Lower bounds on Field Concentrations

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September 19 2007

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Introduction

In heterogeneous media the initiation of failure is a multi-scale phenomena.

If you apply a load at the structural scale, the load is often amplified by the microstructure creating local zones of high field concentration.

We will work with gradient fields associated with intensive quantities given by electric potential inside heterogeneous media.

Field concentrations are measured using the $\mathbf{L}^p$ norm of the gradient field.

Notation

- Ω is a bounded open subset of $\mathbf{R}^n$.
- $\mathbf{Y} = (0,1)^n$ is the unit cube in $\mathbf{R}^n$.
- Let $p \geq 2$ and let q such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

- Let $L^p_n(D) = \left\{ u : D \longrightarrow \mathbf{R}^n : \int_D |u(x)|^p \, dx < \infty \right\}$. This space is a Banach space and its norm is defined by $\|u\|_{L^p_n(D)} = \left(\int_D |u(x)|^p \, dx \right)^{1/p}$.

Notation

- Let $W^{1,p}_{per}(Y)$ be the set of all functions $u \in \mathbf{W}^{1,p}(Y)$ with mean value zero which have the same trace on the opposite faces of Y.
- We consider N-phase materials. The characteristic function for the i-th material $\chi_i(y)$ is Y-periodic and $\sum_{i=1}^{N} \chi_i(y) = 1$.
- Let $A : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be defined by
$A(y, \lambda) = \sum_{i=1}^{N} \chi_i(y) a_i \left|\lambda\right|^{p-2} \lambda$, with $a_i \geq 0$.

Structure Conditions on A

- For every $\lambda \in \mathbf{R}^n$, $A(\cdot, \lambda)$ is Y-periodic and Lebesgue measurable.
- $|A(y, 0)| \leq C_0$ for a.e $y \in \mathbf{R}^n$.
- **Continuity**: For a.e $y \in \mathbf{R}^n$,

$$|A(y, \lambda_1) - A(y, \lambda_2)| \leq C_1 |\lambda_1 - \lambda_2|^\alpha (|\lambda_1| + |\lambda_2| + 1)^{p-1-\alpha}$$

- **Monotoniciy**: For a.e $y \in \mathbf{R}^n$,

$$(A(y, \lambda_1) - A(y, \lambda_2), \lambda_1 - \lambda_2) \geq C_2 |\lambda_1 - \lambda_2|^\beta (|\lambda_1| + |\lambda_2| + 1)^{p-\beta}$$

Properties of A

- For every $\lambda \in \mathbf{R}^n$, $A(\cdot, \lambda)$ is Y-periodic and Lebesgue measurable.
- Have $|A(y, 0)| = 0$ for all $y \in \mathbf{R}^n$.
- **Continuity**

$$|A(y, \lambda_1) - A(y, \lambda_2)| \leq C_1 |\lambda_1 - \lambda_2| (|\lambda_1| + |\lambda_2| + 1)^{p-2} .$$

- **Monotoniciy**

$$(A(y, \lambda_1) - A(y, \lambda_2), \lambda_1 - \lambda_2) \geq C_2 |\lambda_1 - \lambda_2|^p$$

Continuity

$$
\begin{aligned}
|A(y,\lambda_1) - A(y,\lambda_2)| &= \left| \sum_{i=1}^{N} \chi_i(y) a_i \left[|\lambda_1|^{p-2} \lambda_1 - |\lambda_2|^{p-2} \lambda_2 \right] \right| \\
&\leq M \left| |\lambda_1|^{p-2} \lambda_1 - |\lambda_2|^{p-2} \lambda_2 \right| .
\end{aligned}
$$

We work with

$$
\begin{aligned}
& \left| |\lambda_1|^{p-2} \lambda_1 - |\lambda_2|^{p-2} \lambda_2 \right|^2 \\
= \; & |\lambda_1|^{2(p-1)} + |\lambda_2|^{2(p-1)} - 2 |\lambda_1|^{p-2} |\lambda_2|^{p-2} \lambda_1 \cdot \lambda_2 .
\end{aligned}
$$

