Analysis Test Bank
November 15, 2006

The following topics should be reviewed for the Core-1 Analysis Comprehensive Exam. Unless otherwise stated, functions are understood to be defined on the real line/real n-dimensional Euclidean space and are real valued

(a) Advanced calculus topics including:

- Continuous functions, uniform convergence, uniform continuity, Stone-Weierstrass theorem, compactness, Riemann integral. (See the syllabus for Math 4031.)
- Differentiable functions, series and Taylor's theorem, (See the syllabus for Math 4032.)
- Jacobians, inverse and implicit functions, (See the syllabus for Math 4035.)

(b) Lebesgue measure, σ-algebras, Borel sets, measurable functions

(c) Lebesgue integral on $\mathbb{R}$ and $\mathbb{R}^n$, convergence theorems

(d) Tonelli-Fubini Theorem on $\mathbb{R}^m \times \mathbb{R}^n$

(e) Functions of bounded variation, absolutely continuous functions, convex functions, and the fundamental theorem of calculus

(f) Basic properties of Banach spaces.

- $C([a, b])$ with the sup-norm.
- L^p-spaces and the Riesz representation.

Suggested Reading

- D.S. Bridges: Foundations of Real and Abstract Analysis. Springer, 1997
- A. Browder: Mathematical Analysis. Springer, 1996.
- I.P. Natanson: Theory of Functions of a Real Variable, Vol. 1, 1955.
- H.L. Royden: Real Analysis, Macmillan, 1988.
- W. Rudin: Principles of Mathematical Analysis. McGraw-Hill, 1953.
- G.F. Simmons: Introduction to Topology and Modern Analysis. McGraw-Hill, 1963.
- E.M. Stein and R. Shakarchi: Real Analysis-Measure Theory, Integration, & Hilbert Spaces. Princeton U. Press, 2005.
- R.L. Wheeden and A. Zygmund: Measure and Integral: An Introduction to Real Analysis. Marcel Dekker, 1977.

Analysis Test Bank

1 Advanced Calculus

Unless otherwise stated functions are real valued and defined on the real line and/or real n-space.

I.1 (a) A subset S of $\mathbb{R}$ is of type F_σ if S is the countable union of closed sets. Let f be any function from $\mathbb{R}$ to $\mathbb{R}$. Prove that the set of points of discontinuity of f is of type F_σ.

(b) Can a function from $\mathbb{R}$ to $\mathbb{R}$ be continuous on the rationals and discontinuous on the irrationals? What if the roles of the rationals and irrationals are interchanged?

(c) Briefly explain why there are continuous, nowhere differentiable functions on $\mathbb{R}$.

I.2 (a) Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{x}{n}$ $(n \in \mathbb{N})$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent on $\mathbb{R}$, uniformly convergent on $[0, 1]$, but not uniformly convergent on $\mathbb{R}$.

(b) Let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(x) = \frac{1}{1+nx^2}$ $(n \in \mathbb{N})$. Show that $(f_n)_{n\in\mathbb{N}}$ is a bounded subset of $C[0, 1]$ and that no subsequence of $(f_n)_{n\in\mathbb{N}}$ converges in $C[0, 1]$.

I.3 Identify all subsets of $[0, 1]$ on which $\sum_{n=0}^{\infty} x^n$ converges uniformly. Explain.

I.4 Recall that a step function is finite linear combination of characteristic functions of intervals.

(a) Show that every continuous function on $[0, 1]$ is a uniform limit of step functions.

(b) Is the converse true?

I.5 Prove or disprove:

(a) The product of two uniformly continuous functions on $\mathbb{R}$ is also uniformly continuous.

(b) The product of two uniformly continuous functions on $[0, 1]$ is also uniformly continuous.

I.6 Prove or disprove the following two statements:

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n \cos nx$ converges pointwise everywhere on $\mathbb{R}$.

(b) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n \cos nx$ converges to a continuous function on $\mathbb{R}$.

I.7 Let $C_{0,0}[0,1]$ be the space of all continuous real functions f on the interval $[0,1]$ satisfying $f(0) = f(1) = 0$. Let $\mathcal{P}_{0,0}$ be the subspace of polynomials in $C_{0,0}[0,1]$. Show that $\mathcal{P}_{0,0}$ is dense in $C_{0,0}[0,1]$ in the sup norm.

I.8 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Prove that the following are equivalent.

(a) $\lim_{n\to\infty} x_n = a$.

(b) Every subsequence of $(x_n)_{n\in\mathbb{N}}$ contains a subsequence that converges to a.

I.9 Prove: If $f \in C[0,1]$ and $\int_0^1 f(x)e^{-nx}\,dx = 0$ for all $n \in \mathbb{N}_0$, then $f = 0$.

I.10 Let $f_n : [1,\infty) \to \mathbb{R}$ be defined by $f_n(x) := \frac{n+1}{n}e^{-nx}$ $(n \in \mathbb{N})$. Show that the series $\sum_{k=1}^{\infty} f_k$ converges uniformly to a continuous function.

I.11 $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) := (\sin x)^n$ $(n \in \mathbb{N})$. Does $(f_n)_{n\in\mathbb{N}}$ converge uniformly?

I.12 Let $f_n : [0,\infty) \to \mathbb{R}$ be defined by $f_n(x) := (x/n)e^{-(x/n)}$ $(n \in \mathbb{N})$.

(a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f on $[0,a]$ for any non-negative real number a. Does the sequence converge uniformly to f on $[0,\infty)$? Justify your answer.

(b) Show that

$$\lim_{n\to\infty} \int_0^a f_n(x)\,dx = \int_0^a f(x)\,dx,$$

but that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx \neq \int_0^\infty f(x)\,dx.$$

I.13 Does $e^z(x^2+y^2+z^2) - \sqrt{1+z^2} + y = 0$ have a solution $z = f(x,y)$, where f is continuous at $(x = 1, y = 0)$ and $f(1,0) = 0$? Explain carefully.

I.14 Let $f : \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function.

(a) Use Taylor's formula with remainder to show that, given x and h, $f'(x) = (f(x + 2h) - f(x))/2h - hf''(\xi)$ for some ξ.

(b) Assume $f(x) \to 0$ as $x \to \infty$, and that f'' is bounded. Show that $f'(x) \to 0$ as $x \to \infty$.

I.15 Prove or give a counterexample: If f is a uniform limit of polynomials on $[-1, 1]$, then the Maclaurin series of f converges to f in some neighborhood of 0.

I.16 Let $f(x) = x^k \sin(1/x)$ if $x \neq 0$ and $f(0) = 0$.

