Local boundary behavior of harmonic and analytic functions: Abelian theorems for quasiasymptotics

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Summary

The aims of this talk are to give a brief introduction to the concept of quasiasymptotic behavior of distributions and present some new abelian results for harmonic and analytic functions on the upper half-plane admiting distributional boundary values.

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Plan:

- Harmonic and analytic representations of distributions.
- Definition of quasiasymptotics at points.
- Quasiasymptotics of order less than 1, the Poisson kernel and local boundary behavior of harmonic functions
- Quasiasymptotics of other orders and local boundary behavior of harmonic functions.
- Abelian theorems for analytic functions.

Notation

- All of our functions and distributions are over the real line.
- $\mathcal{D}$ and $\mathcal{D}'$ denote the Schwartz spaces of test functions and distributions.
- $\mathcal{S}$ and $\mathcal{S}'$ are the spaces of rapidly decreasing functions and the space of tempered distributions.
- $\mathcal{E}$ and $\mathcal{E}'$ denote the space of all smooth functions and its dual, the space of compactly supported distributions.

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- $\mathcal{E}$ and $\mathcal{E}'$ denote the space of all smooth functions and its dual, the space of compactly supported distributions.
- The Fourier transform in $\mathcal{S}$ is defined as

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \phi(t) e^{ixt} \mathrm{d}t.$$

- The evaluation of a distribution f at a test function ϕ will be denoted by

$$\langle f(x), \phi(x) \rangle .$$

Analytic representations of distributions

Let $f \in \mathcal{D}'$, we say that f is the distributional jump of an analytic function F, analytic for $\Im m\ z \neq 0$, across the real axis if

$$\lim_{y \to 0^+} F(x+iy) - F(x-iy) = f(x),$$

in the weak topology of $\mathcal{D}'$, meaning that $\forall \phi \in \mathcal{D}'$

$$\lim_{y \to 0^+} \int (F(x+iy) - F(x-iy))\phi(x)dx = \langle f(x), \phi(x) \rangle .$$

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- Every distribution admits a representation as the jump of an analytic function.
- Any two analytic representation differ by an entire function. So, we can see distributions as hyperfunctions.

Analytic representations and Cauchy transform

Let $f \in \mathcal{E}'$, that is, f has compact support. Then we can find an analytic representation by using the Cauchy transform,

$$F(z) = \frac{1}{2\pi i} \left\langle f(x), \frac{1}{x-z} \right\rangle$$

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Example: For the delta distribution, we have that

$$F(z) = -\frac{1}{2\pi i z}.$$

More generally, the Cauchy transform of the distribution $\delta^{(k-1)}$ is

$$F(z) = (-1)^k \frac{k!}{2\pi i z^k}.$$

Analytic representations the Fourier transform

Let f be a tempered distribution, then we can find an analyitic representation of f by using the Fourier transform

$$F(z) = \begin{cases} \dfrac{1}{2\pi}\left\langle \hat{f}_{-}(t), e^{-izt} \right\rangle, & \Im m\, z > 0\ , \\ -\dfrac{1}{2\pi}\left\langle \hat{f}_{+}(t), e^{-izt} \right\rangle, & \Im m\, z < 0\ , \end{cases}$$

where $\hat{f} = \hat{f}_{-} + \hat{f}_{+}$ and $\operatorname{supp} \hat{f}_{-} \subseteq (-\infty, 0]$ and $\operatorname{supp} \hat{f}_{+} \subseteq [0, \infty)$.

Harmonic and analytic representations on $\Im m\, z > 0$

Let f be a distribution, we say that a harmonic function $U(z)$, harmonic on $\Im m\, z > 0$ is a harmonic representation of f on the upper half-plane if

$$\lim_{y \to 0^+} U(x+iy) = f(x).$$

- Every distribution admits an harmonic representation.
- If f admits an analytic representation F on the upper semiplane we write as usual $f(x) = F(x+i0)$.
- An analytic function has distributional boundary values if and only if locally satisfies an estimate $F(z) = O(y^{-k})$.