Substract and add $2 |\lambda_1|^{p-1} |\lambda_2|^{p-1}$ to get

Continuity

$$
\begin{aligned}
& \left| |\lambda_1|^{p-2}\lambda_1 - |\lambda_2|^{p-2}\lambda_2 \right|^2 \\
= \; & \left(|\lambda_1|^{p-1} - |\lambda_2|^{p-1} \right)^2 + \left(|\lambda_1|^{p-2}|\lambda_2|^{p-2} \right) 2 \left(|\lambda_1||\lambda_2| - \lambda_1 \cdot \lambda_2 \right) \\
\leq \; & \left| |\lambda_1|^{p-1} - |\lambda_2|^{p-1} \right|^2 + \left(|\lambda_1|^{p-2} + |\lambda_2|^{p-2} \right)^2 |\lambda_1 - \lambda_2|^2 \\
\leq \; & (p-1)^2 \left(|\lambda_1|^{p-2} + |\lambda_2|^{p-2} \right)^2 |\lambda_1 - \lambda_2|^2 \;\; \textbf{(*)} \\
+ \; & \left(|\lambda_1|^{p-2} + |\lambda_2|^{p-2} \right)^2 |\lambda_1 - \lambda_2|^2 \\
\leq \; & K^2 \left[(1 + |\lambda_1| + |\lambda_2|)^{p-2} \right]^2 |\lambda_1 - \lambda_2|^2 .
\end{aligned}
$$

Then we have

$$
|A(y,\lambda_1) - A(y,\lambda_2)| \leq C_1 |\lambda_1 - \lambda_2| \left(|\lambda_1| + |\lambda_2| + 1 \right)^{p-2} .
$$

(*)Convexity

Consider $f(x) = x^{p-1}$, with $p \geq 2$, defined on $\mathbf{R}^+$.
Have f is a convex function and therefore satisfies:

$$\begin{cases} f'(|\lambda_1|)(|\lambda_2| - |\lambda_1|) \leq f(|\lambda_2|) - f(|\lambda_1|), \\ f'(|\lambda_2|)(|\lambda_1| - |\lambda_2|) \leq f(|\lambda_1|) - f(|\lambda_2|); \end{cases}$$

i.e,

$$\begin{cases} (p-1)|\lambda_1|^{p-2}(|\lambda_2| - |\lambda_1|) \leq |\lambda_2|^{p-1} - |\lambda_1|^{p-1}, \\ (p-1)|\lambda_2|^{p-2}(|\lambda_1| - |\lambda_2|) \leq |\lambda_1|^{p-1} - |\lambda_2|^{p-1}. \end{cases}$$

(*)Convexity

Then

$$
\begin{aligned}
(p-1)\,|\lambda_2|^{p-2}\,(|\lambda_1|-|\lambda_2|) &\leq |\lambda_1|^{p-1}-|\lambda_2|^{p-1} \\
&\leq (p-1)\,|\lambda_1|^{p-2}\,(|\lambda_1|-|\lambda_2|)\,.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
\left||\lambda_1|^{p-1}-|\lambda_2|^{p-1}\right| &\leq (p-1)\left(|\lambda_1|^{p-2}+|\lambda_2|^{p-2}\right)||\lambda_1|-|\lambda_2|| \\
&\leq (p-1)\left(|\lambda_1|^{p-2}+|\lambda_2|^{p-2}\right)|\lambda_1-\lambda_2|\,.
\end{aligned}
$$

Preliminary Results: Homogenization Theorem

Set $\epsilon_k = \frac{1}{k} > 0$, $k = 1, 2, \dots$.

$$A^{\epsilon_k}(x, \lambda) = A\left(\frac{x}{\epsilon_k}, \lambda\right)$$

$$\chi_i^{\epsilon_k}(x) = \chi_i\left(\frac{x}{\epsilon_k}\right),$$

for every $x \in \mathbf{R}^n$ and every $\lambda \in \mathbf{R}^n$.
Consider the Dirichlet problem

$$\begin{cases} -div\left(A^{\epsilon_k}\left(x, \nabla u^{\epsilon_k}\right)\right) = f \text{ on } \Omega, \\ u^{\epsilon_k} \in \mathbf{W}_0^{1,p}(\Omega); \end{cases} \tag{1}$$

$f \in \mathbf{W}^{-1,q}(\Omega)$.

Preliminary Results: Homogenization Theorem

Have $u_{\epsilon_k} \rightharpoonup u^H$ in $W^{1,p}(\Omega)$ as $\epsilon_k \to 0$, where u^H is solution of

$$\begin{cases} -div\left(b\left(\nabla u^H\right)\right) = f \text{ on } \Omega, \\ u^H \in \mathbf{W}_0^{1,p}(\Omega); \end{cases} \tag{2}$$

where the monotone map $b : \mathbf{R}^n \to \mathbf{R}^n$(independent of f and Ω) is defined for all $\xi \in \mathbf{R}^n$ by

$$b(\xi) = \int_Y A(y, \xi + \nabla \upsilon(y))dy, \tag{3}$$

where υ is the solution to the cell problem:

$$\begin{cases} \int_Y \left(A(y, \xi + \nabla \upsilon), \nabla w\right) dy = 0 \text{ for every } w \in \mathbf{W}_{per}^{1,p}(Y), \\ \upsilon \in \mathbf{W}_{per}^{1,p}(Y). \end{cases} \tag{4}$$

