Analysis Test Bank
July 15, 2005

The following topics should be reviewed for the Core-1 Analysis Comprehensive Exam. Unless otherwise stated, functions are understood to be defined on the real line and are real valued

I. Advanced calculus topics including:

- Continuous functions, uniform convergence, uniform continuity, Stone-Weierstrass theorem, compactness, Riemann integral. (See the syllabus for Math 4031.)
- Differentiable functions, series and Taylor's theorem, (See the syllabus for Math 4032.)
- Jacobians, inverse and implicit functions, (See the syllabus for Math 4035.)

II. Lebesgue measure, σ-algebras, Borel sets, measurable functions

III. Lebesgue integral, convergence theorems

IV. Functions of bounded variation, absolutely continuous functions, convex functions, and the fundamental theorem of calculus

V. Basic properties of Banach spaces.

- $C([a, b])$ with the sup-norm.
- L^p-spaces and the Riesz representation.

Suggested Reading

- D.S. Bridges: Foundations of Real and Abstract Analysis. Springer, 1997
- A. Browder: Mathematical Analysis. Springer, 1996.
- I.P. Natanson: Theory of Functions of a Real Variable, Vol. 1, 1955.
- H.L. Royden: Real Analysis, Macmillan, 1988.
- W. Rudin: Principles of Mathematical Analysis. McGraw-Hill, 1953.
- G.F. Simmons: Introduction to Topology and Modern Analysis. McGraw-Hill, 1963.
- R.L. Wheeden and A. Zygmund: Measure and Integral: An Introduction to Real Analysis. Marcel Dekker, 1977.

Analysis Test Bank

I. Advanced Calculus

Unless otherwise stated functions are real valued and define on the real line.

I.1
a. A subset S of $\mathbb{R}$ is of type F_σ if S is the countable union of closed sets. Let f be any function from $\mathbb{R}$ to $\mathbb{R}$. Prove that the set of points of discontinuity of f is of type F_σ.
b. Can a function from $\mathbb{R}$ to $\mathbb{R}$ be continuous on the rationals and discontinuous on the irrationals? What if the roles of the rationals and irrationals are interchanged?
c. Briefly explain why there are continuous, nowhere differentiable functions on $\mathbb{R}$.

I.2
a. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{x}{n}$ $(n \in \mathbb{N})$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent on $\mathbb{R}$, uniformly convergent on $[0,1]$, but not uniformly convergent on $\mathbb{R}$.
b. Let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = \frac{1}{1+nx^2}$ $(n \in \mathbb{N})$. Show that $(f_n)_{n\in\mathbb{N}}$ is a bounded subset of $C[0,1]$ and that no subsequence of $(f_n)_{n\in\mathbb{N}}$ converges in $C[0,1]$.

I.3 Identify all subsets of $[0,1]$ on which $\sum_{n=0}^{\infty} x^n$ converges uniformly. Explain.

I.4 Recall that a step function is finite linear combination of characteristic functions of intervals.
a. Show that every continuous function on $[0,1]$ is a uniform limit of step functions.
b. Is the converse true?

I.5 Prove or disprove:
a. The product of two uniformly continuous functions on $\mathbb{R}$ is also uniformly continuous.
b. The product of two uniformly continuous functions on $[0,1]$ is also uniformly continuous.

I.6 Prove or disprove the following two statements:
a. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n \cos nx$ converges pointwise everywhere on $\mathbb{R}$.

b. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n \cos nx$ converges to a continuous function on $\mathbb{R}$.

I.7 Let $C_{0,0}[0,1]$ be the space of all continuous real functions f on the interval $[0,1]$ satisfying $f(0) = f(1) = 0$. Let $\mathcal{P}_{0,0}$ be the subspace of polynomials in $C_{0,0}[0,1]$. Show that $\mathcal{P}_{0,0}$ is dense in $C_{0,0}[0,1]$ in the sup norm.

I.8 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Prove that the following are equivalent.

a. $\lim_{n\to\infty} x_n = a$.

b. Every subsequence of $(x_n)_{n\in\mathbb{N}}$ contains a subsequence that converges to a.

I.9 Prove: If $f \in C[0,1]$ and $\int_0^1 f(x)e^{-nx}\,dx = 0$ for all $n \in \mathbb{N}_0$, then $f = 0$.