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- An analytic function has distributional boundary values if and only if locally satisfies an estimate $F(z) = O(y^{-k})$.
- If $f(x) = F(x+i0) - F(x-i0)$ then $U(z) = F(z) - F(\bar{z})$ for $\Im m\, z > 0$ is a harmonic representation of f.

Harmonic representations and the Poisson kernel

The Poisson kernel for the upper half-plane is defined as

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If f is a distribution with compact support, we can find an explicit harmonic representation of f by evaluating at the Poisson kernel

$$U(z)=\left\langle f(t),\frac{y}{\pi\left((x-t)^2+y^2\right)}\right\rangle,$$

where $z=x+yi$.

Harmonic representations and Fourier Transform

Let f be a tempered distribution. One may use the Fourier transform to obtain harmonic representations. Let $\hat{f}_{\pm}$ two tempered distributions such that $\operatorname{supp}\hat{f}_{-} \subseteq (-\infty, 0]$, $\operatorname{supp}\hat{f}_{+} \subseteq [0, \infty)$ and $\hat{f} = \hat{f}_{-} + \hat{f}_{+}$, then

$$U(z) = \frac{1}{2\pi}\left\langle \hat{f}_{-}(t), e^{-izt} \right\rangle + \frac{1}{2\pi}\left\langle \hat{f}_{+}(t), e^{-i\bar{z}t} \right\rangle$$

is a harmonic representation of f.

First order asymptotic separation of variables at a point

Let $f \in \mathcal{D}'$, we study asymptotic behaviors of the form

$$f(x_0 + \epsilon x) = \rho(\epsilon) g(x) + o(\rho(\epsilon)) \ \ \epsilon \to 0^+,$$

in the weak topology of $\mathcal{D}'$, where $g \in \mathcal{D}'$ and ρ is a positive measurable function. The above relation means that

$$\lim_{\epsilon \to 0^+} \left\langle \frac{f(x_0 + \epsilon x)}{\rho(\epsilon)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle ,$$

It can be shown that if g is assumed to be nonzero, then $\rho(\epsilon) = \epsilon^\alpha L(\epsilon)$, where L is a slowly varying function and g is homogeneous distribution of degree α.

Slowly Varying Functions

Recall that real-valued measurable function defined in some interval of the form $(0, A]$, $A > 0$, is called *slowly varying function at the origin* if L is positive for small arguments and

$$\lim_{\epsilon \to 0^+} \frac{L(a\epsilon)}{L(\epsilon)} = 1,$$

for each $a > 0$.
Similarly one defines slowly varying functions at infinity.

Homogeneous distributions

In one variable, one knows explictly all homogeneous distributions.
If $\alpha \notin \mathbb{Z}_-$, then they are linear combinations of x_-^α and x_+^α, where for $\alpha > -1$

$$\langle x_+^\alpha, \phi(x) \rangle = \int_0^\infty x^\alpha \phi(x) dx,$$

and if $\alpha < -1$, $-n-1 < \alpha < -n$, then
$x_+^\alpha = \Gamma(\alpha+1)/\Gamma(\alpha+n+1) \frac{d}{dx} x_+^{\alpha+n}$. One defines $x_-^\alpha = (-x)_+^\alpha$.

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If $\alpha = -k$, $k \in \mathbb{Z}_+$, then they are linear combinations of $\delta^{k-1}(x)$ and x^{-k}.

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If $\alpha = -k$, $k \in \mathbb{Z}_+$, then they are linear combinations of $\delta^{k-1}(x)$ and x^{-k}.
One also has the homogeneous distributions $(x+i0)^\alpha$ and $(x-i0)^\alpha$, which are the boundary values of the analytic function z^α from the upper and lower half-planes.

Quasiasymptotic behaviors at a point

Let L be slowly varying. We say that $f \in \mathcal{D}'$ has *quasiasymptotic behavior at* x_0 *in* $\mathcal{D}'$ *with respect to* $\epsilon^{\alpha} L(\epsilon)$, $\alpha \in \mathbb{R}$, if for some $g \in \mathcal{D}'$, homogeneous distribution

$$f(x_0 + \epsilon x) = \epsilon^{\alpha} L(\epsilon) g(x) + o(\epsilon^{\alpha} L(\epsilon)), \ \epsilon \to 0^{+} \ \text{ in } \mathcal{D}'.$$

Again, it means that for every $\phi \in \mathcal{D}$,

$$\lim_{\epsilon \to 0^{+}} \left\langle \frac{f(x_0 + \epsilon x)}{\epsilon^{\alpha} L(\epsilon)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle .$$

We also say that f has *quasiasymptotic of order* α *at* x_0 *with respect to* L.

