ON STOCHASTIC INTEGRATION FOR WHITE NOISE DISTRIBUTION THEORY

A Dissertation

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Abstract

This thesis consists of two parts, each part concentrating on a different problem from the theory of Stochastic Integration. Chapter 1 contains the introduction explaining the results in this dissertation in general terms. We use the infinite dimensional space $\mathcal{S}'(\mathbb{R})$, which is the dual of the Schwartz space $\mathcal{S}(\mathbb{R})$, endowed with the gaussian measure μ. The Hilbert space (L^2) is defined as $(L^2) \equiv L^2(\mathcal{S}'(\mathbb{R}), \mu)$, and our results are based on the Gel'fand triple $(\mathcal{S})_\beta \subset (L^2) \subset (\mathcal{S})^*_\beta$. The necessary preliminary background in white noise analysis are well elaborated in Chapter 2.

In the first part of the dissertation we present a generalization of the Itô Formula to anticipating processes in the white noise framework. This is contained in Chapter 3. We first introduce an extension of the Itô integral to anticipating processes called the *Hitsuda Skorokhod integral*. We then use the anticipating Itô formula for processes of the type $\theta(X(t), B(c))$, where $X(t)$ is a Wiener integral and $B(t) = \langle \cdot, 1_{[0,t)} \rangle$ is Brownian motion for $a \leq c \leq b$, which was obtained in [21] as a motivation to obtain our first main result. In this result, we generalize the formula in [21] to processes of the form $\theta(X(t), F)$ where $X(t)$ is a Wiener integral and $F \in \mathcal{W}^{1/2}$. The space $\mathcal{W}^{1/2}$ is a special subspace of (L^2).

Chapter 4 contains the second part of our work. We first state the Clark-Ocone (C-O) formula in the white noise framework. Before the main result we verify the formula for Brownian functionals using the very important white noise tool, the S-transform. Previous proofs of this formula have not used this method. As our second main result, we extend the formula to generalized Brownian functionals of the form $\Phi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : , f_n \rangle$, where $f_n \in L^2(\mathbb{R})^{\hat{\otimes} n}$. The formula for such Φ takes on the same form as that for Brownian functionals. As examples we verify the formula

for the Hermite Brownian functional $:B(t)^n:_t$ using the Itô formula. We also compute the C-O formula for the Donsker's delta function using the result for the Hermite Brownian functional. Finally, the formula is generalized to compositions of tempered distributions with Brownian motion, i.e, to functionals of the form $f(B(t))$ where $f \in \mathcal{S}'(\mathbb{R})$ and $B(t) = \langle \cdot, 1_{[0,t]} \rangle$ is Brownian motion.

Chapter 1. Introduction

In the first part of this dissertation we prove in the white noise framework the anticipating Itô formula

$$\theta(X(t), F) = \theta(X(a), F) + \int_a^t \partial_s^* \left(f(s) \frac{\partial \theta}{\partial x}(X(s), F) \right) ds + \frac{1}{2} \int_a^t f(s)^2 \frac{\partial^2 \theta}{\partial x^2}(X(s), F)\, ds + \int_a^t f(s)(\partial_s F) \frac{\partial^2 \theta}{\partial x \partial y}(X(s), F)\, ds, \tag{1.1}$$

where $f \in L^\infty([a,b])$, $X(t) = \int_a^t f(s) dB(s)$ is a Wiener integral, $F \in \mathcal{W}^{1/2}, \theta \in \mathcal{C}_b^2(\mathbb{R}^2)$ and the integral $\int_a^t \partial_t^* \left(f(s) \frac{\partial \theta}{\partial x}(X(s), F) \right) ds$ is a Hitsuda-Skorokhod integral as introduced by Hitsuda [9] and Skorokhod [31].

As our second result, we study the Clark-Ocone representation formula [4] [27] for Brownian functionals $F \in \mathcal{W}^{1/2}$ given by

$$F = E(F) + \int_T E(\partial_t F | \mathcal{F}_t)\, dB(t). \tag{1.2}$$

Since $E(\partial_t F \mid \mathcal{F}_t)$ is nonanticipating, equation (1.2) can be rewritten as

$$F = E(F) + \int_T \partial_t^* E(\partial_t F | \mathcal{F}_t)\, dt. \tag{1.3}$$

We verify this equation for Brownian functionals using the S-transform. Then we extend this representation to generalized functions Φ in the Kondratiev-Streit space satisfying the conditions $\Phi = \sum_{n=0}^\infty \langle : \cdot^{\otimes n} : , f_n \rangle$, $f_n \in L^2(\mathbb{R})^{\hat{\otimes} n}$.

