

# Topology Syllabus & Review Problems for the Comprehensive Examination

January 1, 2010

**Topics Covered:** The Core-I comprehensive examination in topology will cover the fundamentals of two topics: General Topology of Point Sets and Homotopy Theory and the Fundamental Group. The focus will be on the development of topological invariants. Locally Euclidean spaces (manifolds) and simplicial complexes will be introduced and used in many examples.

## 1 Syllabus for the Topological Spaces and Continuous Maps

The first half of the semester is a quick introduction to the basic concepts and theorems of general topological spaces of point sets, (a.k.a. point-set or general topology). We will study metric and general topological spaces together with continuous maps between them. The central theme will be the development of topological invariants to distinguish topological spaces one from another and homeomorphism is a key concept. The main properties to be studied are open and closed sets, bases, sequences, continuity, homotopy, (path)-connectedness and compactness, as well as their local versions. We also introduce several constructions of spaces, including subspaces, product spaces, function spaces (particularly loop spaces) and quotient (or identification) spaces, including cones and suspensions.

1. *Topology (2nd edition)* by James Munkres, Prentice Hall, 2000: Chapters 2, 3, 4, 5.1

## 2 Syllabus for the Fundamental Group & Covering Spaces

This second half introduces the basics of the fundamental group and covering spaces used in algebraic and geometric topology. Beginning with Poincaré's definition of the fundamental group of a space, various methods of computation are developed (including the Seifert-van Kampen theorem). Fundamental groups of surfaces and simple link complements are computed. The basics of covering spaces are developed and applications include the Brouwer fixed point theorem, the Borsuk-Ulam theorem, and invariance of dimension.

1. Chapters 9, 11 & 13: *Topology (2nd Edition)* by James R. Munkres.
2. Chapter One: *Algebraic Topology* by Allen Hatcher, Cambridge University Press, 2002.

### 3 Review Problems for General Topology

1. Let  $X$  and  $Y$  be topological spaces,  $f : X \rightarrow Y$  a function. Recall that “ $f$  is open” means: if  $U$  is an open set in  $X$  then  $f(U)$  is open in  $Y$ .
  - (a) If  $f$  is continuous, does it follow that  $f$  is open? (Proof or counterexample.)
  - (b) If  $f$  is open, does it follow that  $f$  is continuous? (Proof or counterexample.)
  - (c) Show that if  $X = Y \times Z$  (a product of topological spaces, with the product topology) and  $f$  is the first projection, then  $f$  is open.
  - (d) Under the conditions and notation of (c), if  $F$  is closed in  $X = Y \times Z$ , does it follow that  $f(F)$  is closed in  $Y$ ? (Proof or counterexample.)
2. Let  $f : X \rightarrow Y$  be a quotient map of topological spaces, such that  $Y$  is connected and each set  $f^{-1}(y)$ ,  $y \in Y$ , is a connected subspace of  $X$ . Show that  $X$  is connected.  
 [Recall that a mapping  $f : X \rightarrow Y$  of topological spaces is a *quotient map* if  $f$  is onto (or surjective) and a subset  $U$  of  $Y$  is open if and only if  $f^{-1}(U)$  is open.]
3. Recall that a metric  $d$  on a set  $X$  is called *bounded* if there is a positive real constant  $M$  such that  $d(x, y) \leq M$  for any pair of points  $x, y$  in  $X$ . Show that given any metric  $\delta$  on a set  $X$ , there is a bounded metric  $d$  on  $X$  that induces the same topology as  $\delta$ .
4. Let  $f : X \rightarrow Y$  be a function between topological spaces. The *graph* of  $f$  is defined by

$$G_f = \{(x, y) \in X \times Y \mid y = f(x)\}.$$

- (i) Show that if  $Y$  is Hausdorff and  $f$  is continuous, then  $G_f$  is closed.
  - (ii) Show that if  $Y$  is compact and  $G_f$  is closed, then  $f$  is continuous. (You may use the fact that  $\pi_1 : X \times Y \rightarrow X$  is a closed map when  $Y$  is compact.)
5. Let  $f : X \rightarrow Y$  be a function from a topological space  $X$  to a space  $Y$ . We say that  $f$  is continuous if  $f^{-1}(V)$  is open in  $X$  for every open subset  $V$  of  $Y$ . Show that  $f$  is continuous (in the preceding sense) if and only if for all  $A \subseteq X$ ,  $f(\overline{A}) \subseteq \overline{f(A)}$ .
  6. (i) Show that if  $f : X \rightarrow Y$  is a continuous bijection from a compact space  $X$  to a Hausdorff space  $Y$ , then  $f$  is a homeomorphism.  
 (ii) Give an example of topological spaces  $X$  and  $Y$  and a continuous bijection  $f : X \rightarrow Y$  that is **not** a homeomorphism.