(a) If $k = 2$, show that f is differentiable everywhere but f' fails to be continuous at some point.

(b) If $k = 3$, does f have a second derivative at all points? If so, is f'' a continuous function? Give your reasons.

I.17 Let f be defined on $\mathbb{R}^3$ by $f(x, y, z) = x^2 + 4y^2 - 2yz - z^2$. Show that $f(2, 1, -4) = 0$ and $f_z(2, 1, -4) \neq 0$, and that there exists therefore a differentiable function g in a neighborhood of $(2, 1)$ in $\mathbb{R}^2$, such that $f(x, y, g(x, y)) = 0$. Find $g_x(2, 1)$ and $g_y(2, 1)$. What is the value of $g(2, 1)$?

I.18 Suppose that a function f is defined on $(0, 1]$ and has a finite derivative with $|f'(x)| < 1$. Define $a_n := f(1/n)$ for $n = 1, 2, 3, \ldots$. Show that $\lim_{n\to\infty} a_n$ exists.

I.19 Define a function f on $\mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

(a) Check whether f is infinitely differentiable at 0, and, if so, find $f^{(n)}(0)$, $n = 1, 2, 3, \cdots$. Show details.

(b) Does f have a power series expansion at 0?

(c) Let $g(x) = f(x)f(1-x)$. Show that g is a nontrivial infinitely differentiable function on $\mathbb{R}$ which vanishes outside $(0, 1)$.

I.20 Prove that a function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ if the partial derivatives $f_{x_1}, \ldots, f_{x_n}$ exist and are bounded in a neighborhood of x.

I.21 Let $f_N(x) = \sum_{n=1}^{N} a_n \sin(nx)$ for $a_n, x \in \mathbb{R}$. If $\sum_{n=1}^{\infty} na_n$ converges absolutely, show that $(f_N)_{N\in\mathbb{N}}$ converges uniformly to a function f on $\mathbb{R}$, and that $(f'_N)_{N\in\mathbb{N}}$ converges uniformly to f' on $\mathbb{R}$.

I.22 Let f be a twice continuously differentiable real-valued function on $\mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is a critical point of f if all partial derivatives of f vanish at x (i.e., $\bigtriangledown f(x) = 0$), a critical point x is nondegenerate if the $n \times n$ matrix

$$\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]$$

is nonsingular. Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points. (i.e., the nondegenerate critical points are isolated.)

I.23 Show that a function $f(x) = e^{-x} + 2e^{-2x} + \ldots + ne^{-nx} + \ldots$ is continuous on $(0, \infty)$.

I.24 Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with bounded first and second derivatives. By considering the Taylor expansion of f, show that: $\|f'\|_\infty \leq \frac{1}{h}\|f\|_\infty + h\|f''\|_\infty$ for every $h > 0$. By minimizing over h, show that $\|f'\|_\infty \leq 2\sqrt{\|f\|_\infty \|f''\|_\infty}$, where $\|g\|_\infty$ denotes $\sup_{x\in\mathbb{R}} |g(x)|$.

I.25 Let f be a real-valued differentiable function on an interval (a, b). Show that f is Lipschitz continuous if and only if f has bounded derivative.

2 Lebesgue measure, σ-algebras, Borel sets, measurable functions

Unless otherwise stated, references to measure and integration are Lebesgue measure and integration on real n-space.

II.1 Let $\mathcal{A}$ be a σ-algebra of subsets of $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$ be arbitrary. Prove that the set $\{f^{-1}[S] : S \in \mathcal{A}\}$ is a σ-algebra.

II.2 Suppose that $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue-measurable and that $f : \mathbb{R} \to \mathbb{R}$ is Borel-measurable, that is, $f^{-1}[(a, \infty)]$ is a Borel set for each real number a. Prove that $f \circ g$ is Lebesgue-measurable.

II.3 Prove that the set of points on which a sequence of measurable functions converges is measurable.

II.4 Prove that every Borel set is Lebesgue-measurable.

II.5 Prove that, if there exists a G in G_δ with $E \subset G$ and $m^*(G\backslash E) = 0$, then E is measurable.

II.6 Prove that the Cantor set is a Borel set.

II.7 Prove that any subset of a set of Lebesgue-measure zero is Lebesgue-measurable.

II.8 Prove that every Lebesgue-measurable set is contained in a Borel set with the same measure.

II.9 Prove that every Lebesgue-measurable set contains a Borel set with the same measure.

II.10 Let f be a measurable function that is not almost everywhere infinite. Prove that there exists a subset $S \subset \mathbb{R}$ of positive measure such that f is bounded on S.

II.11 If E is a measurable subset of $[0, 1]$ then there is a measurable subset $A \subset E$ such that $m(A) = \frac{1}{2}m(E)$.

II.12 Let $E \subset \mathbb{R}$ be a measurable set with the property that

$$m(E \cap I) \leq \frac{m(I)}{2},$$

for every open interval I. Prove that $m(E) = 0$.

II.13 Let E be a measurable set in $[0, 1]$ and let $c > 0$. If $m(E \cap I) \geq cm(I)$, for all open intervals $I \subset [0, 1]$ show that $m(E) = 1$.

II.14 Let $A \subset (0, 1)$ be a measurable set and $m(A) = 0$. Show that

$$m\{x^2 : x \in A\} = 0 \quad \text{and} \quad m\{\sqrt{x} : x \in A\} = 0.$$

3 Lebesgue integral, convergence theorems

Unless otherwise stated, references to measure and integration are Lebesgue measure and integration on finite dimensional real space.

III.1 Let $\chi_{[-n,n]}(\cdot)$ denote the characteristic function of the interval $[-n,n]$ ($n \in \mathbb{N}$). Consider the sequence of functions $f_n(x) := \chi_{[-n,n]}(x)\sin(\frac{\pi x}{n})$ ($x \in \mathbb{R}$).

(a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$ and show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{R}$. Does the sequence converge uniformly on $\mathbb{R}$?

(b) Show that

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{n\to\infty}\int_{-\infty}^{\infty} f_n(x)\,dx.$$

Are the assumptions of Lebesgue's dominated convergence theorem satisfied?

III.2 Let f be a positive function on $(0,1]$ such that f is Riemann integrable on $[t,1]$ for all $t \in (0,1)$, but $\lim_{x\to 0^+} f(x) = \infty$. Assume that the improper (Riemann) integral $(R)\int_0^1 f(x)dx$ exists. Show that f is a measurable Lebesgue integrable function and that

$$\int_{[0,1]} f(x)dx \;=\; (R)\int_0^1 f(x)dx.$$

III.3 For each of the following problems, check whether the limit exists. If so, find its value.