An example of quasiasymptotic behavior

We say that $f \in \mathcal{D}'$ has a jump behavior at $x = x_0$ if it has the quasiasymptotic behavior

$$f(x_0 + \epsilon x) = \gamma_- H(-x) + \gamma_+ H(x) + o(1) \;\; as \;\; \epsilon \to 0^+ \; .$$

Here H is the Heaviside function, i.e., the characteristic function of $(0, \infty)$.

In particular if $\gamma = \gamma_- = \gamma +$, we recover the usual Łojasiewicz notion of the value of a distribution at a point, that is

$$f(x_0 + \epsilon x) = \gamma + o(1) \;\; as \;\; \epsilon \to 0^+ \; .$$

The main question

Suppose that

- $U(z)$ is harmonic or analytic for $\Im m\, z > 0$.
- U has distributional boundary values on $\mathbb{R}$, say f is the boundary distribution.
- f has a quasiasymptotic behavior at x_0.

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Suppose that

- $U(z)$ is harmonic or analytic for $\Im z > 0$.
- U has distributional boundary values on $\mathbb{R}$, say f is the boundary distribution.
- f has a quasiasymptotic behavior at x_0.

Would it be possible to obtain the asymptotic behavior of $U(z)$ as $z \to x_0$?

Answer: In most cases it is possible to obtain the angular asymptotic behavior.

The quasiasymptotic behavior of order less than 1

A natural question is the following suppose that $f(x)$ has quasiasymptotic behavior of order α with respect to L, would it be possible to replace ϕ in

$$\lim_{\epsilon \to 0^+} \left\langle \frac{f(x_0+\epsilon x)}{\epsilon^\alpha L(\epsilon)}, \phi(x) \right\rangle = \langle g(x), \phi(x) \rangle \, .$$

by the Poisson kernel?

This would lead directly to the asymptotic behavior of the Poisson harmonic representation.

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This would lead directly to the asymptotic behavior of the Poisson harmonic representation.

Answer: It is possible when $\alpha < 1$ and f has compact support. Since quasiasymptotic is a local property, this is enought for our case because one can always assume that f has compact support

Angular behavior of harmonic functions for $\alpha < 1$

Let $f \in \mathcal{D}'$ have the quasiasymptotic behavior at $x_0 \in \mathbb{R}$ in $\mathcal{D}'$

$$f\left(x_0 + \epsilon x\right) = \epsilon^{\alpha} L(\epsilon)\left(C_- x_-^{\alpha} + C_+ x_+^{\alpha}\right) + o\left(\epsilon^{\alpha} L(\epsilon)\right) \text{ as } \epsilon \to 0^+,$$

where $\alpha < 1$ and $\alpha \notin \mathbb{Z}_-$. Let U be a harmonic representation of f on $\Im m\, z > 0$ Let $\theta = \arg(z - x_0)$.

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where $\alpha < 1$ and $\alpha \notin \mathbb{Z}_-$. Let U be a harmonic representation of f on $\Im m\, z > 0$ Let $\theta = \arg(z - x_0)$.
Then

$$U(z) = \left|z - x_0\right|^{\alpha} \frac{L\left(\left|z - x_0\right|\right)}{\sin \alpha\pi} \left(C_- \sin \alpha\theta + C_+ \sin \alpha\left(\pi - \theta\right)\right)$$

$$+ o\left(\left|z - x_0\right|^{\alpha} L\left(\left|z - x_0\right|\right)\right),$$

as $z \to x_0$ on $\eta \le \theta \le \pi - \eta$.

