Introduction to Multiobjective Optimal Control Problem

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October 17, 2007

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Student Seminar on Control Theory and Optimization Fall 2007

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Motivation

Practical decision problems often involve many factors and can be described by a vector valued decision function whose componets describe several competitive objectives. The comparision between different values of the decision function is determined by a **preference** of the decision maker.

Examples of A Prference Relation

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The preference relation for two vectors $x, y \in \mathbb{R}^m$ in a *weak Pareto* sense is defined by $x \prec y$ if and only if $x_i \leq y_i, i = 1, \ldots, m$, at least one of the inequalities is strict. In other words, $x \prec y$ if and only if $x - y \in K := \{z \in \mathbb{R}^m : z \text{ has nonpositive components }\}$.

The Preference Determined by the Lexicographical Order

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Write $r \prec s$ if there exists an integer $q \in \{0, 1, \ldots, m-1\}$ such that $r_i = s_i,\ i = 1, \ldots, q$, and $r_{q+1} < s_{q+1}$. It is easy to check that satisfies (A1) and (A2) in the definition. But it is not continuous.

Central Question

. Given a preference, is it always possible to define a utility function that determines the preference?

. In the multiobjective optimal control problems, the question amounts to asking whether it is reduce a multiobjective optimal control problem to an optimal control problem with a reasonable single objective function.

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Debreau's Existence Theorem

A preference $\prec$ is determined by a continuous utility function if and only if $\prec$ is continuous in the sense that, for any x, the sets $\{y : x \prec y\}$ and $\{x : y \prec x\}$ are closed.

Multifunctions

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A *multifunction* $F : \mathbb{R}^m \to \mathbb{R}^n$ is a map from $\mathbb{R}^m$ to the subsets of $\mathbb{R}^n$, that is for every $x \in \mathbb{R}^m$, we associate a (potentially empty) set $F(x)$.
Its graph, denoted $Gr(F)$ is defined by

$$Gr(F) = \{(x, y) | y \in F(x)\}.$$

Lipschitz Continuity

A multifunction F is said to be *Lipschitz continuous* if there is a $k \geq 0$ so that for any $x_1, x_2 \in \mathbb{R}^m$ we have

$$F(x_1) \subset F(x_2) + k|x_1 - x_2|B.$$

Sub-Lipschitz Continuity

A multifunction F is said to be *sub-Lipschitzian* in the sense of Lowen and Rockafellar at z if there exist $\beta \geq 0, \epsilon > 0$, and a summale function $\kappa : [a, b] \to \mathbb{R}$ so that for almost all $t \in [a, b]$, for all $N > 0$, for all $x, x' \in z(t) + \epsilon\mathbb{B}$, and $y \in \dot{z}(t) + N\mathbb{B}$, one has

$$d(y, F(t, x)) - d(y, F(t, x')) \leq (\kappa(t) + \beta N)|x - x'|.$$

Proximal Subgradient

Given a lower semicontinuous function $f : X \to \overline{\mathbb{R}}$ and a point in the effective domain of f, that is, the set

$$\mathrm{dom} f := \{x' \in X : f(x') < +\infty\},$$

we say that η is a *proximal subgradient* of f at x if there exists $\sigma \geq 0$ such that

$$f(x') - f(x) + \sigma ||x' - x||^2 \geq \langle \eta, x' - x \rangle$$

for all x' in a neighborhood of x. The set of such η, is referred to as the proximal subdifferential.
The limiting subdifferntial is denoted as

$$\partial_L f(x) := \{lim \eta_i : \eta_i \in N_S^P(x_i), x_i \to x, f(x_i) \to f(x)\}.$$

Limiting Normal Cone

The *limiting normal cone* to S at x is obtained by applying a sequential closure operation to N_S^P :

$$N(S,x) = N_S^L(x) := \{lim \eta_i : \eta_i \in N_S^P(x_i), x_i \to x, x_i \in S.\}$$

Regularity on A Preference

We write $l(r) := \{s \prec r\}$. We say that a *preference* $\prec$ is closed provided that

(A1) for any $r \in \mathbb{R}^m$, $r \in \overline{l(r)}$;

(A2) for any $r \prec s$, $t \in \overline{l(r)}$ implies that $t \prec s$.

We say that $\prec$ is regular at $\bar{r} \in \mathbb{R}^m$ provided that

(A3) for any sequences $r_k, \theta_k \to \bar{r}$ in $\mathbb{R}^m$,

$$\limsup_{k\to\infty} N(\overline{l(r_k)}, \theta_k) \subset N(\overline{l(\bar{r})}, \bar{r}).$$

Formulation of Zhu's Problem

Consider the following multiobjective optimization problem with endpoint constraints,

$$(\mathcal{P}) \begin{cases} \text{Minimize } \phi(y(1)) \\ \text{subject to } \dot{y}(t) \in F(y(t)) \text{ a.e. in } [0,1], \\ y(0) \in \alpha_0,\ y(1) \in E, \end{cases}$$

where $\phi = (\phi_1, \ldots, \phi_m)$ is a Lipschitz vector function on $\mathbb{R}^n$, E is closed and F is a multifunction from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Basic Assumptions

(H1) For every x, $F(x)$ is a nonempty compact convex set.

