A Survey of Hybrid Control Systems

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Motivation

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- We consider systems where the state can change continuously or discretely and the dynamics themselves can change discretely.
- This is a very general "definition" of hybrid systems, so naturally there are several frameworks to consider. We will look at several hybrid control systems.
A *standard control system* is of the form

\[ \dot{x} = f(t, x(t), u(t)), \quad x(0) = x_0, \]

where the function \( u(t) \in U \) is a control function.
Differential Inclusions

- Let $F$ map $\mathbb{R} \times \mathbb{R}^n$ to the subsets of $\mathbb{R}^n$. Then a differential inclusion is of the form

  $$\dot{x}(t) \in F(t, x(t)) \text{a.e., } t \in [a, b]$$

- If we let $F(t, x(t)) = f(t, x(t), U)$, then differential inclusions can subsume the control system formulation.

- Thus, differential inclusions cover a wider array of problems.
Continuous Optimal Control

We want to solve $\mathcal{P}_c$:

$$\min_{u(t) \in U} \ell(x(T)) + \int_0^T L(t, x(t), u(t)) \, dt$$

over all $x(t)$ that satisfy

$$\dot{x} = f(t, x(t), u(t)), \quad x(0) = x_0, \quad x(T) \in C_1$$

This is called the Bolza problem. Note we can replace the dynamics with a differential inclusion, which will sometimes be utilized in the following examples. This will lead us to rewrite $L$ so that it depends instead on $(t, x, \dot{x})$. 
We define the \textit{pseudo-Hamiltonian}, a map $H_p : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ given by

$$H_p(t, x, p, u, \lambda) := \langle p, f(t, x, u) \rangle - \lambda L(t, x, u).$$
Pontryagin Maximum Principle

Let \((x, u)\) solve the optimal control problem. Then there is a \(\lambda \in \{0, 1\}\), and arc \(p\) such that

- The adjoint equation below holds a.e.

\[-\dot{p}(t) \in \partial_x H_p(t, x(t), p(t), u(t), \lambda)\]

- The psuedo Hamiltonian is maximized at \(u(t)\) a.e.; i.e.

\[
\max\{H_p(t, x(t), w(t), \lambda) : w(t) \in U(t)\} = H_p(t, x(t), p(t), u(t), \lambda)
\]
Pontryagin Maximum Principle

Let \((x, u)\) solve the optimal control problem. Then there is a \(\lambda \in \{0, 1\}\), and arc \(p\) such that

- \(\|p\| + \lambda > 0\).
- There is a \(\zeta \in \partial C\ell(x(b))\) so that the following transversality condition holds:

\[
-p(b) - \lambda \zeta \in N_{C_1}^C(x(b)).
\]
Using a differential inclusion notation for our system, we define the *Hamiltonian* in the standard way

\[
H(x, p) = \sup_{v \in F(t, x)} \{ \langle p, v \rangle - L(t, x, v) \}
\]

and define the *value function* as

\[
V(t, x) = \inf(\int_t^T L(s, x(s, u(s)) ds + \ell(x(T))).
\]

Then we can show that

\[
V_t = H(x, \nabla_x V).
\]
Multiprocesses

Our first hybrid control problems are called multiprocesses. A multiprocess is a $k$-tuple comprised of \{\(\tau_0^i, \tau_1^i, y_i(\cdot), w_i(\cdot)\}\) where the first two entries are the endpoints of a closed interval, and

\[
\begin{align*}
    y_i(\cdot) & : [\tau_0^i, \tau_1^i] \rightarrow \mathbb{R}^{n_i}, \\
    w_i(\cdot) & : [\tau_0^i, \tau_1^i] \rightarrow \mathbb{R}^{m_i}
\end{align*}
\]

are absolutely continuous and measurable, respectively. We require

\[
\dot{y}_i = f_i(t, y_i(t), w_i(t))
\]
Optimal Multiprocesses

Essentially, multiprocesses are ordered set of control systems where we are allowed to choose when we switch between the different systems. We use the cost function

$$\ell(\{\tau^i_0, \tau^i_1, y_i(\tau^i_0), y_i(\tau^i_1)\}) + \sum_i \int_{\tau^i_0}^{\tau^i_1} L_i(t, y_i(t), w_i(t)) dt$$