(a) $\lim_{n\to\infty}\int_1^n (1-\frac{x}{n})^n dx$,

(b) $\lim_{n\to\infty}\int_1^{2n} (1-\frac{x}{n})^n dx$.

III.4 (a) Characterize those bounded functions on $[0,1]$ which are Riemann integrable.

(b) Define g on $[0,1]$ by

$$g(x) = \begin{cases} 0 & \text{if x is irrational} \\ 1/q & \text{if } x = p/q \text{ in lowest terms} \end{cases}$$

Is g a Riemann integrable function? Give a proof of your assertion.

III.5 Show that if

$$f(x) = \begin{cases} 1 & \text{if if } x \text{ is rational} \\ -1 & \text{if if } x \text{ is irrational} \end{cases}$$

then f is not Riemann integrable on the interval $[0,1]$. Is f Lebesgue integrable? Explain.

III.6 Show there are no bounded sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ for which $f_n(x) = a_n\sin(2\pi nx) + b_n\cos(2\pi nx)$ converges to 1 almost everywhere on $[0,1]$.

III.7 Let $f(x)$ be a real-valued measurable function on $[0,1]$. Show that

$$\lim_{n\to\infty}\int_0^1 (\cos(\pi f(x)))^{2n}dx = m\{x : f(x) \text{ is an integer}\},$$

where m denotes Lebesgue measure.

III.8 (a) Show that $f(x) = x^{-r}$ is a Lebesgue integrable function on $[0,1]$ if $0 \le r < 1$.

(b) If $0 \le r < 1$ let

$$a_n = \int_0^1 \frac{dx}{n^{-1} + x^r} \text{ (Lebesgue integral).}$$

Compute $\lim_{n\to\infty} a_n$. Be sure to quote the appropriate integration theorems to justify your calculations.

III.9 Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \frac{1}{n}, \text{ if } |x| \le n \\ 0, \text{ if } |x| > n \end{cases}$$

(a) Show that f_n converges to 0 uniformly on $\mathbb{R}$, and that

$$\lim_{n\to\infty}\int_{\mathbb{R}} f_n(x)\,dx = 2$$

while

$$\int_{\mathbb{R}}\left(\lim_{n\to\infty} f_n\right)(x)\,dx \ne 2$$

.

(b) Explain why this example does not contradict the Lebesgue dominated convergence theorem.

III.10 (a) Show that $f(x) = 1/\sqrt{x}$ is Lebesgue integrable on $(0,1)$.

(b) Find $\inf\{\int_0^1 \psi(x)\,dx | \psi$ is a simple function, and $\psi(x) \ge 1/\sqrt{x}$ on $(0,1)\}$. (Simple functions are finite linear combinations of characteristic functions of measurable sets with extended real-valued coefficients.)

III.11 Give an example of a sequence $(f_n)_{n\in\mathbb{N}}$ of bounded, measurable functions on $[0,1)$ such that

$$\lim_{n\to\infty}\int_0^1 |f_n(x)|\,dx = 0$$

but such that f_n converges pointwise nowhere.

III.12 Consider a Lebesgue-measurable function f on $\mathbb{R}$ with $\int_{\mathbb{R}} f(t)^2\,dt < \infty$. show that the function $g(x) = \int_{\mathbb{R}} f(t-x)f(t)\,dt$ is continuous.

III.13 Prove that $\lim_{n\to\infty}\int_{-\infty}^{\infty}(\sin nt)f(t)dt = 0$ for every Lebesgue integrable function f on $\mathbb{R}$. (Hint: Do the problem first for step functions.)

III.14 Let $f_n(x) = \dfrac{n}{x(lnx)^n}$ for $x \geq e$ and $n \in \mathbb{N}$.

(a) For which $n \in \mathbb{N}$ does the Lebesgue integral $\int_e^\infty f_n(x)\,dx$ exist?

(b) Determine $\lim\limits_{n\to\infty} f_n(x)$ for $x > e$.

(c) Does the sequence $(f_n)_{n\in\mathbb{N}}$ satisfy the assumptions of Lebesgue's dominated convergence theorem?

III.15 Define $f(x) = \int_{\mathbb{R}} \cos(xy)g(y)\,dy$ for $x \in I\!\!R$ where g is an integrable function on $I\!\!R$. Show that f is continuous.

III.16 Let $f \in L^\infty[0,1]$ and $\int_0^1 x^n f(x)\,dx = 0$ for $n \in \mathbb{N}$. Show that $f = 0$ a.e.

III.17 Let f be a non-negative Lebesgue-measurable function on $(0,\infty)$ such that f^2 is integrable. Let $F(x) = \int_0^x f(t)\,dt$ where $x > 0$. Show that $\lim_{x\to 0^+}\frac{F(x)}{\sqrt{x}} = 0$.

III.18 Let f be a differentiable function on $[-1,1]$. Prove that $\lim_{\epsilon\to 0}\int_{\epsilon<|x|\leq 1}\frac{1}{x}f(x)\,dx$ exists.

III.19 Let $f_n(x) := \dfrac{x^n}{n!}e^{-x}$ for $n \in \mathbb{N}_0$.

(a) Show that $\lim_{n\to\infty} f_n(x) = 0$ for all $x > 0$.

(b) Show that $f_n \in L^1(0,\infty)$ with $\|f_n\|_1 = 1$ for all $n \in \mathbb{N}_0$.

(c) Show that $\lim_{k\to\infty}\int_0^k \frac{x^n}{n!}(1-\frac{x}{k})^k\,dx = 1$ for all $n \in \mathbb{N}_0$.

III.20 Prove that $\lim_{b\to\infty}\int_0^b \frac{\sin x}{x}\,dx$ exists but that the function $\frac{\sin x}{x}$ is not integrable over $(0,\infty)$.

III.21 Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\, dx$$

III.22 Let f be a continuous nonnegative function on $[a,b]$ where $a<b$. Let $M=\max\{f(x) : a\le x\le b\}$. Show that

$$\lim_{n\to\infty} \left(\int_a^b f(x)^n\, dx\right)^{\frac{1}{n}} = M.$$

III.23 Find and justify the limits:

(a) $\displaystyle \lim_{n\to\infty} \int_0^n \frac{\sin x}{1+nx^2}\, dx$

(b) $\displaystyle \lim_{n\to\infty} \int_0^{e^n} \frac{x}{1+nx^2}\, dx.$

III.24 Let $f_n(x) = \sum_{i=0}^{n-1} \frac{1}{n} f(x+\frac{i}{n})$, where f is a continuous function on $I\!R$. Show that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ converges uniformly on every finite segment $[a,b]$ to the function $F(x) = \int_x^{x+1} f(s)\, ds$.