(H2) F is Lipschitz with rank L_F, i.e. for any x, y,

$$F(x) \subset F(y) + L_F||x - y||\mathbb{B}_{\mathbb{R}^n}.$$

Main Result

Theorem

Let x be a solution to the multiobjective optimal control problem $(\mathcal{P})$. Suppose that the preference $\prec$ is regular at $\phi(x(1))$. Then there exist an absolutely continuous mapping $p : [0,1] \to \mathbb{R}^m$, a vector $\lambda \in N(\overline{l(\phi(x(1)))}, \phi(x(1)))$ with $||\lambda|| = 1$, and a scalar $\lambda_0 = 0$ or 1 satisfying $\lambda_0 + ||p(t)|| \neq 0,\ \forall t \in [0,1]$ such that

$$(\dot{p}(t), \dot{x}(t)) \in \partial_C H(x(t), p(t)) \ \textit{a.e. in}\ [0,1],$$
$$-p(1) \in \lambda_0 \partial \langle \lambda, \phi \rangle (x(1)) + N(E, x(1))$$

Moreover, one can always choose $\lambda_0 = 1$ when $x(1) \in int\ E$.

A Single Objective Problem

When $m = 1$ and $r \prec s$, it yields that $r < s$. The theorem reduces to the classical Hamiltonian necessary conditions for an optimal control problem. Thus, the necessary condtions in the above theorem are true generalizations of the Hamiltonian necessary conditions for single objective optimal control problems.

Multiobjective Dynamic Optimization Problem

$$(\mathcal{P}')\begin{cases}\text{minimize } f(x(a), x(b)),\\ (x(a), x(b)) \in S,\\ \dot{x}(t) \in F(t, x(t)) \text{ a.e. } t \in [a, b].\end{cases}$$

where S is closed and F is a multivalued function which is measurable for each t.

Main Result

Let z be a local solution to the multiobjective optimal control problem ($\mathcal{P}'$). Suppose that F is sub-Lipschitzian at z and that the preference $\prec$ is regular at $f(z(a), z(b))$. Then there exist $p \in W^{1,1}$, $\lambda \geq 0$, and $w \in \tilde{N}(\overline{l(f(z(a), z(b)))}, f(z(a), z(b)))$, with $|\omega| = 1$ such that $(\lambda, p) \neq 0$ and

$$\dot{p}(t) \in \text{co} D^* F(t, z(t), \dot{z}(t)) \text{ a.e. } t \in [a, b],$$
$$(p(a), -p(b)) \in \lambda \partial(\langle \omega, f(\cdot, \cdot) \rangle)(z(a), z(b)) + N(S, (z(a), z(b)),$$
$$\langle p(t), \dot{z}(t) \rangle = H(t, z(t), p(t)) \text{ a.e. } t \in [a, b].$$

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Continued

If in addition F is a convex-valued, then we can replace the first one by the the following one:

$$\dot{p}(t) \in \text{co}\{q : (-q, \dot{z}(t)) \in \partial H(t, z(t), p(t))\} \text{ a.e. } t \in [a, b].$$

Application to a Generalized Pareto Optimal

Let K be a pointed convex cone ($K \cap (-K) = \{0\}$). We difine the preference $\prec$ by $r \prec s$ if and only if $r - s \in K$ and $r \neq s$. A multiobjective optimal control problem with this preference is called a generalized Patreo optimal control problem. This reference is regular at any $r \in \mathbb{R}^m$. Moreover, for any $r \in \mathbb{R}^m$, we have

$$\tilde{N}(\bar{l}(r), r) = k^0 = \{s \in \mathbb{R}^m : \langle s, q \rangle \leq 0 \text{ for all } q \in K.\}$$

Corollary

Let z be a local solution to the generalized Pareto multiobjective optimal control problem $(\mathcal{P}')$. Then there exist $p \in W^{1,1},\ \lambda \geq 0$, and $w \in K^0$, with $|\omega| = 1$ such that $(\lambda, p) \neq 0$ and

$$\begin{gathered} \dot{p}(t) \in \mathrm{co} D^* F(t, z(t), \dot{z}(t)) \text{ a.e. } t \in [a, b], \\ (p(a), -p(b)) \in \lambda \partial(\langle \omega, f(\cdot, \cdot) \rangle)(z(a), z(b)) + N(S, (z(a), z(b))), \\ \langle p(t), \dot{z}(t) \rangle = H(t, z(t), p(t)) \text{ a.e. } t \in [a, b]. \end{gathered}$$

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- S. Bellaassali and A. Jourani, *Necessary optimality conditions in multiobjective dynamic oprimization*. SIAM J. Control Optim. (42), 2004, pp 2043-2061.

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