Since the Hermite Brownian functional plays a very important role in white noise analysis, using the Itô formula, we verify its Clark-Ocone representation. As a specific example of a generalized Brownian functional that meets our conditions, we compute the Clark-Ocone formula for the Donsker's delta function which is

$$\delta(B(t)-a)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{a^2}{2t}}+\int_{-\infty}^{t}\frac{1}{\sqrt{2\pi}}\frac{(a-B(s))}{(t-s)^{\frac{3}{2}}}e^{-\frac{(a-B(s))^2}{2(t-s)}}\,dB(s). \tag{1.4}$$

This formula relies heavily on the representation for the Hermite Brownian functional. Finally, we generalize the Clark-Ocone formula to compositions of tempered distributions with Brownian motion, i.e, to generalized Brownian functionals of the form $f(B(t))$ where $f\in\mathcal{S}'(\mathbb{R})$. As an observation, another generalization of the Clark-Ocone formula was obtained in [6] for regular generalized functions in the space $(\mathcal{S})_1^*$ which is larger than our space $(\mathcal{S})_\beta^*$, $0\le\beta<1$. But we have a stronger result than the one in [6].

Chapter 2. Background

In this chapter the basic concepts and notations from White Noise Analysis of interest to our work are introduced. The proofs for the necessary theorems are not presented in here and the interested reader is provided with the relevant references.

2.1 Concept and Notations

Let E be a real separable Hilbert space with norm $|\cdot|_0$. Let A be a densely defined self-adjoint operator on E, whose eigenvalues $\{\lambda_n\}_{n\geq 1}$ satisfy the following conditions:

- $1 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$.
- $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty$.

For any $p \geq 0$, let $\mathcal{E}_p$ be the completion of E with respect to the norm $|f|_p := |A^p f|_0$. We note that $\mathcal{E}_p$ is a Hilbert space with the norm $|\cdot|_p$ and $\mathcal{E}_p \subset \mathcal{E}_q$ for any $p \geq q$. The second condition on the eigenvalues above implies that the inclusion map $i : \mathcal{E}_{p+1} \longrightarrow \mathcal{E}_p$ is a Hilbert-Schmidt operator.

Let

$$\mathcal{E} = \text{the projective limit of } \{\mathcal{E}_p\,; p \geq 0\},$$

$$\mathcal{E}' = \text{the dual space of } \mathcal{E}.$$

Then the space $\mathcal{E} = \bigcap_{p\geq 0} \mathcal{E}_p$ equipped with the topology given by the family $\{|\cdot|_p\}_{p\geq 0}$ of semi-norms is a nuclear space. Hence $\mathcal{E} \subset E \subset \mathcal{E}'$ is a Gel'fand triple with the following continuous inclusions:

$$\mathcal{E} \subset \mathcal{E}_q \subset \mathcal{E}_p \subset E \subset \mathcal{E}_p' \subset \mathcal{E}_q' \subset \mathcal{E}', \qquad q \geq p \geq 0.$$

We used the Riesz Representation Theorem to identify the dual of E with itself.

It can be shown that for all $p \geq 0$, the dual space $\mathcal{E}_p'$ is isomorphic to $\mathcal{E}_{-p}$, which is the completion of the space E with respect to the norm $|f|_{-p} = |A^{-p}f|_0$.

Minlo's theorem allows us to define a unique probability measure μ on the Borel subsets of $\mathcal{E}'$ with the property that for all $f \in \mathcal{E}$, the random variable $\langle \cdot, f\rangle$ is normally distributed with mean 0 and variance $|f|_0^2$. We are using $\langle\ ,\ \rangle$ to denote the duality between $\mathcal{E}'$ and $\mathcal{E}$. This means that the characteristic functional of μ is given by

$$\int_{\mathcal{E}'} e^{i\langle x,\xi\rangle} d\mu(x) = e^{-\frac{1}{2}|\xi|_0^2}, \qquad \forall \xi \in \mathcal{E}. \tag{2.1}$$

The probability space $(\mathcal{E}', \mu)$ is called the white noise space. The space $L^2(\mathcal{E}', \mu)$ will be denoted by (L^2); i.e., (L^2) is the set of functions $\varphi : \mathcal{E}' \longrightarrow \mathbb{C}$ such that φ is measurable and $\int_{\mathcal{E}'} |\varphi(x)|^2 d\mu(x) < \infty$. If we denote by E_c the complexification of E, the Wiener-Itô Theorem allows us to associate to each $\varphi \in (L^2)$ a unique sequence $\{f_n\}_{n\geq 0}$, $f_n \in E_c^{\hat{\otimes} n}$ and express φ as $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$ where $I_n(f_n)$ is a multiple Wiener integral of order n (see [11]). This decomposition is similar to what is referred to as the "Fock-Space decomposition" as shown in [26].