7. Let  $Y = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$  and let  $Z$  be the graph of the function  $y = \sin(\pi/x)$  for  $0 < x \leq 1$ . Is the set  $X = Y \cup Z$  connected or disconnected in the standard topology on  $\mathbb{R}^2$ ? Prove your answer.
8. Prove that for  $A \subset X$ ,  $\overline{A} \setminus \text{Int } A = \overline{A} \cap \overline{X \setminus A}$  by showing each set is contained in the other.
9. Prove that a product of (finitely or infinitely many) connected spaces is connected.
10. Let  $p: X \rightarrow Y$  be a continuous surjection. Show that if  $X$  is compact and  $Y$  is Hausdorff then  $p$  is a quotient map.
11. Let  $f: S^1 \rightarrow \mathbb{R}$  be continuous, where  $S^1$  is the unit circle in  $\mathbb{R}^2$ .
- Show that there is a point  $z \in S^1$  such that  $f(z) = f(-z)$ .
  - Show that  $f$  is not surjective.
12. Let  $X$  be a  $T_0$  space. (That is, for any two distinct points of  $X$ , there is an open set containing exactly one of them.) Suppose that, for any  $x \in X$  and closed subset  $A$  of  $X$  not containing  $x$ , there are disjoint open sets  $U \ni x$  and  $V \supseteq A$ . Prove that  $X$  is Hausdorff.
13. Let  $F: X \rightarrow Y$  be a continuous function between topological spaces  $X, Y$ , and let  $A$  be a subset of  $X$ .
- If  $A$  is compact, prove that  $f(A)$  is compact.
  - If  $A$  is connected, prove that  $f(A)$  is connected.
14. (a) Let  $X$  be a Hausdorff space and  $A$  be a compact subset of  $S$ . Prove that  $A$  is closed.
- (b) Let  $f: X \rightarrow Y$  be a continuous map from a compact space  $S$  to a Hausdorff space  $Y$ . Prove that  $f$  is a closed map.
15. Let  $X, Y$  be topological spaces,  $X = A \cup B$  where  $A, B$  are closed subsets of  $X$ , and  $f: X \rightarrow Y$  be a function such that the restrictions  $f|_A$  and  $f|_B$  are continuous with respect to the relative topologies on  $A, B$ , respectively. Prove that  $f$  is continuous.
16. Let  $X$  be a topological space, let  $E$  be a connected subspace, and let  $A$  be a subset of  $X$  sandwiched between  $E$  and its closure  $\overline{E}$ :

$$E \subset A \subset \overline{E}.$$

Prove that  $A$  is a connected subspace of  $X$ .

17. Prove that a closed subset of a compact space is compact.
18. Let  $S$  denote the following union of three point-sets in the Euclidean plane  $\mathbf{R}^2$ :

$$S = \{(t, 0) : 0 < t \leq 1\} \cup \{(\frac{1}{n}, s) : n = 1, 2, 3, \dots \text{ and } 0 \leq s \leq 1\} \cup \{(0, \frac{1}{2})\}.$$

State whether  $S$  is connected or not, and prove your assertion.

