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Control Theory Seminar, Fall 2007
1 Basic concepts
   - Extended-valued functions
   - Real case
   - First and second order conditions
   - Examples

2 Applications
References

We work in a n-dimensional real Euclidean space $E$.

Sets will be indicated with capital letters.

Points and vectors will be lower case.

For scalars we use greek characters.
Convex functions

Let $C \subset E$ be a convex set. A function $f : C \rightarrow \mathbb{R}$ is convex if

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

for all $x, y \in C$ and $0 < \alpha < 1$. 

![Diagram showing convex function with a line segment connecting two points on the graph, illustrating the convexity property.]
Geometric interpretation

Let $C \subset E$ be a convex set. A function $f : C \to \mathbb{R}$ is convex if and only if the set

$$\text{epi } f = \{(x, r) \mid r \geq f(x)\}$$

is convex as a subset of $E \times \mathbb{R}$. 

![Convex function and nonconvex function](image)
Extended definition of convex function

**Definition**

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- Simplifies notation.
- The supremum of a set of functions might take infinite values, even if all the functions in the set are finite.
- Allows penalization and exclusion in optimization problems.
Properness

Definition

A extended-valued function $\tilde{f}$ is called proper provided

- $\tilde{f}$ is not identically $+\infty$
- $\tilde{f}(x) > -\infty$, for all $x$.

Proper functions help us avoid undefined expressions such as $+\infty - \infty$. 
Extension of a finite-valued convex function on $C$ as an extended-valued convex function on $E$

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in C, \\ +\infty & \text{otherwise.} \end{cases}$$
Restriction of a extended-valued proper convex function on $E$ to a finite-valued convex function

Take

$$C = \text{dom } \tilde{f} = \{ x \mid \tilde{f}(x) < +\infty \}$$

and define

$$f : C \to \mathbb{R}, \quad f = \tilde{f}|_C$$
Jensen’s inequality

Let $f : E \rightarrow (-\infty, +\infty]$ be a function. Then $f$ is convex if and only if

$$f\left(\sum_{i=1}^{m} \lambda_i x_i\right) \leq \sum_{i=1}^{m} \lambda_i f(x_i)$$

whenever $\lambda_i \geq 0$, for all $i$, $\sum_{i=1}^{m} \lambda_i = 1$. 
For $g$ a real-valued function on an interval $I$.

**Proposition**

$g$ is convex on $I$ if and only if, for all $x_0 \in I$, the slope-function

$$x \mapsto \frac{f(x) - f(x_0)}{x - x_0}$$

is increasing in $I \setminus \{x_0\}$.

**Proposition**

If $g$ is convex on $I$, then $g$ is continuous on the interior of $I$. 
Convex functions on the real line

Proposition

If $g$ is convex on $I$, then $g$ admits finite left and right derivatives at each $x_0$ in the interior of $I$. 

First order condition

**Theorem**

Let $f : E \rightarrow [\neg \infty, +\infty]$ be a differentiable function. Then $f$ is convex if and only if $\text{dom } f$ is a convex set and

$$f(y) \geq f(x) + \langle \nabla f(x), y - x \rangle$$

for every $x, y \in \text{dom } f$. 

![Graph showing the first order condition]

**Introduction to convex sets II: Convex Functions**
Corollary

Let $f : E \rightarrow [-\infty, +\infty]$ be a differentiable convex function. Then $x \in \text{dom } f$ is a global minimizer if and only if $\nabla f(x) = 0$

Corollary

Let $f : E \rightarrow [-\infty, +\infty]$ be a differentiable convex function. Then the mapping $\nabla f$ is monotone, i.e.,

$$\langle \nabla f(y) - \nabla f(x), y - x \rangle \geq 0, \quad x, y \in \text{dom } f$$
(1) Show the result for $g : \mathbb{R} \to [-\infty, +\infty]$.