I.10 Let $f_n : [1,\infty) \to \mathbb{R}$ be defined by $f_n(x) := \frac{n+1}{n}e^{-nx}$ $(n \in \mathbb{N})$. Show that the series $\sum_{k=1}^{\infty} f_k$ converges uniformly to a continuous function.

I.11 $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) := (\sin x)^n$ $(n \in \mathbb{N})$. Does $(f_n)_{n\in\mathbb{N}}$ converge uniformly?

I.12 Let $f_n : [0,\infty) \to \mathbb{R}$ be defined by $f_n(x) := (x/n)e^{-(x/n)}$ $(n \in \mathbb{N})$.

a. Determine $f(x) = \lim_{n\to\infty} f_n(x)$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $[0,a]$ for any non-negative real number a. Does the sequence converge uniformly to f on $[0,\infty)$? Justify your answer.

b. Show that

$$f(x) = \lim_{n\to\infty} \int_0^a f_n(x)\,dx = \int_0^a f(x)\,dx,$$

but that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx \neq \int_0^\infty f(x)\,dx.$$

I.13 Does $e^z(x^2 + y^2 + z^2) - \sqrt{1+z^2} + y = 0$ have a solution $z = f(x,y)$, where f is continuous at $(x = 1, y = 0)$ and $f(1,0) = 0$? Explain carefully.

I.14 Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function.

a. Use Taylor's formula with remainder to show that, given x and h, $f'(x) = (f(x+2h) - f(x))/2h - hf''(\xi)$ for some ξ.

b. Assume $f(x) \to 0$ as $x \to \infty$, and that f'' is bounded. Show that $f'(x) \to 0$ as $x \to \infty$.

I.15 Prove or give a counterexample: If f is a uniform limit of polynomials on $[-1,1]$, then the Maclaurin series of f converges to f in some neighborhood of 0.

I.16 Let $f(x) = x^k \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$.

a. If $k = 2$, show that f is differentiable everywhere but f' fails to be continuous at some point.

b. If $k = 3$, does f have a second derivative at all points? If so, is f'' a continuous function? Give your reasons.

I.17 Let f be defined on $\mathbb{R}^3$ by $f(x, y, z) = x^2 + 4y^2 - 2yz - z^2$. Show that $f(2, 1, -4) = 0$ and $f_z(2, 1, -4) \neq 0$, and that there exists therefore a differentiable function g in a neighborhood of $(2, 1)$ in $\mathbb{R}^2$, such that $f(x, y, g(x, y)) = 0$. Find $g_x(2, 1)$ and $g_y(2, 1)$. What is the value of $g(2, 1)$?

I.18 Suppose that a function f is defined on $(0, 1]$ and has a finite derivative with $|f'(x)| < 1$. Define $a_n := f(1/n)$ for $n = 1, 2, 3, \ldots$. Show that $\lim_{n\to\infty} a_n$ exists.

I.19 Define a function f on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x > 0 \\ 0 & , \text{if } x \leq 0 \end{cases}$$

a. Check whether f is infinitely differentiable at 0, and, if so, find $f^{(n)}(0)$, $n = 1, 2, 3, \cdots$. Show details.

b. Does f have a power series expansion at 0?

c. Let $g(x) = f(x)f(1-x)$. Show that g is a nontrivial infinitely differentiable function on $\mathbb{R}$ which vanishes outside $(0, 1)$.

I.20 Prove that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ if the partial derivatives $f_{x_1}, \ldots, f_{x_n}$ exist and are bounded in a neighborhood of x.

I.21 Let $f_N(x) = \sum_{n=1}^{N} a_n \sin(nx)$ for $a_n, x \in \mathbb{R}$. If $\sum_{n=1}^{\infty} n a_n$ converges absolutely, show that $(f_N)_{N\in\mathbb{N}}$ converges uniformly to a function f on $\mathbb{R}$, and that $(f'_N)_{N\in\mathbb{N}}$ converges uniformly to f' on $\mathbb{R}$.

I.22 Let f be a twice continuously differentiable real-valued function on $\mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is a critical point of f if all partial derivatives of f vanish at x (i.e., $\triangledown f(x) = 0$), a critical point x is nondegenerate if the $n \times n$ matrix

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]$$

is nonsingular. Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points. (i.e., the nondegenerate critical points are isolated.)

I.23 Show that a function $f(x) = e^{-x} + 2e^{-2x} + \ldots + ne^{-nx} + \ldots$ is continuous on $(0, \infty)$.