19. Let  $C$  be a subset of a topological space  $X$ .
- Prove that if  $C$  is connected, then the closure of  $C$  is connected.
  - Give an example where  $C$  is connected but the interior of  $C$  is not connected. (You may wish to take  $X = \mathbf{R}^2$ .)
20. Let  $p : X \rightarrow Y$  be a quotient map, and  $f : X \rightarrow Z$  a continuous function such that  $f(x_1) = f(x_2)$  whenever  $p(x_1) = p(x_2)$ . Show that there is a unique function  $g : Y \rightarrow Z$  such that  $g \circ p = f$ , and that  $g$  is continuous.
21. Prove that a non-empty connected subset of a topological space  $X$  that is both open and closed is a connected component of  $X$ .
22. Prove that a compact Hausdorff space is normal.
23. Let  $p : X \rightarrow Y$  be a closed, continuous surjection. Prove that if  $Y$  is compact and  $p^{-1}(y)$  is compact for every  $y \in Y$ , then  $X$  is compact. [Hint: if  $U$  is an open set containing  $p^{-1}(y)$ , there is a neighborhood  $V$  of  $y$  such that  $p^{-1}(V) \subseteq U$ .]
24. Let  $f : [-1, 1] \rightarrow [-1, 1]$  be a continuous function. Prove that there is a point  $x_0$  of  $[-1, 1]$  such that  $f(x_0) = x_0$ . [Hint: consider  $g(x) = (x - f(x))/|x - f(x)|$ .]
25. Give an example of a connected space that is not path-connected, and prove that it has the stated properties.
26. Prove that a path-connected space is connected. Then, prove that a connected, locally path-connected space is path-connected.
27. (a) Show that a metric space is normal.  
(b) Show that a compact Hausdorff space is normal.
28. Let  $\{X_\alpha \mid \alpha \in J\}$  be a family of topological spaces, and let  $X = \prod_{\alpha \in J} X_\alpha$  with the product topology. Let  $\pi_\alpha : X \rightarrow X_\alpha$  be the projection, and let  $f : Y \rightarrow X$  be a function from a topological space  $Y$  to  $X$ . Prove that  $f$  is continuous if and only if the composite  $\pi_\alpha \circ f$  is continuous for each  $\alpha \in J$ .
29. Let  $C$  be a connected subset of a topological space  $X$ . Prove or disprove:

- (a) The closure  $\overline{C}$  of  $C$  is connected.
  - (b) The interior  $C^\circ$  of  $C$  is connected.
30. (a) Show that if  $f: X \rightarrow Y$  is a continuous bijection from a compact space  $X$  to a Hausdorff space  $Y$ , then  $f$  is a homeomorphism.
- (b) Give an example of topological spaces  $X$  and  $Y$  and a continuous bijection  $f: X \rightarrow Y$  that is **not** a homeomorphism.

## 4 Review Problems for Homotopy & the Fundamental Group

1. Show that the following three conditions (on a topological space  $X$ ) are equivalent:
  - (a) Every map  $S^1 \rightarrow X$  is homotopic to a constant map.
  - (b) Every map  $S^1 \rightarrow X$  extends to a map  $D^2 \rightarrow X$ .
  - (c) The fundamental group,  $\pi_1(X, x_0)$ , is trivial for all  $x_0 \in X$ .
2. Let  $\{f_i, g_i : i = 0, 1\}$  be four closed paths based at  $x_0$ . Define the concept of path-homotopy (denoted  $\simeq$ ) and also the operation of path multiplication (denoted  $f_0 \circ f_1$ ). Show that the following cancellation property holds for path-homotopy, if  $f_0 \circ g_0 \simeq f_1 \circ g_1$  and  $g_0 \simeq g_1$  then  $f_0 \simeq f_1$ .
3. Prove the Brouwer fixed point theorem in dimension two: Every continuous map  $f: D^2 \rightarrow D^2$  has a fixed point.
4. Let  $X$  be a locally path connected space with an open cover,  $\{U, V\}$ , consisting of two connected open sets such that  $U$  and  $V$  are both contractible and  $U \cap V$  is connected. Compute its fundamental group. State carefully any basic results you use.
5. Let  $X = S^1 \vee S^1$ , the join of two circles joined at a single point. Compute the fundamental group of  $X$  from the fact that the fundamental group of  $S^1$  is isomorphic to the integers,  $\mathbb{Z}$ . Define *covering space* and explicitly construct a 3-fold covering space of  $X$ .
6. A subset  $X$  of  $\mathbb{R}^n$  is *convex* if the line segment between any two points of  $X$  is contained in  $X$ ; it is *star-convex* if there is some point  $x_0$  in  $X$  such that the line segment from  $x_0$  to any other point of  $X$  is contained in  $X$ . Give an example of a star-convex set that is not convex, and prove that any star-convex set is contractible.
7. Let  $f: X \rightarrow Y$  be continuous. Let  $x_1$  and  $x_2$  be points of  $X$ , and let  $y_1 = f(x_1)$  and  $y_2 = f(x_2)$ . There are induced homomorphisms  $f_{1*}: \pi_1(X, x_1) \rightarrow \pi_1(Y, y_1)$  and  $f_{2*}: \pi_1(X, x_2) \rightarrow \pi_1(Y, y_2)$ . Show that if  $X$  is path-connected then there are