Preliminary Results: Corrector Theory

- $Y^i_{\epsilon_k} = \epsilon_k(i + Y)$, where $i \in \mathbf{Z}^n$.
- $I_{\epsilon_k} = \left\{ i \in \mathbf{Z}^n : Y^i_{\epsilon_k} \subset \Omega \right\}$.
- $\Omega_k = \cup Y^i_{\epsilon_k}$, for $i \in I_{\epsilon_k}$.
- Let $\varphi \in \mathbf{L}^p_n(\Omega)$ and $M_{\epsilon_k}\varphi : \mathbf{R}^n \to \mathbf{R}^n$ be a funtion defined by

$$M_{\epsilon_k}(\varphi)(x) = \sum_{i \in I_{\epsilon_k}} \chi_{Y^i_{\epsilon_k}}(x) \frac{1}{\left| Y^i_{\epsilon_k} \right|} \int_{Y^i_{\epsilon_k}} \varphi(y) dy. \qquad (5)$$

Preliminary Results: Corrector Theory

- $\|M_{\epsilon_k}(\varphi) - \varphi\|_{\mathbf{L}^p_n(\Omega)} \to 0$.
- $M_{\epsilon_k}(\varphi) \to \varphi$ a.e. on Ω.
- By Jensen's inequality: $\|M_{\epsilon_k}(\varphi)\|_{\mathbf{L}^p_n(\Omega)} \leq \|\varphi\|_{\mathbf{L}^p_n(\Omega)}$.
- $P : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ defined by

$$P(x, \xi) = \xi + \nabla \upsilon(x)$$

  where υ is the unique solution of (4).
- $P(\cdot, \xi)$ is Y-periodic and $P_{\epsilon_k}(x, \xi) = P(\frac{x}{\epsilon_k}, \xi)$ is ϵ_k-periodic in x.
- $P(\cdot, \xi) \rightharpoonup \xi$ in $\mathbf{L}^p_n(\Omega)$.

Preliminary Results: Corrector Theory

Taking $\varphi = \nabla u^H$ in (5), we get:

$$M_{\epsilon_k}(\nabla u^H)(x) = \sum_{i \in I_{\epsilon_k}} \chi_{Y^i_{\epsilon_k}}(x) \frac{1}{\left|Y^i_{\epsilon_k}\right|} \int_{Y^i_{\epsilon_k}} \nabla u^H(y) dy.$$

Therefore by the Corrector Theorem (*Theorem 2.1* of [1]), we have

$$\left\| P\left(\frac{x}{\epsilon_k}, M_{\epsilon_k}(\nabla u^H)(x)\right) - \nabla u^{\epsilon_k}(x) \right\|_{\mathbf{L}^p_n(\Omega)} \to 0, \qquad (6)$$

Preliminary Results: Young Measures

- A function $\psi(x, \lambda) : \Omega \times \mathbf{R}^m \to \mathbf{R}$ that is measurable in x and continuous in λ is called a *Caratheodory function*.
- A Young measure υ is a family $\{\upsilon_x\}_{x \in \Omega}$ of probability measures associated with a sequence $\{f_j\}_{j=1}^{\infty}$, $f_j : \Omega \subseteq \mathbf{R}^n \longrightarrow \mathbf{R}^n$ such that the support of $\upsilon_x \subset \mathbf{R}^n$ and they depend measurably on $x \in \Omega$, i.e, for all $\varphi : \mathbf{R}^n \longrightarrow \mathbf{R}$, the function

$$\bar{\varphi}(x) = \int_{\mathbf{R}^m} \varphi(\lambda) d\upsilon_x(\lambda) = \langle \varphi, \upsilon_x \rangle$$

is measurable.

Preliminary Results: Young Measures

(**THEOREM 6.2 from Pedregal's book**)
Let $\Omega \subset \mathbf{R}^N$ be a measurable set and let $z_j : \Omega \longrightarrow \mathbf{R}^m$ be measurable functions such that

$$\sup_j \int_\Omega g(|z_j|)dx < \infty,$$

where $g : [0, \infty) \longrightarrow [0, \infty]$ is continuous, non-decreasing function such that $\lim_{t\to\infty} g(t) = \infty$. There exists a subsequence, not relabeled, and a family of probability measures, $\upsilon = \{\upsilon_x\}_{x\in\Omega}$ such that whenever $\{\psi(x, z_j(x))\}$ is weakly convergent in $\mathbf{L}^1(\Omega)$, for all Caratheodory function $\psi(x, \lambda) : \Omega \times \mathbf{R}^m \to \mathbf{R}^* = \mathbf{R} \cup \{+\infty\}$, the weak limit is the function