Example

If f has a value at x_0 in the sense of Łojasiewicz, say $f(x_0) = \gamma$, then for any harmonic representation U

$$\lim_{z \longrightarrow x_0} U(z) = \gamma \quad \text{angularly.}$$

Angular behavior for $\alpha \in \mathbb{Z}_-$

Let $f \in \mathcal{D}'$ have the quasiasymptotic

$$f(x_0 + \epsilon x) = \frac{L(\epsilon)}{\epsilon^k}\left(\gamma\delta^{(k-1)}(x) + \beta x^{-k}\right) + o\left(\frac{L(\epsilon)}{\epsilon^k}\right) \text{ as } \epsilon \to 0^+$$

in $\mathcal{D}'$. Then if U is a harmonic representation of f on $\Im m\, z > 0$, it has the angular asymptotic behavior

$$U(z) = L(|z - x_0|)\left(\frac{(-1)^k(k-1)!\gamma}{\pi}\Im m\left(\frac{1}{z - x_0}\right) + \beta\Re e\left(\frac{1}{z - x_0}\right)\right)$$

$$+o\left(\frac{L(|z - x_0|)}{|z - x_0|^k}\right)$$

as $z \to x_0$ on any sector $\eta < \arg(z - x_0) < \pi - \eta$, where $0 < \eta \leq \frac{\pi}{2}$.

Angular behavior of harmonic functions for $\alpha > 1$

Let $f \in \mathcal{D}'$ have the quasiasymtotic behavior

$$f(x_0 + \epsilon x) = \epsilon^{\alpha} L(\epsilon) \left(C_- x_-^{\alpha} + C_+ x_+^{\alpha} \right) + o\left(\epsilon^{\alpha} L(\epsilon)\right) \text{ as } \epsilon \to 0^+,$$

in $\mathcal{D}'$. Suppose U is a harmonic representation of f on $\Im m\, z > 0$. Then if $\alpha > 1$, $\alpha \notin \mathbb{Z}$, there are constants $a_1, \ldots, a_n$, $n < \alpha$, such that U has the angular asymptotic behavior

$$U(z) = \sum_{j=1}^{n} a_j \left| z - x_0 \right|^j \sin j\theta + C_- \left| z - x_0 \right|^{\alpha} L\left(\left| z - x_0 \right| \right) \frac{\sin \alpha\theta}{\sin \alpha\pi}$$

$$+ C_+ \left| z - x_0 \right|^{\alpha} L\left(\left| z - x_0 \right| \right) \frac{\sin \alpha(\pi - \theta)}{\sin \alpha\pi} + o\left(\left| z - x_0 \right|^{\alpha} L\left(\left| z - x_0 \right| \right) \right),$$

as $z \to x_0$ on sectors of the form $\eta < \theta < \pi - \eta$, here $\theta = \arg(z - x_0)$.

Angular behavior for $\alpha > 1$

- In this case the results were obtained by asymptotic properties of the Fourier transform and Fourier harmonic representations.
- It is possible to obtain partial results when $\alpha \in \mathbb{Z}_+$.
  - For even integers, we only obtain the radial behavior.
  - On the other hand, for odd integers one gets the radial behavior of the conjugate harmonic.

Local boundary behavior of analytic functions

Let $f \in \mathcal{D}'$ have the quasiasymptotic behavior in $\mathcal{D}'$

$$f(x_0 + \epsilon x) = \epsilon^{\alpha} L(\epsilon) g(x) + o\left(\epsilon^{\alpha} L(\epsilon)\right) \text{ as } \epsilon \longrightarrow 0^{+}.$$

Suppose that $f(x) = F(x + i0)$, for F analytic on $\Im m\, z > 0$.

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Suppose that $f(x) = F(x + i0)$, for F analytic on $\Im m\, z > 0$.

Then there is a constant such that $g(x) = C(x + i0)^{\alpha}$.

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Suppose that $f(x) = F(x + i0)$, for F analytic on $\Im m\, z > 0$.

Then there is a constant such that $g(x) = C(x + i0)^{\alpha}$.

Moreover,

$$F(z) \sim C L\left(|z - x_0|\right)(z - x_0)^{\alpha}$$

as $z \to x_0$ on any sector $\eta < \arg(z - x_0) < \eta - \pi$.

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