We will require, however, that we be given a set $\Lambda$ such that

$$\{\tau^i_0, \tau^i_1, y_i(\tau^i_0), y_i(\tau^i_1)\} \subset \Lambda$$

Our problem is then to find a multiprocess that minimizes the cost function with the above endpoint constraining satisfied.
An Example of a Multiprocess

Consider the situation of harvesting a renewable resource. The standard dynamics are given by

$$\dot{x}(t) = F(x(t)) - \sigma x(t)u(t).$$

We restrict $u$ to the interval $[0, E]$. Then a standard profit function which we wish to maximize, including the discount constant $\delta$ is

$$\int_0^T e^{-\delta t} [\pi x(t) - c] u(t) dt.$$
An Example of a Multiprocess

We turn to the situation where there are two species $x_1$, and $x_2$ which we wish to switch between once in order to optimize profits. We introduce the following cost function

$$\phi_0 e^{-\delta \tau} - \int_0^{\tau} e^{-\delta t} (\pi_1 x_1(t) - c_1) u(t) dt - \int_{\tau}^{T} e^{-\delta t} (\pi_2 x_2(t) - c_2) u(t) dt$$

with switching time $\tau$ and initial condition will be $x_2(\tau) = z(\tau)$ where

$$\dot{z}(t) = F_2(z(t)), \quad z(0) = x_0^2.$$
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Multiprocess Maximum Principle

Suppose \( \{ T_0^i, T_1^i, x_i(\cdot), u_i(\cdot) \} \) is a minimizing multiprocess. Then, under basic assumptions, like those for the PMP, there are real numbers \( \lambda \geq 0, h_0^i, h_1^i \), and absolutely continuous functions \( p_i(\cdot) : [T_0^i, T_1^i] \to \mathbb{R}^{ni} \) such that

\[
\lambda + \sum_i |p_i(T_1^i)| = 1
\]

and the following hold true

- \( -\dot{p}_i(t) \in \partial C x H_i(t, x(t), u_i(t), p_i(t), \lambda), \text{ a.e. } t \in [T_0^i, T_1^i] \),

- \( H_i(t, x(t), u_i(t), p_i(t), \lambda) = \max_{w \in U_i} H_i(t, x(t), w, p_i(t), \lambda) \text{ a.e. } t \in [T_0^i, T_1^i] \),
Multiprocess Maximum Principle

Suppose \( \{T_0^i, T_1^i, x_i(\cdot), u_i(\cdot)\} \) is a minimizing multiprocess. Then, under basic assumptions, like those for the PMP, there are real numbers \( \lambda \geq 0, h_0^i, h_1^i \), and absolutely continuous functions \( p_i(\cdot) : [T_0^i, T_1^i] \to \mathbb{R}^{n_i} \) such that

\[ \lambda + \sum_i |p_i(T_1^i)| = 1 \]

and the following hold true:

- \( h_0^i \in \text{co ess}_{t \to T_0^i} \left[ \sup_{w \in U_i} H_i(t, x(T_0^i), w, p_i(T_0^i), \lambda) \right], \)
- \( h_1^i \in \text{co ess}_{t \to T_1^i} \left[ \sup_{w \in U_i} H_i(t, x(T_1^i), w, p_i(T_1^i), \lambda) \right], \)
- \( \{-h_0^i, h_1^i, p(T_0^i), -p(T_1^i)\} \in \mathcal{N}_\Lambda^{C} + \lambda \partial^{C} f. \)
Our Example Revisited

The tranversality conditions state that

\[ h_0^2 = h_1^1 + p_2(\tau)\dot{z}(\tau) + \delta \phi_0 e^{-\delta t} \]