I.24 Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with bounded first and second derivatives. By considering the Taylor expansion of f, show that: $\|f'\|_\infty \leq \frac{1}{h}\|f\|_\infty + h\|f''\|_\infty$ for every $h > 0$. By minimizing over h, show that $\|f'\|_\infty \leq 2\sqrt{\|f\|_\infty \|f''\|_\infty}$, where $\|g\|_\infty$ denotes $\sup_{x\in\mathbb{R}} |g(x)|$.

I.25 Let f be a real-valued differentiable function on an interval (a, b). Show that f is Lipschitz continuous if and only if f has bounded derivative.

II. Lebesgue measure, σ-algebras, Borel sets, measurable functions

Unless otherwise stated, references to measure and integration are Lebesgue measure and integration on the real line.

II.1 Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be arbitrary. Prove that the set $\{f^{-1}[S] : S \in \mathcal{A}\}$ is a σ-algebra.

II.2 Suppose that $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue-measurable and that $f : \mathbb{R} \to \mathbb{R}$ is Borel-measurable, that is, $f^{-1}[(a, \infty)]$ is a Borel set for each real number a. Prove that $f \circ g$ is Lebesgue-measurable.

II.3 Prove that the set of points on which a sequence of measurable functions converges is measurable.

II.4 Prove that every Borel set is Lebesgue-measurable.

II.5 Prove that, if there exists a G in G_δ with $E \subset G$ and $m^*(G \setminus E) = 0$, then E is measurable.

II.6 Prove that the Cantor set is a Borel set.

II.7 Prove that any subset of a set of Lebesgue-measure zero is Lebesgue-measurable.

II.8 Prove that every Lebesgue-measurable set is contained in a Borel set with the same measure.

II.9 Prove that every Lebesgue-measurable set contains a Borel set with the same measure.

II.10 Let f be a measurable function that is not almost everywhere infinite. Prove that there exists a subset $S \subset \mathbb{R}$ of positive measure such that f is bounded on S.

II.11 If E is a measurable subset of $[0,1]$ then there is a measurable subset $A \subset E$ such that $m(A) = \frac{1}{2}m(E)$.

II.12 Let $E \subset \mathbb{R}$ be a measurable set with the property that

$$m(E \cap I) \leq \frac{m(I)}{2},$$

for every open interval I. Prove that $m(E) = 0$.

II.13 Let E be a measurable set in $[0,1]$ and let $c > 0$. If $m(E \cap I) \geq cm(I)$, for all open intervals $I \subset [0,1]$ show that $m(E) = 1$.

II.14 Let $A \subset (0,1)$ be a measurable set and $m(A) = 0$. Show that

$$m\{x^2 : x \in A\} = 0 \quad \text{and} \quad m\{\sqrt{x} : x \in A\} = 0.$$

III. Lebesgue integral, convergence theorems

Unless otherwise stated, references to measure and integration are Lebesgue measure and integration on the real line.

III.1 Let $\chi_{[-n,n]}(\cdot)$ denote the characteristic function of the interval $[-n,n]$ $(n \in \mathbb{N})$. Consider the sequence of functions $f_n(x) := \chi_{[-n,n]}(x)\sin(\frac{\pi x}{n})$ $(x \in \mathbb{R})$.

a. Determine $f(x) = \lim_{n\to\infty} f_n(x)$ and show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{R}$. Does the sequence converge uniformly on $\mathbb{R}$?

b. Show that

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)\,dx.$$

Are the assumptions of Lebesgue's dominated convergence theorem satisfied?

III.2 Let f be a positive function on $(0,1]$ such that f is Riemann integrable on $[t,1]$ for all $t \in (0,1)$, but $\lim_{x\to 0^+} f(x) = \infty$. Assume that the improper (Riemann) integral $(R)\int_0^1 f(x)dx$ exists. Show that f is a measurable Lebesgue integrable function and that

$$\int_{[0,1]} f(x)dx \;=\; (R)\int_0^1 f(x)dx.$$

III.3 For each of the following problems, check whether the limit exists. If so, find its value.

a. $\lim_{n\to\infty} \int_1^n (1 - \frac{x}{n})^n dx$,

b. $\lim_{n\to\infty} \int_1^{2n} (1 - \frac{x}{n})^n dx$.

