G.E.A.U.X. MATH @ LSU: TOPOLOGY IV: CONNECTEDNESS

Definition 0.1. Let X be a topological space. A *separation* of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be *connected* if there does not exist a separation of X.

Proposition 0.2. *A space X is connected if and only if the only subsets of X that are both open and closed in X are the empty set and X itself.*

Proof. For if A is a nonempty proper subset of X that is both open and closed in X, then the sets $U = A$ and $V = X - A$ constitute a separation of X for they are open, disjoint, and nonempty, and their union is X. Conversely, if U and V form a separation of X, then U is nonempty and different from X, and it is both open and closed in X. $\square$

Lemma 0.3. *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no separation of Y.*

Proof. Suppose first that A and B form a separation of Y. Then A is both open and closed in Y. The closure of A in Y is the set $\overline{A} \cap Y$ (where $\overline{A}$ denotes the closure of A in X). Since A is closed in Y, $A = \overline{A} \cap Y$; or to say the same thing, $\overline{A} \cap B = \emptyset$. Since $\overline{A}$ is the union of A and its limit points, B contains no limit points of A. A similar argument shows that A contains no limit points of B.

Conversely, suppose that A and B are disjoint nonempty sets whose union is Y, neither of which contains a limit point of the other. Then $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$; therefore, we conclude that $\overline{A} \cap Y = A$ and $\overline{B} \cap Y = B$. Thus both A and B are closed in Y, and since $A = Y - B$ and $B = Y - A$, they are open in Y as well. $\square$

Example 0.4. (1) $Y = [-1, 0) \cup (0, 1] \subset \mathbb{R}$
(2) $Y = [-1, 1] \subset \mathbb{R}$
(3) $\mathbb{Q} \subset R$
(4) $Y = \{x \times y | y = 0\} \cup \{x \times y | x > 0 \text{ and } y = 1/x\} \subset \mathbb{R}^2$

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Lemma 0.5. *If the sets C and D form a separation of X, and if Y is a connected subspace of X, then Y lies entirely within either C or D.*

Proof. Since C and D are both open in X, the sets $C \cap Y$ and $D \cap Y$ are open in Y. These two sets are disjoint and their union is Y; if they were both nonempty, they would constitute a separation of Y. Therefore, one of them is empty. Hence Y must lie entirely in C or in D. $\square$

Theorem 0.6. *The union of a collection of connected subspaces of X that have a point in common is connected.*

Proof. Let $\{A_\alpha\}$ be a collection of connected subspaces of a space X; let p be a point of $\cap A_\alpha$. We prove that the space $Y = \cup A_\alpha$ is connected. Suppose that $Y = C \cup D$ is a separation of Y. The point p is in one of the sets C or D; suppose $p \in C$. Since A_α is connected, it must lie entirely in either C or D, and it cannot lie in D because it contains the point p of C. Hence $A_\alpha \subset C$ for every α, so that $\cup A_\alpha \subset C$, contradicting the fact that D is nonempty. $\square$

Theorem 0.7. *Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is also connected.*

Proof. Let A be connected and let $A \subset B \subset \overline{A}$. Suppose that $B = C \cup D$ is a separation of B. By Lemma 0.5, the set A must lie entirely in C or in D; suppose that $A \subset C$. Then $\overline{A} \subset \overline{C}$; since $\overline{C}$ and D are disjoint, B cannot intersect D. This contradicts the fact that D is a nonempty subset of B. $\square$

Theorem 0.8. *A finite cartesian product of connected spaces is connected.*

Proof. Proceed by induction on the number of connected spaces in the cartesian product, but only the base case $X \times Y$ is presented below.

Choose a "base point" $a \times b \in X \times Y$. Note that $X \times b$ is connected, as is $a \times Y$. The space $T_x := (X \times b) \cup (x \times Y)$ is connected for each x because these share the point $x \times b$. Then the union $\cup T_x = X \times Y$ is connected because each connected space T_x contains the point $a \times b$. $\square$

1. Exercises

Exercise 1. *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. If $\mathcal{T}$ is coarser than $\mathcal{T}'$, what does connectedness of X in one topology say about connectedness in the other?*

Exercise 2. *Let $\{A_\alpha\}$ be a collection of connected subspaces of X; let A be a connected subspace of X.*

(1) *If $A_n \cap A_{n+1} \neq \emptyset$ for all n, show $\cup A_n$ is connected.*
(2) *If $A \cap A_\alpha \neq \emptyset$ for all α, show $A \cup (\cup A_\alpha)$ is connected.*

Exercise 3. *Let $A \subset X$. Show that if C is a connected subspace of X that intersects both A and $X - A$, then C intersects the boundary of A.*

Exercise 4. *Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) - (A \times B)$ is connected.*

Exercise 5. *Let $Y \subset X$; let X and Y be connected. Show that if A and B form a separation of $X - Y$, then $Y \cup A$ and $Y \cup B$ are connected.*

References

[1] James Munkries, Topology, Second Ed., Chapter 3 Section 23, pp.148-152