The (L^2)-norm $\| \varphi \|_0$ of φ is given by

$$\| \varphi \|_0^2 = \sum_{n=0}^{\infty} n! \mid f_n \mid_0^2,$$

where $| \cdot |_0$ denotes the $E_c^{\hat{\otimes} n}$-norm induced from the norm $| \cdot |_0$ on E. For any $p \geq 0$, let $| \cdot |_p$ be the $\mathcal{E}_{p,c}^{\hat{\otimes} n}$-norm induced from the norm $| \cdot |_p$ on $\mathcal{E}_p$ and define

$$\| \varphi \|_p = \left(\sum_{n=0}^{\infty} n! \mid f_n \mid_p^2 \right)^{\frac{1}{2}}.$$

Let

$$(\mathcal{E}_p) = \{ \varphi \in (L^2)\,;\, \| \varphi \|_p < \infty \}.$$

If $0 < p \le q$, then $(\mathcal{E}_q) \subset (\mathcal{E}_p)$ with the property that for any $q \ge 0$, there exists $p > q$ such that the inclusion map $I_{p,q} : (\mathcal{E}_p) \hookrightarrow (\mathcal{E}_q)$ is a Hilbert-Schmidt operator and $\| I_{p,q} \|_{HS}^2 \le (1 - \| i_{p,q} \|_{HS}^2)^{-1}$ where $i_{p,q}$ is the inclusion map from $\mathcal{E}_p$ into $\mathcal{E}_q$ as noted earlier on.

Analogous to the way $\mathcal{E}$ was defined, we also define

$$(\mathcal{E}) = \text{the projective limit of } \{(\mathcal{E}_p)\,; p \ge 0\},$$

$$(\mathcal{E})^* = \text{the dual space of } (\mathcal{E}).$$

With the above result, $(\mathcal{E}) = \bigcap_{p \ge 0} (\mathcal{E}_p)$ with the topology generated by the family $\{\| \cdot \|_p\,; p \ge 0\}$ of norms. It is a nuclear space forming the infinite dimensional Gel'fand triple $(\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*$. Moreover we have the following continuous inclusions

$$(\mathcal{E}) \subset (\mathcal{E}_q) \subset (\mathcal{E}_p) \subset (L^2) \subset (\mathcal{E}_{-p}) \subset (\mathcal{E}_{-q}) \subset (\mathcal{E})^*, \qquad q \ge p \ge 0.$$

The elements in $(\mathcal{E})$ are called <u>test functions</u> on $\mathcal{E}'$ while the elements in $(\mathcal{E})^*$ are called <u>generalized functions</u> on $\mathcal{E}'$. The bilinear pairing between $(\mathcal{E})$ and $(\mathcal{E})^*$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. If $\varphi \in (L^2)$ and $\psi \in (\mathcal{E})$ then $\langle\langle \varphi, \psi \rangle\rangle = (\varphi, \overline{\psi})$, where $(\cdot, \cdot)$ is the inner product on the complex Hilbert space (L^2).

An element $\varphi \in (\mathcal{E})$ has a unique representation as $\varphi = \sum_{n=0}^{\infty} I_n(f_n),\ f_n \in \mathcal{E}_c^{\hat{\otimes} n}$ with the norm

$$\|\varphi\|_p^2 = \sum_{n=0}^{\infty} n! |f_n|_p^2 < \infty, \qquad \forall p \ge 0.$$

Similarly, an element $\phi \in (\mathcal{E})^*$ can be written as $\phi = \sum_{n=0}^{\infty} I_n(F_n),\ F_n \in (\mathcal{E}_c')^{\hat{\otimes} n}$ with the norm

$$\| \phi \|_{-p}^2 = \sum_{n=0}^{\infty} n! \mid F_n \mid_{-p}^2,$$

The bilinear pairing between ϕ and φ is then represented as

$$\langle\langle \phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle.$$

Kondratiev and Streit (see Chapter 4 in [21]) constructed a wider Gel'fand triple than the one above in the following way:

Let $0 \leq \beta < 1$ be a fixed number. For each $p \geq 0$, define

$$\|\varphi\|_{p,\beta} = \left(\sum_{n=0}^{\infty} (n!)^{1+\beta} |(A^p)^{\otimes n} f_n|_0^2 \right)^{1/2}$$

and let

$$(\mathcal{E}_p)_\beta = \{\varphi \in (L^2); \|\varphi\|_{p,\beta} < \infty\}.$$

It should be noted here that $(\mathcal{E}_0)_\beta \neq (L^2)$ unless $\beta = 0$. Similarly to the above setting, for any $p \geq q \geq 0$, we have $(\mathcal{E}_p)_\beta \subset (\mathcal{E}_q)_\beta$ and the inclusion map $(\mathcal{E}_{p+\frac{\alpha}{2}})_\beta \hookrightarrow (\mathcal{E}_p)_\beta$ is a Hilbert-Schmidt operator for some positive constant α.