Then, using knowledge from the PMP for the one system case, we know that \( u \) will be maximal on both sides of \( \tau \) so

\[ h_0^2 = p_2(\tau)[F_2(z(\tau)) - \sigma_2 z(\tau)E] + e^{-\delta \tau} [\pi_2 z(\tau) - c_2]E \]

and

\[ h_1^1 = e^{-\delta \tau} [\pi_1 x_1(\tau) - c_1]E. \]
Our Example Revisited

We then get the following implicit statement on the switching time using only necessary conditions

\[ [\pi_2 x_2(\tau) - c_2]E = [\pi_1 x_1(\tau) - c_1]E + \delta \phi_0 \]
\[ + e^{\delta \tau} p_2(\tau) \sigma_2 x_2(\tau) E. \]
SGP Systems

We now turn to a new hybrid system which can be seen as generalizing, in some ways, autonomous multiprocesses by removing the ordering of the switches. We are given the following data

- A finite set $Q$,
- A family of smooth manifolds $M = \{M_q\}_{q \in Q}$ and sets $U' = \{U'_q\}_{q \in Q}$,
- Functions $f_q : M_q \times U_q \to TM_q$ with $f_q(x, u) \in T_x M_q$ for each $(x, u) \in MQ \times U'_q$,.
SGP Systems

- $U = \{U_q\}_{q \in Q}$, a family of sets of maps from $\mathbb{R}$ into $U'_q$.
- A family of intervals $J = \{J_q\}_{q \in Q}$ where $J_q \subset \mathbb{R}^+$.
- A subset $S$ of

$$\{(q, x, q', x', u(\cdot), \tau) : q, q' \in Q, x \in M_q, x' \in M_{q'}, u(\cdot) \in U_{q'}, \tau \in J_{q'}\}.$$
We define a solution as a triple \( X(t) = (q(t), x(t), \tau(t)) \) where there is a \( \{t_i\} \) partition of \([0, T]\) such that

- If \( x_i(\cdot) = x|_{(t_i, t_{i+1})} \), then \( \dot{x}_i(t) = f_{q_i}(x(t), u(t)) \)
- \( (x_i(t_i), x_{i+1}(t_i)) \in S_{q_i, q_{i+1}} \) where

\[
S_{q_i, q_{i+1}} := \left\{ (x, x') \in M_{q_i} \times M_{q_{i+1}} : (q, x, q', x', u(\cdot), \tau) \in S \right. \\
\left. \text{for some } u(\cdot) \in U_{q_{i+1}}, \tau \in J_{q_{i+1}} \right\}
\]
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SGP Systems

Also, we need

\[ u_{i+1} \in U_{q_i, x_i(t_i), q_{i+1}, x_{i+1}(t_i)} \]

where

\[ U_{q_i, x_i(t_i), q_{i+1}, x_{i+1}(t_i)} := \{ u(\cdot) \in U_{q_{i+1}} : (q_i, x_i, q_{i+1}, x_{i+1}, u(\cdot), \tau) \in S \text{ for some } \tau \in J_{q_{i+1}} \} \]
SGP Cost Functions

The cost function associated with this problem is of the following form

\[
C(X) = \sum_{j=1}^{\nu} \int_{t_{j-1}}^{t_j} L_{q_j}(x_j(t), u_j(t)) \, dt \\
+ \sum_{j=1}^{\nu-1} \Phi_{q_j, q_{j+1}}(x_j(t_j), x_{j+1}(t_j)) \\
+ \phi_{q_1, q_\nu}(x_1(t_0), x_\nu(t_\nu)).
\]
Let $X$ be a solution to the above hybrid problem. Then there is an adjoint pair $(p, \lambda)$ with $p = \{p_1, p_2, \ldots, p_\nu\}$, $\lambda \in \mathbb{R}^+$ such that

1. $-\dot{p}(t) = \partial_x H_i(t, x(t), p(t), u(t), \lambda)$
2. $-\dot{p}(t) = \partial_x H_i(t, x(t), p(t), u(t), \lambda)$
3. The Hamiltonian is maximized for this adjoint pair
Let $X$ be a solution to the above hybrid problem. Then there is an adjoint pair $(p, \lambda)$ with $p = \{p_1, p_2, \ldots, p_\nu\}$, $\lambda \in \mathbb{R}^+$ such that