III.25 Let $f\in L^1(0,\infty)$, and suppose that $\int_0^\infty x|f(x)|\, dx<\infty$. Prove that the function

$$g(y) = \int_0^\infty e^{-xy} f(x) dx$$

is differentiable at every $y\in(0,\infty)$.

III.26 Let $f\in L^1(I\!R)$ with respect to Lebesgue measure, and suppose that

$$\int_{\mathbb{R}} |x||f(x)|\, dx<\infty.$$

Show that the function

$$g(y) = \int_{\mathbb{R}} e^{ixy} f(x) dx$$

is differentiable at every $y\in I\!R$.

III.27 Prove that, if f is a real-valued Lebesgue-integrable function on $\mathbb{R}$, then

$$\lim_{x\to 0} \int |f(x+t)-f(t)|\, dt = 0.$$

III.28 Let $f \in L^1(\mathbb{R})$.

(a) Prove: $\lim_{n\to\infty} \int_0^{1/n} f(x)dx = 0$.

(b) Prove or disprove: $\lim_{n\to\infty} \int_n^{\infty} f(x)\,dx = 0$.

III.29 Give an example of a sequence of uniformly bounded measurable functions f_n on $[0,1]$ such that $m\{x|f_n(x) \neq 0\} \to 0$ as $n \to \infty$, but the sequence $f_n(x)$ does not converge for any $x \in [0,1]$.

III.30 Let $n \geq 3$ and

$$f_n(x) = \begin{cases} n^2x, \text{ for } 0 \leq x < \dfrac{1}{n} \\ 2n - n^2x, \text{ for } \dfrac{1}{n} \leq x \leq \dfrac{2}{n} \\ 0, \text{ for } \dfrac{2}{n} < x \leq 1 \end{cases}$$

Sketch the graphs of f_3 and f_4. Prove that, if g is a continuous real-valued function on $[0,1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = g(0).$$

(Hint: First show that $\int_0^1 f_n(x)\,dx = 1$.)

III.31 Assume that the real-valued measurable function $f(t,x)$ and its partial derivative $\frac{\partial}{\partial t}f(t,x)$ are bounded on $[0,1]^2$. Show that for $t \in (0,1)$

$$\frac{d}{dt}\left[\int_0^1 f(t,x)\,dx\right] = \int_0^1 \frac{\partial}{\partial t}f(t,x)\,dx.$$

III.32 Prove that if f_n is Lebesgue integrable on $[0,1]$ for each $n \in \mathbb{N}$, and

$$\sum_{n=1}^{\infty} \int_0^1 |f_n(x)|\,dx < \infty,$$

then $\sum_{n=1}^{\infty} f_n(x)$ is convergent a.e., and

$$\int_0^1 \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_0^1 f_n(x)dx.$$

III.33 Let $f \in L^1(\mathbb{R})$. Prove that $\lim_{n\to\infty} \int_0^n e^{-nx} f(x)\,dx = 0$.

III.34 Assume that $A \geq 0$, $B > 0$, and f is continuous and nonnegative on $[a, b]$. Assume that $f(t) \leq A + B \int_a^t f(s)\,ds$ for $a \leq t \leq b$. Prove that $f(t) \leq Ae^{B(t-a)}$ for $a \leq t \leq b$.

III.35 Let f be a nonnegative integrable function on $[0, 1]$. Suppose that for every $n \in \mathbb{N}$

$$\int_0^1 (f(x))^n\,dx = \int_0^1 f(x)\,dx.$$

Show that f is almost everywhere the characteristic function for some measurable set.

III.36 Provide an example of a sequence $\{f_n\}$ of measurable functions on $[0, 1]$ such that $f_n \to f$ *a.e.* and $f_n \geq 0$, but $\liminf \int f_n \neq \int f$.

III.37 Let $1 \leq p < \infty$. Suppose $f_n \in L^p([0, 1])$, $\|f_n\|_p \leq 1$, and $f_n \to f$ a.e..

(a) Show that $f \in L^p([0, 1])$ and $\|f\|_p \leq 1$.

(b) Let $1 < p < \infty$ and $g \in L^q([0, 1])$ where $\frac{1}{p} + \frac{1}{q} = 1$. Prove that

$$\int_0^1 f_n g \to \int_0^1 fg.$$

(c) Give an example to show that the conclusion in item b would be false if $p = 1$.

4 Functions of bounded variation, absolutely continuous functions, convex functions, and the fundamental theorem of calculus

Unless otherwise stated, references to measure and integration are Lebesgue measure and integration on finite dimensional real space.

IV.1 If f is continuous on an interval $[a, b]$ and has a bounded derivative in (a, b), show that f is of bounded variation on $[a, b]$. Is the boundedness of f' necessary for f to be of bounded variation? Justify your answer.

IV.2 Let $f(x) = x^2 \sin(\frac{1}{x})$, $g(x) = x^2 \sin(\frac{1}{x^2})$ for $x \neq 0$, and $f(x) = g(x) = 0$ for $x = 0$.

(a) Show that f and g are differentiable everywhere (including at $x = 0$).

(b) Show that f is bounded variation on the interval $[-1, 1]$, but g is not.

(c) Let $f(x) = x \sin(1/x)$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$. Is f of bounded variation on $[-1, 1]$?

IV.3 Show that the product of two absolutely continuous functions on a closed finite interval $[a, b]$ is absolutely continuous.

IV.4 A real-valued function f on an interval I for which there exists a constant C such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all x and y in I is called a *Lipschitz function*.

(a) Show that a Lipschitz function is absolutely continuous.

(b) Show that an absolutely continuous function f on an interval is Lipschitz if and only if f' is essentially bounded.

IV.5 A function $f : [0, 1] \to L^1[0, 1]$ is called Lipschitz continuous if there exists $M > 0$ such that $\|f(t) - f(s)\|_1 \leq M|t - s|$ for all $t, s \in [0, 1]$. It is called differentiable at a point $s \in (0, 1)$ if the differential quotients $\frac{f(t)-f(s)}{(t-s)}$ converge in $L^1[0, 1]$ as $t \to s$. Let $f : [0, 1] \to L^1[0, 1]$ be given by $f(t) = \chi_{[0,t]}$, where $\chi_{[0,t]}$ denotes the characteristic function of the interval $[0, t]$. Show that f is Lipschitz continuous and nowhere differentiable.