III.4 a. Characterize those bounded functions on $[0,1]$ which are Riemann integrable.

b. Define g on $[0,1]$ by

$$g(x) = \begin{cases} 0 & \text{if x is irrational} \\ 1/q & \text{if } x = p/q \text{ in lowest terms} \end{cases}$$

Is g a Riemann integrable function? Give a proof of your assertion.

c. Show that if

$$f(x) = \begin{cases} 1 & \text{if if } x \text{ is rational} \\ -1 & \text{if if } x \text{ is irrational} \end{cases}$$

then f is not Riemann integrable on the interval $[0,1]$. Is f Lebesgue integrable? Explain.

III.5 Show there are no bounded sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ for which $f_n(x) = a_n \sin(2\pi nx) + b_n \cos(2\pi nx)$ converges to 1 almost everywhere on $[0,1]$.

III.6 Let $f(x)$ be a real-valued measurable function on $[0,1]$. Show that

$$\lim_{n\to\infty} \int_0^1 (\cos(\pi f(x)))^{2n} dx = m\{x : f(x) \text{ is an integer}\},$$

where m denotes Lebesgue measure.

III.7 a. Show that $f(x) = x^{-r}$ is a Lebesgue integrable function on $[0,1]$ if $0 \le r < 1$.

b. If $0 \le r < 1$ let

$$a_n = \int_0^1 \frac{dx}{n^{-1} + x^r} \text{ (Lebesgue integral).}$$

Compute $\lim_{n\to\infty} a_n$. Be sure to quote the appropriate integration theorems to justify your calculations.

III.8 Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \frac{1}{n}, \text{ if } |x| \le n \\ 0, \text{ if } |x| > n \end{cases}$$

a. Show that f_n converges to 0 uniformly on $\mathbb{R}$, and that

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, dx = 2$$

while

$$\int_{\mathbb{R}} \left(\lim_{n\to\infty} f_n\right)(x)\, dx \neq 2$$

.

b. Explain why this example does not contradict the Lebesgue dominated convergence theorem.

III.9 a. Show that $f(x) = 1/\sqrt{x}$ is Lebesgue integrable on $(0,1)$.

b. Find $\inf\{\int_0^1 \psi(x)\, dx | \psi$ is a simple function, and $\psi(x) \geq 1/\sqrt{x}$ on $(0,1)\}$. (Simple functions are finite linear combinations of characteristic functions of measurable sets with extended real-valued coefficients.)

III.10 Give an example of a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded, measurable functions on $[0,1)$ such that

$$\lim_{n\to\infty} \int_0^1 |f_n(x)|\, dx = 0$$

but such that f_n converges pointwise nowhere.

III.11 Consider a Lebesgue-measurable function f on $\mathbb{R}$ with $\int_{\mathbb{R}} f(t)^2\, dt < \infty$. show that the function $g(x) = \int_{\mathbb{R}} f(t-x)f(t)\, dt$ is continuous.

III.12 Prove that $\lim_{n\to\infty} \int_{-\infty}^{\infty} (\sin nt) f(t) dt = 0$ for every Lebesgue integrable function f on $\mathbb{R}$. (Hint: Do the problem first for step functions.)

III.13 Let $f_n(x) = \dfrac{n}{x(lnx)^n}$ for $x \geq e$ and $n \in \mathbb{N}$.

a. For which $n \in \mathbb{N}$ does the Lebesgue integral $\int_e^\infty f_n(x)\, dx$ exist?

b. Determine $\lim\limits_{n\to\infty} f_n(x)$ for $x > e$.

c. Does the sequence $(f_n)_{n\in\mathbb{N}}$ satisfy the assumptions of Lebesgue's dominated convergence theorem?

III.14 Define $f(x) = \int_{\mathbb{R}} \cos(xy)g(y)\, dy$ for $x \in I\!R$ where g is an integrable function on $I\!R$. Show that f is continuous.

III.15 Let $f \in L^\infty[0,1]$ and $\int_0^1 x^n f(x)\, dx = 0$ for $n \in \mathbb{N}$. Show that $f = 0$ a.e.

III.16 Let f be a non-negative Lebesgue-measurable function on $(0,\infty)$ such that f^2 is integrable. Let $F(x) = \int_0^x f(t)\, dt$ where $x > 0$. Show that $\lim_{x\to 0^+} \frac{F(x)}{\sqrt{x}} = 0$.

III.17 Let f be a differentiable function on $[-1,1]$. Prove that $\lim_{\epsilon\to 0}\int_{\epsilon<|x|\le 1}\frac{1}{x}f(x)\,dx$ exists.