Let

$$(\mathcal{E})_\beta = \text{the projective limit of } \{(\mathcal{E}_p)_\beta; p \geq 0\},$$

$$(\mathcal{E})_\beta^* = \text{the dual space of } (\mathcal{E})_\beta.$$

Then $(\mathcal{E})_\beta$ is a nuclear space and we have the Gel'fand triple $(\mathcal{E})_\beta \subset (L^2) \subset (\mathcal{E})_\beta^*$ with the following continuous inclusions:

$$(\mathcal{E})_\beta \subset (\mathcal{E}_p)_\beta \subset (L^2) \subset (\mathcal{E}_p)_\beta^* \subset (\mathcal{E})_\beta^*, \qquad p \geq 0,$$

where the norm on $(\mathcal{E}_p)_\beta^*$ is given by

$$\|\varphi\|_{-p,-\beta} = \left(\sum_{n=0}^{\infty} (n!)^{1-\beta} |(A^{-p})^{\otimes n} f_n|_0^2 \right)^{1/2}.$$

Thus we have the following relationships with the earlier Gel'fand triple first discussed:

$$(\mathcal{E})_\beta \subset (\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^* \subset (\mathcal{E})_\beta^*.$$

An even wider Gel'fand triple was introduced in [5] to create what is known as the *Cochran-Kuo-Sengupta* (CKS) space. The interested reader is referred to that paper for its construction.

2.2 Hermite Polynomials, Wick Tensors, and Multiple Wiener Integrals

The Hermite polynomial of degree n with parameter σ^2 is defined by

$$: x^n :_{\sigma^2} = (-\sigma^2)^n e^{\frac{x^2}{2\sigma^2}} D_x^n e^{-\frac{x^2}{2\sigma^2}}.$$

These polynomials have a generating function given by

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} : x^n :_{\sigma^2} = e^{tx - \frac{1}{2}\sigma^2 t^2}. \tag{2.2}$$

The following formulas are also helpful:

$$: x^n :_{\sigma^2} = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!(-\sigma^2)^k x^{n-2k}, \tag{2.3}$$

$$x^n = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!! \sigma^{2k} : x^{n-2k} :_{\sigma^2}, \tag{2.4}$$

where $(2k-1)!! = (2k-1)(2k-3)\cdots 3.1$ with $(-1)!! = 1$.

The trace operator is the element $\tau \in (\mathcal{E}_c')^{\hat{\otimes} 2}$ defined by

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \; \xi, \eta \in \mathcal{E}_c.$$

Let $x \in \mathcal{E}'$. The Wick tensor $: x^{\otimes n} :$ of an element x is defined as

$$: x^{\otimes n} : \; = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!(-1)^k x^{\otimes(n-2k)} \hat{\otimes} \tau^{\otimes k},$$

where τ is the trace operator. The following formula similar to equation (2.3) is also important for Wick tensors, i.e.,

$$x^{\otimes n} = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!! : x^{\otimes(n-2k)} : \hat{\otimes} \tau^k.$$

For $x \in \mathcal{E}'$ and $\xi \in \mathcal{E}$, the following equalities related to Wick tensors hold:

$$\langle : x^{\otimes n} : , \xi^{\otimes n} \rangle =: \langle x, \xi \rangle^n :_{|\xi|_0^2},$$

$$\| \langle : x^{\otimes n} : , \xi^{\otimes n} \rangle \|_0 = \sqrt{n!} |\xi|_0^n.$$

In order to make mathematical computations concerning multiple Wiener integrals easier, they are expressed in terms of Wick tensors. This is achieved via two statements as follows (see Theorem 5.4 [21]).