IV.6 Let f be a nonnegative real function on $[0, 1]$ and let $I = \int_0^1 f(x)\, dx$. Show that

$$\sqrt{1 + I^2} \leq \int_0^1 \sqrt{1 + f^2(x)}\, dx \leq 1 + I.$$

IV.7 Suppose f is a nonnegative integrable function on $[0, 1]$. Prove that

$$\sqrt{\int_0^1 f(t)\, dt} \geq \int_0^1 \sqrt{f(t)}\, dt.$$

IV.8 (a) Provide an example of a function on $[0, 1]$ that is not absolutely continuous but is of bounded variation.

(b) Provide examples of two different continuous functions on $[0, 1]$ that have the same derivative *a.e.* and that are both equal to zero at 0.

IV.9 Suppose F is absolutely continuous on $[0,1]$ and that $g \in L^1([0,1])$, with $\int_0^1 g = 0$. Prove the "integration by parts" law:

$$\int_0^1 F(x)\, g(x)\, dx \;=\; -\int_0^1 \left[F'(x) \int_0^x g\right]\, dx$$

IV.10 (a) Provide an example of a function of unbounded variation on $[0,1]$ that has a derivative equal to zero at almost all $x \in [0,1]$.

(b) Provide an example of a function that is absolutely continuous on $[0,1]$ but has an unbounded derivative.

IV.11 Prove that, if f is differentiable a.e. on $[0,1]$ and f' is not in $L^1([0,1])$, then f is not of bounded variation on $[0,1]$.

IV.12 Prove that, if f is absolutely continuous on $[0,1]$, then the total variation of f on $[0,1]$ is equal to $\int_0^1 |f'|$.

IV.13 Suppose that f is a real-valued function of bounded variation on $[0,1]$.
Prove that

(a) f has a right- and left-hand limit at each point in $(0,1)$;

(b) f can have only countable many points of discontinuity;

(c) If, in addition to being of bounded variation on $[0,1]$, f is absolutely continuous on $[0,T]$ for each $T < 1$, then there exists an absolutely continuous function g on $[0,1]$ that coincides with f on $[0,1)$.

5 Basic properties of Banach spaces

Unless otherwise stated, references to measure and integration are Lebesgue measure and integration on finite dimensional real space.

V.1 Let $C([a,b])$ be the space of real continuous functions on a closed interval $[a,b]$ equipped with the sup norm. Let $\mathcal{M} = \{f \in C([a,b]) : f(x) > 0 \text{ for all } x \in [a,b]\}$. Show that $\mathcal{M}$ is an open subset of $C([a,b])$.

V.2 Let X be the normed linear space obtained by putting the norm $\|f\|_1 = \int_0^1 |f(t)|\, dt$ on the set of real continuous functions on $[0,1]$.

(a) Show that X is not a Banach space.

(b) Show that the linear functional $\Lambda f = f(1/2)$ is not bounded.

V.3 Show that $L_p[0,1]$ is separable for $1 \le p < \infty$, but not separable for $p = \infty$.

V.4 Show that $L^p(0,1) \subset L^q(0,1)$ for any $p > q \ge 1$. Here the integrability is with respect to the Lebesgue measure. Is the inclusion map for $L^p(0,1)$ into $L^q(0,1)$ continuous?

V.5 Prove or disprove the equality $L^\infty[0,1] = \cap_{1 \le p < \infty} L^p[0,1]$.

V.6 Let $f \in L_p(I\!R), 1 \le p < \infty$. Show that $\displaystyle\int_{|x|>n} |f(x)|^p \, dx \to 0$ for $n \to \infty$.

V.7 Let $g_n = n\chi_{[0,n^{-3}]}$. Show that $\int_0^1 f(x) g_n(x)\, dx \to 0$ as $n \to \infty$ for all $f \in L^2[0,1]$, but not all $f \in L^1[0,1]$.

V.8 Provide an example of the following:

(a) A nonzero bounded linear functional on $L^p([0,1])$, $1 < p < \infty$.

(b) A nonzero bounded linear functional on ℓ^∞.

V.9 Describe precisely how the dual of ℓ^1 is represented concretely.

V.10 Why is the dual of L^∞ not equal to L^1, in other words, why is L^1 not reflexive?

V.11 What is the completion of the space of continuous functions on $[0,1]$ in the p-norm ($1 \le p < \infty$)? In the ∞-norm?

V.12 Let $p + q = pq$. For $g \in L^q(E)$, define $\hat{g} \in (L^p(E))^*$ as $\hat{g}(f) = \int_E gf$. Prove that $\|\hat{g}\| = \|g\|_{L^q(E)}$.

V.13 Prove that the linear space of finite sequences is dense in ℓ^p for $1 \le p < \infty$, but it is not dense in ℓ^∞.

V.14 Prove that $L^\infty(\mathbb{R}) \cap L^p(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

V.15 Let $1 < p < q < r < \infty$. If $f \in L^p(\mathbb{R})$ and $f \in L^r(\mathbb{R})$, then $f \in L^q(\mathbb{R})$.

V.16 Prove that ℓ^∞ is not separable, that is, it has no countable dense set.

6 Sundry Problems

Unless otherwise stated, references to measure and integration are Lebesgue measure and integration on real n-space, m is Lebesgue measure on the appropriate dimensional real space and m_* is outer Lebesgue measure.

VI.1 Let E_1 and E_2 be countable sets. Show $E_1 \times E_2$ is countable.

VI.2 Let $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}^d$. Show

$$|d(x_1, y_1) - d(x_2, y_2)| \leq d(x_1, x_2) + d(y_1, y_2).$$

VI.3 Let $d = d_1 + d_2$ where d_1 and d_2 are natural numbers. Let $B_r(x)$ be the open ball of radius r with center x. Let $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$. Thus $(x, y) \in \mathbb{R}^d$. Let $\epsilon > 0$. Show

$$B_{\frac{\epsilon}{\sqrt{2}}}(x) \times B_{\frac{\epsilon}{\sqrt{2}}}(y) \subseteq B_\epsilon(x, y) \subseteq B_\epsilon(x) \times B_\epsilon(y).$$

In a similar vein, show that if $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, then

$$B_{\frac{\epsilon}{\sqrt{d}}}(x_1) \times B_{\frac{\epsilon}{\sqrt{d}}}(x_2) \times \cdots \times B_{\frac{\epsilon}{\sqrt{d}}}(x_d) \subseteq B_\epsilon(x) \subseteq B_\epsilon(x_1) \times B_\epsilon(x_2) \times \cdots \times B_\epsilon(x_d).$$

VI.4 Show every subsequence of a convergent sequence in $\mathbb{R}^d$ converges.