isomorphisms  $\phi: \pi_1(X, x_1) \rightarrow \pi_1(X, x_2)$  and  $\psi: \pi_1(Y, y_1) \rightarrow \pi_1(Y, y_2)$  such that the following diagram commutes.

$$\begin{array}{ccc} \pi_1(X, x_1) & \xrightarrow{f_{1*}} & \pi_1(Y, y_1) \\ \downarrow \phi & & \downarrow \psi \\ \pi_1(X, x_2) & \xrightarrow{f_{2*}} & \pi_1(Y, y_2) \end{array}$$

8. Let  $X$  be the subspace of  $\mathbb{R}^2$  that is the union of two circles of radius 1 centered at  $(-2, 0)$  and  $(2, 0)$  and the line segment from  $(-1, 0)$  to  $(1, 0)$ . State the Seifert-van Kampen Theorem, and use it to find the fundamental group of  $X$ .
9. Let  $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  be a covering map, where  $X$  is path-connected and locally path-connected, and  $\tilde{X}$  is simply-connected. (That is,  $\tilde{X}$  is the universal covering space of  $X$ ). Prove that the group of covering transformations of  $\tilde{X}$  is isomorphic to the fundamental group of  $X$ .
10. Let  $X$  be the join of two circles (*Figure-8 space*). Exhibit both a regular and an irregular 3-fold covering of  $X$ .
11. Recall that a (non-empty) subset  $X$  of  $\mathbb{R}^n$  is *convex* if the line segment between any two points in  $X$  is contained in  $X$ . If  $X$  is convex, show that  $X$  is simply connected.
12. State the Seifert-Van Kampen Theorem. Use this theorem to show that the  $n$ -sphere  $\mathbb{S}^n$  is simply connected for  $n \geq 2$ .
13. Let  $p: E \rightarrow B$  be a covering space of a path-connected space  $B$ , and let  $x$  and  $y$  be two points in  $B$ . Show that the sets  $p^{-1}(x)$  and  $p^{-1}(y)$  have the same cardinality.
14. Let  $X, Y$ , and  $Z$  be path connected, locally path connected spaces, and let  $p: X \rightarrow Z$ ,  $q: X \rightarrow Y$ , and  $r: Y \rightarrow Z$  be continuous maps with  $p = r \circ q$ . If  $p: X \rightarrow Z$  and  $r: Y \rightarrow Z$  are covering maps, prove that  $q: X \rightarrow Y$  is also a covering map.
15. Suppose that  $X$  and  $Y$  are topological spaces, and let  $x_0 \in X$  and  $y_0 \in Y$ . Prove that  $\pi_1(X \times Y, x_0 \times y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$ .
16. Compute the fundamental group of  $S^2$  with  $n$  points removed, where  $n$  is a positive integer. Also compute the fundamental group of  $\mathbb{R}^3$  with the three coordinate axes removed.
17. Let  $X = S^1 \vee S^1$ , the wedge of two circles. Exhibit both a regular and an irregular 4-fold covering of  $X$ . Does  $X$  have an irregular 2-fold covering? Explain.
18. Let  $p: (E, e_0) \rightarrow (B, b_0)$  be a covering map, where  $B$  and  $E$  are path-connected and locally path-connected. Let  $H = p_*(\pi_1(E, e_0))$  be the image of the fundamental

group of  $E$  in  $\pi_1(B, b_0)$ . Prove that the group of covering transformations of  $E$  is isomorphic to  $N(H)/H$ , where  $N(H)$  is the normalizer of  $H$ .