(2) Use the fact that $f : E \to [-\infty, +\infty]$ is convex if and only if the real function $g$ defined by

$$g(t) = f(ty + (1 - t)x), \; ty + (1 - t)x \in \text{dom } f$$

is convex.
Second order condition

**Theorem**

Let $f$ be a twice continuously differentiable real-valued function on an open interval $(\alpha, \beta)$. Then $f$ is convex if and only if its second derivative is nonnegative throughout $(\alpha, \beta)$.

**Theorem**

Let $f : E \to [-\infty, +\infty]$ be a twice continuously differentiable function. Then $f$ is convex if and only if $\text{dom } f$ is a convex set and the Hessian matrix $\nabla^2 f(x) \succeq 0$ for all $x \in \text{dom } f$. 
Examples of convex functions in the real line

- $g(x) = \exp(\alpha x), \ x \in \mathbb{R}$
- $g(x) = x^p, 1 \leq p < \infty, \ x \leq 0$
- $g(x) = |x|^p, 1 \leq p < \infty$
- $g(x) = -x^p, 0 \leq p < 1, \ x \leq 0$
- $g(x) = x^p, -\infty < p < 0, \ x > 0$
- $g(x) = (\alpha^2 - x^2)^{-1/2}, \alpha > 0, |x| < \alpha$
- $g(x) = -\log(x), \ x > 0$
- Negative entropy $g(x) = x \log(x), \ x > 0$
Examples of convex functions in $\mathbb{R}^n$

- Any norm
- $f(x) = \max \{x_1, x_2, \ldots, x_n\}$
- Log-sum-exp $f(x) = \log(\exp(x_1) + \exp(x_2) + \cdots + \exp(x_n))$
- Geometric mean $f(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}$
- Indicator function of a convex set $C$, $\delta(\cdot \mid C)$

\[
\delta(x \mid C) = \begin{cases} 
0 & \text{if } x \in C, \\
+\infty & \text{otherwise}.
\end{cases}
\]

We have

\[
\inf_{x \in C} f(x) = \inf_{x \in E} \left( f(x) + \delta(x \mid C) \right)
\]
Operations that preserve convexity

Suppose $f, f_1, \ldots, f_m$ are convex functions on $E$

- $h(x) = \lambda_1 f_1 + \cdots + \lambda_m f_m$, $\lambda_i$ are positive scalars.
- $h(x) = \sup \{ f_1(x), \ldots, f_n(x) \}$.
- $h(x) = f(Ax)$, $A$ linear transformation.
- Inf-convolution
  $h(x) = (f_1 \ast f_2)(x) = \inf_{y \in E} \{ f_1(x - y) + f_2(y) \}$, $f_1, f_2$ proper
Convexity of $-\log(x)$ ensures that, for $0 < \theta < 1$, $a, b \geq 0$

$$a^\theta b^{1-\theta} \leq \theta a + (1 - \theta)b$$

A particular selection for $a$ and $b$ helps proving Hölder’s inequality: for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

$$\sum_{i=1}^{n} |x_i y_i| \leq \left( \sum_{i=1}^{n} x_i^p \right)^{1/p} \left( \sum_{i=1}^{n} y_i^q \right)^{1/q}, \text{ where } 1/p + 1/q = 1$$
Proposition

For a convex function $f$ on $E$, the level sets

$$\{x \mid f(x) < \alpha\} \text{ and } \{x \mid f(x) \leq \alpha\}$$

are convex for every $\alpha$.

Note: reverse does not hold!

Corollary

For an arbitrary family $\{f_i\}$ of convex functions on $E$ and real numbers $\alpha_i$, $i \in I$, the set

$$\{x \mid f_i(x) \leq \alpha_i, \ i \in I\}$$

is convex.
Existence of global minimizers

Proposition

Let $D \subset E$ be nonempty and closed, and that all the level sets of the continuous function $f : D \rightarrow \mathbb{R}$ are bounded. Then $f$ has a global minimizer.

Proposition

For a convex $C \subset E$, a convex function $f : C \rightarrow \mathbb{R}$ has bounded level sets if and only if it satisfies the growth condition

$$\liminf_{\|x\| \rightarrow \infty} \frac{f(x)}{\|x\|} > 0$$