VI.5 Let E be a nonempty subset of $\mathbb{R}$. Let $c \in \mathbb{R}$. Show $\sup E = c$ if and only if $x \leq c$ for all $x \in E$ and for each $\epsilon > 0$, there is an $x \in E$ with $x + \epsilon > c$.

VI.6 Show every open subset U of $\mathbb{R}$ can be written has a countable disjoint union $\cup_{i \in I}(a_i, b_i)$ of disjoint open intervals.

VI.7 Let E be a subset of $\mathbb{R}^d$. Recall a subset S of E is said to be dense in E if for each $x \in E$ and each $\epsilon > 0$, one has $B_\epsilon(x) \cap S \neq \emptyset$. Using the countability of the set of points in $\mathbb{R}^d$ with rational entries, show every subset E of $\mathbb{R}^d$ has a countable dense subset.

VI.8 Let F_1 and F_2 be disjoint compact subsets of $\mathbb{R}^d$. Show $d(F_1, F_2) > 0$.

VI.9 Show if F_1 and F_2 are nonempty compact subsets of $\mathbb{R}^d$, there are points $x_1 \in F_1$ and $x_2 \in F_2$ with $d(x_1, x_2) = d(F_1, F_2)$.

VI.10 Show the Cantor ternary set is totally disconnected; that is show it contains no nonempty open interval.

VI.11 Show the Cantor ternary set consists of all x in $[0, 1]$ that can be written in form

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

where each a_k is either 0 or 2.

VI.12 Let f be a continuous nonnegative function on the closed interval $a \leq x \leq b$ where $a < b$. Let F be the closed subset $\{(x, y) \mid a \leq x \leq b,\ 0 \leq y \leq f(x)\}$ of $\mathbb{R}^2$. Show $m_*(F)$ is the Riemann integral $\int_a^b f(x)\, dx$.

VI.13 Show the set of irrationals is a G_δ set but is not an F_σ set. **Hint:** Show $\mathbb{Q}$ is not a G_δ for otherwise you could obtain a decreasing sequence G_n of dense open sets that have empty intersection. Then use the decomposition of each G_n into a disjoint countable union of open intervals.

VI.14 Using the fact that the rationals in any closed interval $a \le x \le b$ where $a < b$ is not a G_δ set, give an example of a Borel subset of $\mathbb{R}$ which is not an F_σ or a G_δ set.

VI.15 Let $\delta_1, \delta_2, \ldots, \delta_d$ be nonzero real numbers. Let E be a subset of $\mathbb{R}^d$. Define $\phi(x_1, x_2, \ldots, x_d) = (\delta_1 x_1, \delta_2 x_2, \ldots, \delta_d x_d)$.
Show

$$m_*(\phi(E)) = |\delta_1 \delta_2 \cdots \delta_d| m_*(E).$$

Then show E is Lesbesgue measurable if and only if $\phi(E)$ is Lebesgue measurable and then

$$m(\phi(E)) = |\delta_1 \delta_2 \cdots \delta_d| m(E).$$

VI.16 Recall that an extended real valued function $f : \mathbb{R}^d \to [-\infty, \infty]$ is measurable if $f^{-1}[-\infty, \alpha)$ is measurable for each $\alpha \in \mathbb{R}$. Show $f^{-1}(\alpha) = \{x \mid f(x) = \alpha\}$ is measurable for $\alpha \in [-\infty, \infty]$.

VI.17 Show a subset $E \subseteq \mathbb{R}^d$ is Lebesgue measurable if and only if for each $\epsilon > 0$ there is a measurable set W with $m_*(E \Delta W) < \epsilon$.

VI.18 Show the composition of two Borel measurable functions is Borel measurable.

VI.19 Show there is a sequence of Lebesgue measurable functions $f_n : [0,1] \to \mathbb{R}$ which converge pointwise everywhere to a function $f(x)$ having the property that there is not a subset A of $[0,1]$ of Lebesgue measure 0 such that f_n converges uniformly to f on the set $[0,1] - A$.

VI.20 Let $Q = [0,1]^d$ be the unit cube in $\mathbb{R}^d$ and suppose $f : Q \to \mathbb{R}$ is a continuous function. Show the Lebesgue measure of the compact set $\Gamma = \{(x, f(x)) \mid x \in Q\}$ in $\mathbb{R}^{d+1}$ is zero. **Hint:** Use uniform continuity.

VI.21 Suppose E_n is a sequence of measurable sets with $\sum m(E_n) < \infty$. Set $F_N = \cup_{n=N}^{\infty} E_n$.

(a) Show that
$$\lim_{N\to\infty} m(F_N) = 0.$$

(b) One defines $\limsup E_n$ by
$$\limsup_{n\to\infty} E_n = \cap_{N=1}^{\infty} F_N.$$
Show the $\limsup E_n$ consists of all points p which are in infinitely many of the E_n.

(c) Note (a) shows $m(\limsup E_n) = 0$ if $\sum_{n=1}^{\infty} m(E_n) < \infty$. Give an example of a sequence of measurable sets E_n where $m(E_n) \to 0$ as $n \to \infty$, but $m(\limsup E_n) \neq 0$.

VI.22 Suppose f_n is a sequence of real valued Lebesgue measurable functions on $\mathbb{R}^d$. Show if for each $\epsilon > 0$, $m\{x \mid |f_n(x)| \geq \epsilon\} \to 0$ as $n \to \infty$, then

(a) there is a subsequence f_{n_k} of f_n such that $f_{n_k}(x) \to 0$ as $k \to \infty$ for a.e. x.

(b) Show for each $\epsilon > 0$, there is a measurable subset E_ϵ of $\mathbb{R}^d$ and a subsequence f_{n_k} such that $m(E_\epsilon) < \epsilon$ and f_{n_k}converges uniformly to 0 off E_ϵ.

VI.23 Show if E_1 is a G_δ set in $\mathbb{R}^{d_1}$ and E_2 is a G_δ in $\mathbb{R}^{d_2}$, then $E_1 \times E_2$ is a G_δ set in $\mathbb{R}^{d_1+d_2}$.

VI.24 Let E and F be subsets of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$.

(a) Show if E has Lebesgue measure 0 in $\mathbb{R}^{d_1}$ and F is a bounded set in $\mathbb{R}^{d_2}$, then $E \times F$ is Lebesgue measurable and has Lebesgue measure 0 in $\mathbb{R}^{d_1+d_2}$.

(b) Show if E has Lebesgue measure 0, then $E \times F$ is Lebesgue measurable in $\mathbb{R}^{d_1+d_2}$ and has Lebesgue measure 0.