- The switching condition below holds

$$(-p_i(t_i), p_{i+1}(t_i)) - \lambda \nabla \Phi_{q_i, q_{i+1}}(x_i(t_i), x_{i+1}(t_i)) \in TS_{q_i, q_{i+1}}^C$$
Hybrid Maximum Principle

Let $X$ be a solution to the above hybrid problem. Then there is an adjoint pair $(p, \lambda)$ with

$$p = \{p_1, p_2, \ldots, p_\nu\}, \quad \lambda \in \mathbb{R}^+$$

such that

- if $t_i - t_{i-1} \in \text{Int}(J_{q_i})$, then $\sup H_i = \sup H_\nu = 0$
- if $t_i - t_{i-1}$ is the left endpoint of a nontrivial $J_{q_i}$, then $\sup H_i \leq 0$
- if $t_i - t_{i-1}$ is the right endpoint of a nontrivial $J_{q_i}$, then $\sup H_i \geq 0$
Stratifications of $\mathbb{R}^n$

Assume that we have a finite set of disjoint embedded submanifolds $M_j \subset \mathbb{R}^n$ whose union is $\mathbb{R}^n$. Furthermore if $M_j \cap \overline{M_k} \neq \emptyset$, then $M_j \subset \overline{M_k}$. 


Infinite horizon problem

Suppose our cost function is of the form

$$\int_0^\infty e^{-\beta t} L(x(t), u(t)) dt$$

with dynamics

$$\dot{x}(t) = f(x(t), u(t)) \quad x(0) = x_0.$$
Assumptions

We assume that on each submanifold $M_i$ our controls are a compact set $U_i$, we have a continuous $f_i: M_i \times U_i \rightarrow \mathbb{R}^n$ and a cost $L_i$ so that $f_i$ is Lipschitz in the state variable, $L_i$ is nonnegative and continuous. Finally, $f(x, u) = f_i(x, a)$ and $L(x, u) = L - i(x, u)$ when $x \in M_i$. 
If we define a multifunction in an analogous manner, and take its convex, semicontinuous regularization, it can be shown that the value function solves the Hamilton-Jacobi equation.
Hybrid Time Domains

An alternative model of hybrid dynamics uses differential inclusions. First, we define a *compact hybrid time domain* as

$$
\bigcup_{j=0}^{J-1} ([t_j, t_{j+1}] \times \{j\}).
$$

And a *hybrid time domain* $E$ is such that for any $T, J$ then $E \cap [0, T] \times \{0, 1, \ldots, J\}$ is a compact hybrid time domain.
Dynamics of Teel System

We give our hybrid dynamics as

\[ \dot{x} \in F(x) \text{ when } x \in C \]
\[ x^+ \in G(x) \text{ when } x \in D \]

We refer to \( F \) and \( C \) as the \textit{flow map} and \textit{flow set}. Similarly, \( G \) and \( D \) are the \textit{jump map} and \textit{jump set}. 
A hybrid arc is a hybrid time domain and a function $x$ such that

$$\dot{x}((t, j) \in F(x(t, j))) \text{ if } x(t, j) \in C$$

on the interval $(t_j, t_{j+1})$ and

$$x(t, j + 1) \in G(x(t, j))$$

if $x(t, j) \in D$ and $(t, j), (t, j + 1) \in \text{dom } x$. 
Example: The Bouncing Ball

A bouncing ball can be modeled easily with this framework. If $x$ is the height of the ball above the floor, let

$$y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix},$$

$$f(y) = \begin{bmatrix} y_2 \\ -g \end{bmatrix}$$

and

$$C = \{ y_1 > 0 \text{ or } y_1 = 0 \text{ and } y_2 > 0 \}.$$
Example: The Bouncing Ball

Then we model the jump condition by setting

$$D = \{ y_1 = 0 \text{ and } y_2 \leq 0 \}$$

and

$$G(y) \begin{bmatrix} 0 \\ -\mu y_2 \end{bmatrix}.$$ where $\mu \in (0, 1)$ is a dissipation factor.