Some properties of the real line

Keng Wiboonton

Theorem 1 *Every open set of $\mathbb{R}$ can be written as a countable union of mutually disjoint open intervals. (Zorn's lemma is used in the proof.)*

In general, for $n > 1$, open sets in $\mathbb{R}^n$ cannot be written as a countable union of mutually disjoint open intervals. (A subset I of $\mathbb{R}^n$ is an interval if $I = I_1 \times \ldots \times I_n$ where $I_1, \ldots, I_n$ are intervals in $\mathbb{R}$.)

Exercise. Prove that the open unit disc $B(0;1) := \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$ in the plane cannot be as a countable union of mutually disjoint open intervals.

Theorem 2 *(**Heine-Borel**) A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

Theorem 3 *(**Bolzano-Weierstrass**) Every bounded, infinite set of real numbers has a limit point. (Recall that a point is said to be a limit point of A if it is the limit of a sequence of distinct terms from A.)*

Theorem 4 *Every connected subset of $\mathbb{R}$ is an interval.*

Problems

Problem 1. Let $A \subseteq \mathbb{R}$ be uncountable.
(a) Show that A has at least one limit point.
(b) Show that A has uncountably many limit points.

Problem 2. Let $E \subseteq \mathbb{Q}$ be the set of x whose decimal expansion is of the form $x = 0.d_1d_2 \ldots d_N$ for some $N \in \mathbb{N}$, and where $d_1, \ldots, d_N \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ (so $d_j \neq 0$ and $d_j \neq 9$ for j = 1, ..., N). Show that any compact subset of E is finite.
Can we drop the hypothesis that $d_j \neq 0$ and $d_j \neq 9$?

Problem 3. Prove that there exists no continuous bijection from $(0, 1)$ to $[0, 1]$.