(c) Show if E and F are Lebesgue measurable, then $E \times F$ is Lebesgue measurable. (**Hint:** Take G_δ sets E' and F' containing E and F with $m(E' - E) = m(F' - F) = 0$ and consider $E' \times F'$.

VI.25 Show if $F \subseteq O \subseteq \mathbb{R}^d$ and F is compact and O is open, then there is a continuous function f such that $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}^d$ and $f(x) = 1$ on F and $f(x) = 0$ for $x \notin O$. **Hint:** Consider $f(x) = \min\{\frac{d(x,O^c)}{d(F,O^c)}, 1\}$.

VI.26 Let $f : \mathbb{R}^d \to [0, 1]$ be measurable. Set

$$E_{k,n} = \{x \mid |x| \leq n \text{ and } \frac{k}{n} \leq f(x) < \frac{k+1}{n}\}$$

where $k \in \{0, 1, 2, \ldots, n\}$. Show $E_{k,n}$ is measurable. Then for each k choose $F_{k,n} \subseteq E_{k,n} \subseteq O_{k,n}$ with $F_{k,n}$ compact, $O_{k,n}$ open and $m(O_{k,n} - F_{k,n}) < \frac{1}{2^n(n+1)}$. Now find continuous functions $f_{k,n}$ with $0 \leq f_{k,n} \leq 1$, $f_{k,n} = \frac{k}{n}$ on $F_{k,n}$, and $f = 0$ off $O_{k,n}$. Set $f_n = \sum_{k=0}^{n} f_{k,n}$. Each f_n is continuous. Show $f_n(x) \to f(x)$ as $n \to \infty$ for a.e. x.

VI.27 Let $f \geq 0$ be a Lebesgue measurable function on $\mathbb{R}^d$. Define $\mu(E) = \int_E f\, dm$.

(a) Show μ is a measure on the Lebesgue measurable sets.

(b) Show $\mu(E) = 0$ whenever $m(E) = 0$.

(c) Show μ is purely infinite on the set $E = \{x \mid f(x) = \infty\}$. (Note μ is purely infinite on E if and only if $\mu(F) = 0$ or $\mu(F) = \infty$ on any measurable subset F of E.)

(d) Show μ is σ-finite on the set $E = \{x \mid f(x) < \infty\}$. Note this means there is a sequence E_n of measurable sets with $E = \cup_{n=1}^{\infty} E_n$ and $\mu(E_n) < \infty$ for all n.

VI.28 It is known the ball of radius 1 in $\mathbb{R}^{2m}$ has volume (i.e., Lebesgue measure)

$$\frac{\pi^m}{m!}.$$

Find the volume of any ball in $\mathbb{R}^{2m}$ which has radius $r > 0$.

VI.29 Show if f is a Lebesgue measurable function on $\mathbb{R}^d$, then there exists a Borel measurable function h on $\mathbb{R}^d$ such that $f = h$ a.e. **Hint:** Let $q \in \mathbb{Q}$ and let $E_q = \{x \mid f(x) < q\}$. The set E_q is Lebesgue measurable. Choose a Borel set $B_q \supseteq E_q$ with $m(B_q - E_q) = 0$. Set $F_q = \cup_{q' < q} B_q$. Note $F_q \subseteq F_{q'}$ if $q < q'$. Define $h(x) = \inf\{q \mid x \in F_q\}$. Show h is a Borel function and equals f a.e. **Hint:** Show $f = h$ on the complement of $\cup_{q \in \mathbb{Q}}(B_q - E_q)$ and show h is a Borel function.

VI.30 Let $\langle a_n \rangle$ and $\langle b_n \rangle$ be sequences in $[-\infty, \infty]$. Show

(a) $\limsup a_n = -\liminf(-a_n)$

(b) $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$

VI.31 Let F be a measurable subset of $\mathbb{R}$. Let E be the set of x in F such that there is a $\delta > 0$ with $m((x - \delta, x + \delta) \cap F) = 0$. Show that E has measure 0.

VI.32 Determine

$$\lim_{n\to\infty}\int_1^\infty \sin\left(\frac{x}{n}\right)\frac{n^3}{1+n^2x^3}\,dx.$$

VI.33 Determine

$$\lim_{n\to\infty}\int_0^1 \sin\left(\frac{x}{n}\right)\frac{n^3}{1+n^2x^3}\,dx.$$

VI.34 Show there are no countably infinite σ-algebras.

VI.35 Give an example of a sequence of measurable real valued functions f_n such that

$$\liminf \int f_n(x)\,dx < \int \liminf f_n(x)\,dx.$$

Also give a sequence where $f_n \geq 0$ and

$$\int \liminf f_n(x)\,dx < \liminf \int f_n(x)\,dx.$$

VI.36 Prove the function

$$\phi(x) = \int_0^\infty \frac{t^{x-1}}{1+t}\,dt$$

is continuous for $0 < x < 1$. Is $\phi(x)$ differentiable on $0 < x < 1$?

VI.37 Let f be a real valued function on $\mathbb{R}\times\mathbb{R}$ and suppose

$$x \mapsto f(x,y)$$

is continuous for each y and there is a dense subset E of $\mathbb{R}$ such that

$$y \mapsto f(x,y)$$

is Lebesgue measurable for $x \in E$. Show f is Lebesgue measurable.

VI.38 Show if $p(x_1, x_2)$ is a nonzero polynomial in two variables, the set of points $x \in \mathbb{R}^2$ with $p(x) = 0$ has Lebesgue measure 0.

VI.39 Recall if f is Lebesgue measurable on $\mathbb{R}^d$ and E is a Lebesgue measurable subset of $\mathbb{R}^d$, then $\int_E f\,dm$ is defined if $f\chi_E$ is integrable and then $\int_E f\,dm = \int f\chi_E\,dm$. Show if $\int_Q f\,dm = 0$ for all closed cubes Q, then $f = 0$ a.e.

VI.40 Suppose $f \in L^1(\mathbb{R}^d)$. Show if $\int_E f\,dm \geq 0$ for all Lebesgue measurable sets E, then $f(x) \geq 0$ a.e. x.

VI.41 Give an example of a sequence of Lebesgue integrable function f_n on $\mathbb{R}^d$ where $f_n \to f$ pointwise, f is integrable, $\int f_n \, dm \to 1$ and $\int f \, dm = 0$.

VI.42 Give an example of a sequence of Lebesgue integrable functions f_n where $f_n \to f$ pointwise and $\int f_n \, dm \to 0$ and $\int f \, dm = 1$.

VI.43 Show if f is an integrable function on $\mathbb{R}^d$ and one has

$$\int_{\mathbb{R}^d} fh \, dm = 0$$

for every bounded continuous function h on $\mathbb{R}^d$, then $f = 0$ a.e.