$$\bar{\psi}(x) = \int_{\mathbf{R}^m} \psi(x, \lambda) d\upsilon_x(\lambda).$$

Preliminary Results: Young Measures

By taking $g(t) = t^p$ for $p \geq 1$ we get that *every bounded sequence in* $\mathbf{L}^p(\Omega)$ *contains a subsequence that generates a parametrized measure* in the sense of the previous theorem.
In order to identify the parametrized measure associated to a particular sequence of functions z_j (obtained perhaps in some constructive way or using some scheme), it is enough to check

$$\lim_{j\to\infty} \int_\Omega \xi(x)\varphi(z_j(x))dx = \int_\Omega \xi(x) \int_{\mathbf{R}^m} \varphi(\lambda)d\upsilon_x(\lambda)dx$$

for ξ and φ belonging to dense, countable subsets of $\mathbf{L}^1(\Omega)$ and $\mathbf{C}_0(\mathbf{R}^m)$, respectively.

Preliminary Results: Young Measures

(**LEMMA 6.3 from Pedregal's book**)
Let $\{z_j\}$ and $\{w_j\}$ bounded sequences in $\mathbf{L}^p(\Omega)$.

- If $|\{z_j \neq w_j\}| \to 0$, the parametrized measure for both sequences is the same.
- If $\|w_j - zj\|_{\mathbf{L}^p(\Omega)} \to 0$, then $\{z_j\}$ and $\{w_j\}$ share the same parametrized measure.

Preliminary Results: Young Measures

(**THEOREM 6.11 from Pedregal's book**)
If $\{z_j\}$ is a sequence of measurable functions with associated parametrized measure $\upsilon = \{\upsilon_x\}_{x \in \Omega}$, then for all Caratheodory function $\psi \geq 0$ and $E \subset \Omega$ measurable we have

$$\int_E \int_{\mathbf{R}^n} \psi(x, \lambda) d\upsilon_x(\lambda) dx \leq \liminf_{j \to \infty} \int_E \psi(x, z_j(x)) dx.$$

Before Main Result

By (6) and previous lemma:

$$\left\{ P\left(\frac{x}{\epsilon_k}, M_{\epsilon_k}(\nabla u^H)(x)\right)\right\}_{k\geq 0} \text{ and } \{\nabla u^{\epsilon_k}(x)\}_{k\geq 0}$$

have the same Young Measure, i.e. we have $\nu = \{\nu_x\}_{x\in\Omega}$ such that

$$\begin{aligned} &\int_\Omega \zeta(x)\int_{\mathbf{R}^n}\phi(\lambda)d\nu_x(\lambda)dx \\ =\ &\lim_{k\to\infty}\int_\Omega \zeta(x)\phi\left(P\left(\frac{x}{\epsilon}, M_{\epsilon_k}(\nabla u^H)(x)\right)\right)dx \\ =\ &\lim_{k\to\infty}\int_\Omega \zeta(x)\phi\left(\nabla u^{\epsilon_k}(x)\right)dx, \end{aligned}$$

for all $\phi \in \mathbf{C}_0(\mathbf{R}^n)$ and for all $\zeta \in \mathbf{C}_0^\infty(\mathbf{R}^n)$.

Main Result

Using the notation above, we have:

$$\lim_{k\to\infty}\int_\Omega \zeta(x)\phi\left(P\left(\frac{x}{\epsilon_k}, M_{\epsilon_k}\left(\nabla u^H\right)(x)\right)\right)dx$$
$$= \int_\Omega \zeta(x)\int_Y \phi(P(y,\nabla u^H(x)))dydx.$$

Therefore

$$\int_\Omega \zeta(x)\int_{\mathbf{R}^n}\phi(\lambda)d\nu_x(\lambda)dx = \int_\Omega \zeta(x)\int_Y \phi(P(y,\nabla u^H(x)))dydx.$$

for all $\phi \in \mathbf{C}_0(\mathbf{R}^n)$ and for all $\zeta \in \mathbf{C}_0^\infty(\mathbf{R}^n)$.

Main Result

Have for all Caratheodory function $\psi \geq 0$ and $D \subset \Omega$ measurable

$$\begin{aligned} \int_D \int_{\mathbf{R}^n} \psi(x,\lambda) d\upsilon_x(\lambda) dx &= \int_D \int_Y \psi(x,(P(y,\nabla u^H(x)))) dy dx \\ &\leq \liminf_{k\to\infty} \int_D \psi(x,\nabla u^\epsilon(x)) dx. \end{aligned}$$

In particular

$$\int_D \int_Y \left| P(y,\nabla u^H(x)) \right|^p dy dx \leq \liminf_{k\to\infty} \int_D |\nabla u^\epsilon(x)|^p \, dx.$$

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