III.18 Let $f_n(x) := \dfrac{x^n}{n!}e^{-x}$ for $n\in\mathbb{N}_0$.

a. Show that $\lim_{n\to\infty} f_n(x) = 0$ for all $x>0$.

b. Show that $f_n\in L^1(0,\infty)$ with $\|f_n\|_1 = 1$ for all $n\in\mathbb{N}_0$.

c. Show that $\lim_{k\to\infty}\int_0^k \frac{x^n}{n!}(1-\frac{x}{k})^k\,dx = 1$ for all $n\in\mathbb{N}_0$.

III.19 Prove that $\lim_{b\to\infty}\int_0^b \frac{\sin x}{x}\,dx$ exists but that the function $\frac{\sin x}{x}$ is not integrable over $(0,\infty)$.

III.20 Compute the following limit and justify your calculations:

$$\lim_{n\to\infty}\int_0^\infty\left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right)\,dx$$

III.21 Let f be a continuous nonnegative function on $[a,b]$ where $a<b$. Let $M = \max\{f(x) : a\le x\le b\}$. Show that

$$\lim_{n\to\infty}\left(\int_a^b f(x)^n\,dx\right)^{\frac{1}{n}} = M.$$

III.22 Find and justify the limits:

a. $\displaystyle\lim_{n\to\infty}\int_0^n \frac{\sin x}{1+nx^2}\,dx$

b. $\displaystyle\lim_{n\to\infty}\int_0^{e^n} \frac{x}{1+nx^2}\,dx.$

III.23 Let $f_n(x) = \sum_{i=0}^{n-1}\frac{1}{n}f(x+\frac{i}{n})$, where f is a continuous function on $\mathbb{R}$. Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly on every finite segment $[a,b]$ to the function $F(x) = \int_x^{x+1} f(s)\,ds$.

III.24 Let $f\in L^1(0,\infty)$, and suppose that $\int_0^\infty x|f(x)|\,dx < \infty$. Prove that the function

$$g(y) = \int_0^\infty e^{-xy}f(x)dx$$

is differentiable at every $y\in(0,\infty)$.

III.25 Let $f\in L^1(\mathbb{R})$ with respect to Lebesgue measure, and suppose that

$$\int_{\mathbb{R}} |x||f(x)|\,dx < \infty.$$

Show that the function

$$g(y) = \int_{\mathbb{R}} e^{ixy} f(x)dx$$

is differentiable at every $y \in I\!R$.

III.26 Prove that, if f is a real-valued Lebesgue-integrable function on $\mathbb{R}$, then

$$\lim_{x \to 0} \int |f(x+t) - f(t)|\, dt = 0.$$

III.27 Let $f \in L^1(I\!R)$.

a. Prove: $\lim_{n\to\infty} \int_0^{1/n} f(x)dx = 0$.
b. Prove or disprove: $\lim_{n\to\infty} \int_n^{\infty} f(x)\, dx = 0$.

III.28 Give an example of a sequence of uniformly bounded measurable functions f_n on $[0,1]$ such that $m\{x | f_n(x) \neq 0\} \to 0$ as $n \to \infty$, but the sequence $f_n(x)$ does not converge for any $x \in [0,1]$.

III.29 Let $n \geq 3$ and

$$f_n(x) = \begin{cases} n^2 x, \text{ for } 0 \leq x < \dfrac{1}{n} \\ 2n - n^2 x, \text{ for } \dfrac{1}{n} \leq x \leq \dfrac{2}{n} \\ 0, \text{ for } \dfrac{2}{n} < x \leq 1 \end{cases}$$

Sketch the graphs of f_3 and f_4. Prove that, if g is a continuous real-valued function on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x) g(x) dx = g(0).$$

(Hint: First show that $\int_0^1 f_n(x)\, dx = 1$.)

III.30 Assume that the real-valued measurable function $f(t,x)$ and its partial derivative $\frac{\partial}{\partial t} f(t,x)$ are bounded on $[0,1]^2$. Show that for $t \in (0,1)$

$$\frac{d}{dt}\left[\int_0^1 f(t,x)\, dx\right] = \int_0^1 \frac{\partial}{\partial t} f(t,x)\, dx.$$

III.31 Prove that if f_n is Lebesgue integrable on $[0,1]$ for each $n \in \mathbb{N}$, and

$$\sum_{n=1}^{\infty} \int_0^1 |f_n(x)|\, dx < \infty,$$

then $\sum_{n=1}^{\infty} f_n(x)$ is convergent a.e., and

$$\int_0^1 \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_0^1 f_n(x)dx.$$

III.32 Let $f \in L^1(\mathbb{R})$. Prove that $\lim_{n\to\infty} \int_0^n e^{-nx} f(x)\,dx = 0$.