VI.44 Let f be a Lebesgue integrable function on $\mathbb{R}^d$ and let Q be a closed cube. Let x_n be a sequence in $\mathbb{R}^d$ with $|x_n| \to \infty$ as $n \to \infty$. Show

$$\int_{x_n+Q} f \, dm \to 0 \text{ as } n \to \infty.$$

VI.45 Let f be a real valued function on $\mathbb{R}^d$ such that the set of x at which f is discontinuous has measure 0. Show f is Lebesgue measurable.

VI.46 (a) Show if $h : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is continuous and the preimage $h^{-1}(E)$ of any Borel subset E in $\mathbb{R}^{d_2}$ has Lebesgue measure 0 in $\mathbb{R}^{d_1}$, then $h^{-1}(E)$ is Lebesgue measurable in $\mathbb{R}^{d_1}$ for each Lebesgue measurable subset E of $\mathbb{R}^{d_2}$. Use this to show $f \circ h$ is Lebesgue measurable whenever $f : \mathbb{R}^{d_2} \to \mathbb{R}$ is Lebesgue measurable.

(b) Now show if $f : \mathbb{R}^d \to \mathbb{R}$ is Lebesgue measurable function, then $H(x, y) = f(x - y)$ is Lebesgue measurable on $\mathbb{R}^d \times \mathbb{R}^d$.

VI.47 Suppose f and h are L^1 functions on $\mathbb{R}^d$. Show $H(x, y) = f(x - y)h(y)$ is an L^1 function on $\mathbb{R}^{2d}$. Use this to show the function

$$f * h(x) = \int_{\mathbb{R}^d} f(x - y)h(y) \, dy$$

is defined almost everywhere x and gives an integrable function on $\mathbb{R}^d$. Then show $|f*h|_1 \leq |f|_1 |h|_1$.

VI.48 Let $f : [0, 1] \to \mathbb{R}$ be Lebesgue measurable and suppose $H(x, y) = f(x) - f(y)$ is integrable on $[0, 1]^2$. Show f is integrable on $[0, 1]$ and determine the integral of H.

VI.49 Suppose f is integrable on $\mathbb{R}^d$. Prove

$$H(\alpha) = m\{x \mid f(x) > \alpha\} - m\{x \mid f(x) < -\alpha\}$$

is integrable for $\alpha \geq 0$ and show

$$\int_{[0,\infty)} H \, dm_1 = \int_{\mathbb{R}^d} f \, dm.$$

VI.50 Let $1 < p < \infty$. Suppose $1 \leq q \leq \infty$ where $q \neq p$. Give an example of a measurable function f on $\mathbb{R}$ such that $f \in L^p(\mathbb{R})$ but $f \notin L^q(\mathbb{R})$.

VI.51 Suppose $1 \leq p \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Let $f \in L^p(\mathbb{R})$ where $1 \leq p < \infty$. Find a function $g \in L^q(\mathbb{R})$ so that $|g|_q = 1$ and $\int fg \, dm = |f|_p$.

(b) Give an example of an $f \in L^\infty(\mathbb{R})$ for which there is no $g \in L^1(\mathbb{R})$ such that $|g|_1 = 1$ and $\int fg \, dm = |f|_\infty$.

VI.52 Let Q be a closed cube in $\mathbb{R}^d$. Show $L^q(Q) \subseteq L^p(Q)$ whenever $1 \leq p \leq q \leq \infty$.

VI.53 Let Q be a unit cube in $\mathbb{R}^d$. Show if $f \in L^p(Q)$, then $|f|_p \leq |f|_q$ if $1 \leq p \leq q \leq \infty$.

VI.54 Let f and g be functions in $L^2(\mathbb{R}^d)$. Suppose $|x_n| \to \infty$ as $n \to \infty$. Show

$$\int f(x) g(x + x_n) \, dm(x) \to 0 \text{ as } n \to \infty.$$

VI.55 Let $f \in L^1(\mathbb{R}^2)$. Show $\int_0^1 f(x, y + n) \, dm(y)$ exists for a.e. x and defines a function H_n in $L^1(\mathbb{R})$. Then determine if the sequence H_n has a limit in $L^1(\mathbb{R})$.

VI.56 Let $1 \leq p < \infty$. Suppose $f_n \in L^p(\mathbb{R})$ and $f \in L^p(\mathbb{R})$ and

$$|f_n - f|_p \to 0 \text{ as } n \to \infty.$$

Show there is a subsequence $f_{n(k)}$ such that $f_{n(k)}(x)$ converges a.e. to $f(x)$ as $n \to \infty$.

VI.57 Show if $f : \mathbb{R}^d \to \mathbb{R}$ is a function satisfying for each $\epsilon > 0$, there is a Lebesgue measurable subset W with $m(W) < \epsilon$ and $f|_{W^c}$ is continuous on W^c, then f is Lebesgue measurable.

VI.58 Let F and G be functions on $[a, b]$ into $\mathbb{R}$ and let c be a constant. If F and G have bounded variation, show $V_a^b(F + G) \leq V_a^b F + V_a^b G$ and $V_a^b cF = |c| V_a^b F$.

VI.59 Show f has bounded variation if and only if f^+ and f^- have bounded variation. **Hint:** Show $|f|$ has bounded variation.

VI.60 Let $f : [a, b] \to \mathbb{R}$ be monotonically increasing. Show the set of points in $[a, b]$ where f is discontinuous is countable.

VI.61 Show if f is absolutely continuous on $[a, b]$, then the total variation $V_a^x f$ where $a \leq x \leq b$ is given by $V_a^x f = \int_a^b |f'(x)|\, dm(x)$. **Hint:** First show $F'(x) \geq |f'(x)|$ a.e by using $V_x^{x+h} f \geq |f(x+h) - f(x)|$ and then conclude $\int_a^c |f'(x)|\, dx \leq \int_a^c F'(x)\, dx \leq V_a^c f$. Then show $V_a^c f \leq \int_a^c |f'(x)|\, dx$.

VI.62 Show $f : [a, b] \to \mathbb{R}$ is absolutely continuous if and only if f^+ and f^- are absolutely continuous on $[a, b]$.

VI.63 Let $V_a^x g$ denote the total variation of g from a to x. Show if f is absolutely continuous on $[a, b]$, then $V_a^x f = V_a^x |f|$ for $a \leq x \leq b$ and then give an example of a function f of bounded variation where $V_a^x f$ and $V_a^x |f|$ are different. You may use $V_a^x f = \int_a^x |f'|\, dm$ if f is absolutely continuous.