Some properties of the real line

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**Theorem 1** Every open set of \( \mathbb{R} \) can be written as a countable union of mutually disjoint open intervals. (Zorn’s lemma is used in the proof.)

In general, for \( n > 1 \), open sets in \( \mathbb{R}^n \) cannot be written as a countable union of mutually disjoint open intervals. (A subset \( I \) of \( \mathbb{R}^n \) is an interval if \( I = I_1 \times \ldots \times I_n \) where \( I_1, \ldots, I_n \) are intervals in \( \mathbb{R} \).)

**Exercise.** Prove that the open unit disc \( B(0; 1) := \{(x, y) \in \mathbb{R}^2| x^2 + y^2 < 1 \} \) in the plane cannot be as a countable union of mutually disjoint open intervals.

**Theorem 2** *(Heine-Borel)* A subset of \( \mathbb{R} \) is compact if and only if it is closed and bounded.

**Theorem 3** *(Bolzano-Weierstrass)* Every bounded, infinite set of real numbers has a limit point. (Recall that a point is said to be a limit point of \( A \) if it is the limit of a sequence of distinct terms from \( A \).)

**Theorem 4** Every connected subset of \( \mathbb{R} \) is an interval.

**Problems**

**Problem 1.** Let \( A \subseteq \mathbb{R} \) be uncountable.
(a) Show that \( A \) has at least one limit point.
(b) Show that \( A \) has uncountably many limit points.

**Problem 2.** Let \( E \subseteq \mathbb{Q} \) be the set of \( x \) whose decimal expansion is of the form \( x = 0.d_1d_2\ldots d_N \) for some \( N \in \mathbb{N}, \) and where \( d_1, \ldots, d_N \in \{1, 2, 3, 4, 5, 6, 7, 8\} \) (so \( d_j \neq 0 \) and \( d_j \neq 9 \) for \( j = 1, \ldots, N \)). Show that any compact subset of \( E \) is finite.
Can we drop the hypothesis that \( d_j \neq 0 \) and \( d_j \neq 9 \)?

**Problem 3.** Prove that there exists no continuous bijection from \((0, 1)\) to \([0, 1]\).