III.33 Assume that $A \geq 0$, $B > 0$, and f is continuous and nonnegative on $[a,b]$. Assume that $f(t) \leq A + B\int_a^t f(s)\,ds$ for $a \leq t \leq b$. Prove that $f(t) \leq Ae^{B(t-a)}$ for $a \leq t \leq b$.

III.34 Let f be a nonnegative integrable function on $[0,1]$. Suppose that for every $n \in \mathbb{N}$

$$\int_0^1 (f(x))^n\,dx = \int_0^1 f(x)\,dx.$$

Show that f is almost everywhere the characteristic function for some measurable set.

III.35 Provide an example of a sequence $\{f_n\}$ of measurable functions on $[0,1]$ such that $f_n \to f$ *a.e.* and $f_n \geq 0$, but $\liminf \int f_n \neq \int f$.

III.36 Let $1 \leq p < \infty$. Suppose $f_n \in L^p([0,1])$, $\|f_n\|_p \leq 1$, and $f_n \to f$ a.e..

a. Show that $f \in L^p([0,1])$ and $\|f\|_p \leq 1$.

b. Let $1 < p < \infty$ and $g \in L^q([0,1])$ where $\frac{1}{p} + \frac{1}{q} = 1$. Prove that

$$\int_0^1 f_n g \to \int_0^1 fg.$$

c. Give an example to show that the conclusion in item b would be false if $p = 1$.

IV. Functions of bounded variation, absolutely continuous functions, convex functions, and the fundamental theorem of calculus

Unless otherwise stated, references to measure and integration are Lebesgue measure and integration on the real line.

IV.1 If f is continuous on an interval $[a,b]$ and has a bounded derivative in (a,b), show that f is of bounded variation on $[a,b]$. Is the boundedness of f' necessary for f to be of bounded variation? Justify your answer.

IV.2 Let $f(x) = x^2 \sin(\frac{1}{x})$, $g(x) = x^2 \sin(\frac{1}{x^2})$ for $x \neq 0$, and $f(x) = g(x) = 0$ for $x = 0$.

a. Show that f and g are differentiable everywhere (including at $x = 0$).
b. Show that f is bounded variation on the interval $[-1, 1]$, but g is not.
c. Let $f(x) = x \sin(1/x)$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. Is f of bounded variation on $[-1, 1]$?

IV.3 Show that the product of two absolutely continuous functions on a closed finite interval $[a, b]$ is absolutely continuous.

IV.4 A real-valued function f on an interval I for which there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all x and y in I is called a *Lipschitz function.*

a. Show that a Lipschitz function is absolutely continuous.
b. Show that an absolutely continuous function f on an interval is Lipschitz if and only if f' is essentially bounded.

IV.5 A function $f : [0, 1] \to L^1[0, 1]$ is called Lipschitz continuous if there exists $M > 0$ such that $\|f(t) - f(s)\|_1 \leq M|t - s|$ for all $t, s \in [0, 1]$. It is called differentiable at a point $s \in (0, 1)$ if the differential quotients $\frac{f(t)-f(s)}{(t-s)}$ converge in $L^1[0, 1]$ as $t \to s$. Let $f : [0, 1] \to L^1[0, 1]$ be given by $f(t) = \chi_{[0,t]}$, where $\chi_{[0,t]}$ denotes the characteristic function of the interval $[0, t]$. Show that f is Lipschitz continuous and nowhere differentiable.

IV.6 Let f be a nonnegative real function on $[0, 1]$ and let $I = \int_0^1 f(x)\, dx$. Show that

$$\sqrt{1 + I^2} \leq \int_0^1 \sqrt{1 + f^2(x)}\, dx \leq 1 + I.$$

IV.7 Suppose f is a nonnegative integrable function on $[0, 1]$. Prove that

$$\sqrt{\int_0^1 f(t)\, dt} \geq \int_0^1 \sqrt{f(t)}\, dt.$$

IV.8 a. Provide an example of a function on $[0, 1]$ that is not absolutely continuous but is of bounded variation.
b. Provide examples of two different continuous functions on $[0, 1]$ that have the same derivative *a.e.* and that are both equal to zero at 0.

