + \|u^\epsilon(s)\|_{\mathbb{H}^1}^2) ds \\ &+ 2\sqrt{\epsilon} \left| \int_0^t (w^\epsilon(s), \sigma(s, u^\epsilon(s))dW(s)) \right|. \end{aligned}$$

Taking inner product of (4.28) with ζ^ϵ , and adding with the above,

$$\begin{aligned} & \|w^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|\zeta^\epsilon(t)\|_{\mathbb{L}^2}^2 + \alpha \int_0^t \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \leq K \int_0^t [\|w^\epsilon(s)\|_{\mathbb{L}^2}^2 + \|\zeta^\epsilon(s)\|_{\mathbb{L}^2}^2] ds \\ &+ \frac{4C_1}{\alpha} \int_0^t \|v^\epsilon(s)\|_0^2 \|w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds + \int_0^t \|\sigma(s, u(s))(v^\epsilon(s) - v(s))\|_{\mathbb{L}^2}^2 ds \\ &+ \epsilon C_2 \int_0^t (1 + \|\nabla u^\epsilon(s)\|_{\mathbb{L}^2}^2) ds + 2\sqrt{\epsilon} \left| \int_0^t (w^\epsilon(s), \sigma(s, u^\epsilon(s))dW(s)) \right|, \end{aligned}$$

with $K = \max\{\frac{\alpha}{4} + \epsilon C_2 + L + 1, \frac{2\mu^2 + 4}{\alpha} + L\}$. Taking supremum over t upto $T \wedge \tau_{N, \epsilon}$, then taking expectation of the above inequality, and making use of (4.30),

$$\begin{aligned} & \mathbf{E} \left\{ \sup_{0 \leq t \leq T \wedge \tau_{N, \epsilon}} [\|w^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|\zeta^\epsilon(t)\|_{\mathbb{L}^2}^2] \right\} + \alpha \mathbf{E} \int_0^{T \wedge \tau_{N, \epsilon}} \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\ &\leq K \mathbf{E} \int_0^{T \wedge \tau_{N, \epsilon}} [\|w^\epsilon(s)\|_{\mathbb{L}^2}^2 + \|\zeta^\epsilon(s)\|_{\mathbb{L}^2}^2] ds + \frac{4C_1}{\alpha} \mathbf{E} \int_0^{T \wedge \tau_{N, \epsilon}} \|v^\epsilon(s)\|_0^2 \|w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\ &+ \mathbf{E} \int_0^{T \wedge \tau_{N, \epsilon}} \|\sigma(s, u(s))(v^\epsilon(s) - v(s))\|_{\mathbb{L}^2}^2 ds + \epsilon C_2 \mathbf{E} \int_0^{T \wedge \tau_{N, \epsilon}} (1 + \|\nabla u^\epsilon(s)\|_{\mathbb{L}^2}^2) ds \\ &+ 2\sqrt{2\epsilon} C_2 \left(\mathbf{E} \left\{ \sup_{0 \leq t \leq T \wedge \tau_{N, \epsilon}} \|w^\epsilon(t)\|_{\mathbb{L}^2}^2 \right\} + N + NT + T \right). \end{aligned}$$

If $\epsilon > 0$ is chosen so that $\epsilon < \frac{\alpha}{4C_1} \wedge \frac{\alpha}{32C_2^2}$, then applying Gronwall's inequality,

$$\begin{aligned} & \mathbf{E} \left\{ \sup_{0 \leq t \leq T \wedge \tau_{N, \epsilon}} [\|w^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|\zeta^\epsilon(t)\|_{\mathbb{L}^2}^2] \right\} + \frac{\alpha}{2} \mathbf{E} \int_0^{T \wedge \tau_{N, \epsilon}} \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \\ &\leq C \left\{ \mathbf{E} \int_0^{T \wedge \tau_{N, \epsilon}} \|\sigma(s, u(s))(v^\epsilon(s) - v(s))\|_{\mathbb{L}^2}^2 ds + \epsilon C_2 T \right\} \end{aligned}$$

$$+2\sqrt{2}\epsilon C_2(N + NT + T) \left\} \exp \left((K + 1)T + \frac{4C_1}{\alpha} \mathbb{E} \int_0^T \|v^\epsilon(s)\|_0^2 ds \right).$$

It can be easily shown as before that $T \wedge \tau_{N,\epsilon} \rightarrow T$ as $N \rightarrow \infty$. Hence as $\epsilon \rightarrow 0$, due to the weak convergence of $v^\epsilon \rightarrow v$, we conclude that

$$\mathbb{E} \left\{ \sup_{0 \leq t \leq T} [\|w^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|\zeta^\epsilon(t)\|_{\mathbb{L}^2}^2] \right\} + \frac{\alpha}{2} \mathbb{E} \int_0^T \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \rightarrow 0, \quad (4.31)$$

which implies that for any arbitrary $\delta > 0$,

$$\mathbb{P} \left(\left\{ \sup_{0 \leq t \leq T} [\|w^\epsilon(t)\|_{\mathbb{L}^2}^2 + \|\zeta^\epsilon(t)\|_{\mathbb{L}^2}^2] + \frac{\alpha}{2} \int_0^T \|\nabla w^\epsilon(s)\|_{\mathbb{L}^2}^2 ds \geq \delta \right\} \rightarrow 0 \text{ as } \epsilon \rightarrow 0, (4.32)$$

thus resulting in the weak convergence (convergence in distribution) of $w^\epsilon \rightarrow 0$ in $\mathbb{C}([0, T]; \mathbb{L}^2(\mathcal{O})) \cap \mathbb{L}^2(0, T; \mathbb{H}_0^1(\mathcal{O}))$ and $\zeta^\epsilon \rightarrow 0$ in $\mathbb{C}([0, T]; \mathbb{L}^2(\mathcal{O}))$. Thus the large deviation principle for the stochastic tidal dynamics equation (2.21)-(2.22) is established. \square

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