19. Let  $X$  be path-connected and locally path-connected, and suppose that the fundamental group of  $X$  is finite. Prove that any continuous function  $f: X \rightarrow S^1$  is null-homotopic.
20. Let  $X = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0; y = 0\} \cup \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1; z = 0\}$ . Calculate the fundamental group of  $\mathbb{R}^3 \setminus X$ .
21. Show that a retract of a contractible space is contractible.
22. Let  $A$  be a path-connected subspace of a space  $X$  and  $a_0 \in A$ . Show that the inclusion induces a surjection from  $\pi_1(A, a_0)$  to  $\pi_1(X, a_0)$  if and only if every path in  $X$  with endpoints in  $A$  is path-homotopic to a path in  $A$ .
23. Let  $X = S^1 \vee S^1$ , the wedge of two circles. Give an example of a regular 3-fold cover of  $X$ . Give an example of an irregular 3-fold cover of  $X$ . Is there an irregular 2-fold cover of  $X$ ? Note the terminology used by Hatcher for regular and irregular is normal and non-normal.
24. Derive a presentation of the fundamental group of a closed surface of genus two. Show this group is non-abelian.
25. a) Complete the following definition: Two topological spaces  $X$  and  $Y$  are homotopy equivalent if .....
- b) Let  $X, Y, Z$  be topological spaces. Prove that if  $X$  is homotopy equivalent to  $Y$ , and  $Y$  is homotopy equivalent to  $Z$ , then  $X$  is homotopy equivalent to  $Z$ .
26. State the Seifert-van Kampen theorem and use it to compute the fundamental group of the projective plane,  $\mathbb{R}P^2$ .
27. Let  $X$  be Euclidean 3-space with the following two lines deleted,  $X = \mathbb{R}^3 \setminus \{(t, 0, 1), (t, 0, -1) \mid t \in \mathbb{R}\}$ . Compute the fundamental group of  $X$ . (Be sure to justify your computation).
28. Prove that  $\mathbb{R}^2$  is not homeomorphic to  $\mathbb{R}^n$  for  $n > 2$ .
29. Give an example of a space whose fundamental group is a cyclic group of order six. Provide a proof that the fundamental group is cyclic.
30. For each pair of spaces  $(X, A)$  in the following list, determine whether a retraction  $r: X \rightarrow A$  exists or not. Justify your answer. If the retraction exists, sketch a construction of such a retraction.
  - (a)  $X = \mathbb{R}$  and  $A = [0, 1]$ , a closed interval.

- (b)  $X = \mathbb{R}^2 \setminus (0, 0)$ , the punctured plane and  $A = (0, 1)$ , a single point.
- (c)  $X = D^2 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ ,  $A = S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ .
- (d)  $X$  is the Möbius band and  $A$  is the boundary circle.
31. Prove that there is no open cover,  $\{U, V\}$ , of the real projective plane,  $\mathbb{R}P^2$ , with two connected open sets such that  $U$  and  $V$  are both contractible and  $U \cap V$  is connected.
32. State precisely the Seifert-van Kampen theorem and use it to compute the fundamental group of the connected sum of two projective planes,  $\mathbb{R}P^2$  (equivalently, the space obtained by identifying two Möbius bands along the boundary circle). Describe all the regular covering spaces.
33. Let  $p : \tilde{X} \rightarrow X$  be a covering space with path-connected cover. Explicitly define the right action of  $\pi_1(X, x)$  on the fiber  $p^{-1}(x)$  for a given point  $x \in X$ . Show that this is a group action which satisfies the transitive property.
34. Let  $f$  be a continuous function from the real projective plane  $P^2$  to the circle  $S^1$ . Show that the induced homomorphism of fundamental groups is trivial, and use this to prove that  $f$  is null-homotopic.
35. Let  $p : E \rightarrow X$  be the covering map indicated in Figure below. Determine the group of covering transformations. Is this covering regular?

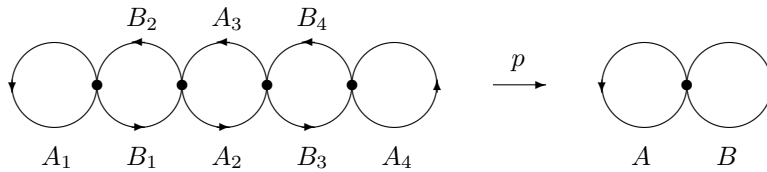


Figure 1: A covering map