IV.9 Suppose F is absolutely continuous on $[0,1]$ and that $g \in L^1([0,1])$, with $\int_0^1 g = 0$. Prove the "integration by parts" law:

$$\int_0^1 F(x)\, g(x)\, dx \quad = \quad -\int_0^1 \left[F'(x) \int_0^x g\right] dx$$

IV.10 a. Provide an example of a function of unbounded variation on $[0,1]$ that has a derivative equal to zero at almost all $x \in [0,1]$.

b. Provide an example of a function that is absolutely continuous on $[0,1]$ but has an unbounded derivative.

IV.11 Prove that, if f is differentiable a.e. on $[0,1]$ and f' is not in $L^1([0,1])$, then f is not of bounded variation on $[0,1]$.

IV.12 Prove that, if f is absolutely continuous on $[0,1]$, then the total variation of f on $[0,1]$ is equal to $\int_0^1 |f'|$.

IV.13 Suppose that f is a real-valued function of bounded variation on $[0,1]$.

Prove that

a. f has a right- and left-hand limit at each point in $(0,1)$;

b. f can have only countable many points of discontinuity;

c. If, in addition to being of bounded variation on $[0,1]$, f is absolutely continuous on $[0,T]$ for each $T < 1$, then there exists an absolutely continuous function g on $[0,1]$ that coincides with f on $[0,1)$.

V. Basic properties of Banach spaces

Unless otherwise stated, references to measure and integration are Lebesgue measure and integration on the real line.

V.1 Let $C([a,b])$ be the space of real continuous functions on a closed interval $[a,b]$ equipped with the sup norm. Let $\mathcal{M} = \{f \in C([a,b]) : f(x) > 0 \text{ for all } x \in [a,b]\}$. Show that $\mathcal{M}$ is an open subset of $C([a,b])$.

V.2 Let X be the normed linear space obtained by putting the norm $\|f\|_1 = \int_0^1 |f(t)|\, dt$ on the set of real continuous functions on $[0,1]$.

a. Show that X is not a Banach space.

b. Show that the linear functional $\Lambda f = f(1/2)$ is not bounded.

V.3 Show that $L_p[0,1]$ is separable for $1 \leq p < \infty$, but not separable for $p = \infty$.

V.4 Show that $L^p(0,1) \subset L^q(0,1)$ for any $p > q \geq 1$. Here the integrability is with respect to the Lebesgue measure. Is the inclusion map for $L^p(0,1)$ into $L^q(0,1)$ continuous?

V.5 Prove or disprove the equality $L^\infty[0,1] = \cap_{1 \leq p < \infty} L^p[0,1]$.

V.6 Let $f \in L_p(I\!R), 1 \leq p < \infty$. Show that $\displaystyle\int_{|x|>n} |f(x)|^p \, dx \to 0$ for $n \to \infty$.

V.7 Let $g_n = n\chi_{[0,n^{-3}]}$. Show that $\int_0^1 f(x) g_n(x) \, dx \to 0$ as $n \to \infty$ for all $f \in L^2[0,1]$, but not all $f \in L^1[0,1]$.

V.8 Provide an example of the following:

a. A nonzero bounded linear functional on $L^p([0,1])$, $1 < p < \infty$.

b. A nonzero bounded linear functional on ℓ^∞.

V.9 Describe precisely how the dual of ℓ^1 is represented concretely.

V.10 Why is the dual of L^∞ not equal to L^1, in other words, why is L^1 not reflexive?

V.11 What is the completion of the space of continuous functions on $[0,1]$ in the p-norm $(1 \leq p < \infty)$? In the ∞-norm?

V.12 Let $p + q = pq$. For $g \in L^q(E)$, define $\hat{g} \in (L^p(E))^*$ as $\hat{g}(f) = \int_E gf$. Prove that $\|\hat{g}\| = \|g\|_{L^q(E)}$.

V.13 Prove that the linear space of finite sequences is dense in ℓ^p for $1 \leq p < \infty$, but it is not dense in ℓ^∞.

V.14 Prove that $L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

V.15 Let $1 < p < q < r < \infty$. If $f \in L^p(\mathbb{R})$ and $f \in L^r(\mathbb{R})$, then $f \in L^q(\mathbb{R})$.

V.16 Prove that ℓ^∞ is not separable, that is, it has no countable dense set.