1. Let $h_1, h_2, \cdots \in E$ be orthogonal and let $n_1 + n_2 + \cdots = n$. Then for almost all $x \in \mathcal{E}'$, we have

$$\langle : x^{\otimes n} : , h_1^{\otimes n_1} \hat{\otimes} h_2^{\otimes n_2} \hat{\otimes} \cdots \rangle =: \langle x, h_1 \rangle^{n_1} :_{|h_1|_0^2} : \langle x, h_2 \rangle^{n_2} :_{|h_2|_0^2} \cdots .$$

2. Let $f \in E_c^{\hat{\otimes} n}$. Then for almost all $x \in \mathcal{E}'$,

$$I_n(f)(x) = \langle : x^{\otimes n} : , f \rangle$$

where $I_n(f)(x)$ is a multiple Wiener integral of order n. With this relationship, we are able to write test functions and generalized functions in terms of Wick tensors as follows: Any element $\varphi \in (L^2)$ can be expressed as

$$\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , f_n \rangle, \ \mu - \text{a.e. for } x \in \mathcal{E}', \ f_n \in E_c^{\hat{\otimes} n}.$$

Suppose $\varphi \in (\mathcal{E})$. Since $\mathcal{E}_c^{\hat{\otimes} n}$ is dense in $E_c^{\hat{\otimes} n}$, we have

$$\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , f_n \rangle, \qquad f_n \in \mathcal{E}_c^{\hat{\otimes} n}.$$

For an element $\phi \in (\mathcal{E}_p)^*$, it follows that

$$\phi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , F_n \rangle, \qquad F_n \in (\mathcal{E}_p')^{\hat{\otimes} n}.$$

2.3 The White Noise Differential Operator

Let $y \in \mathcal{E}'$ and $\varphi = \langle : x^{\otimes n} : , f \rangle \in (\mathcal{E})_\beta$. It can be shown that the directional derivative

$$\lim_{t \to 0} \frac{\varphi(x + ty) - \varphi(x)}{t} = n\langle : x^{\otimes(n-1)} :, y\hat{\otimes}_1 f \rangle,$$

where $y\hat{\otimes}_1 \cdot : E_c^{\hat{\otimes} n} \longrightarrow E_c^{\hat{\otimes}(n-1)}$ is the unique continuous and linear map such that

$$y\hat{\otimes}_1 g^{\otimes n} = \langle y, g \rangle g^{\otimes(n-1)}, \quad g \in E_c.$$

This shows that the function φ has Gâteaux derivative $D_y\varphi$. For general $\varphi(x) = \sum_{n=0}^{\infty} \langle : x^{\otimes n} : , f_n \rangle, \; f_n \in \mathcal{E}_c^{\hat{\otimes} n}$, we define an operator D_y on $(\mathcal{E})_\beta$ as

$$D_y\varphi(x) \equiv \sum_{n=1}^{\infty} n\langle : x^{\otimes(n-1)} :, y\hat{\otimes}_1 f_n \rangle.$$

It can be shown that D_y is a continuous linear operator on $(\mathcal{E})_\beta$ (see section 9.1, [21]). By the duality between $(\mathcal{E})_\beta^*$ and $(\mathcal{E})_\beta$ the adjoint operator D_y^* of D_y can be defined by

$$\langle\langle D_y^* \Phi, \varphi \rangle\rangle = \langle\langle \Phi, D_y\varphi \rangle\rangle, \; \Phi \in (\mathcal{E})^*, \; \varphi \in (\mathcal{E}).$$

Now let $\mathcal{E}$ be the Schwartz space of all infinitely differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\forall n, \; k \in \mathbb{N}$,

$$\sup_{x \in \mathbb{R}} \left| x^n \left(\frac{d^k}{dx^k} \right) f(x) \right| < \infty$$

If we take $y = \delta_t$, the Dirac delta function at t, then

1. $\partial_t \equiv D_{\delta_t}$ is called the white noise differential operator, or the Hida differential operator , or the annihilation operator,

2. $\partial_t^* \equiv D_{\delta_t}^*$ is called the creation operator, and

3. For $\varphi \in (\mathcal{S})_\beta, \; \dot{B}(t)\varphi \equiv \partial_t\varphi + \partial_t^*\varphi$ is called white noise multiplication.

2.4 Characterization Theorems and Convergence of Generalized Functions

Let $h \in E_c$. The <u>renormalization</u> $: e^{\langle \cdot, h \rangle} :$ of $e^{\langle \cdot, h \rangle}$ is defined by

$$: e^{\langle \cdot, h \rangle} : = \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} : , h^{\otimes n} \rangle. \tag{2.5}$$

Moreover,

$$: e^{\langle \cdot, h \rangle} : = e^{\langle \cdot, h \rangle - \frac{1}{2}\langle h, h \rangle}. \tag{2.6}$$

If $\xi \in \mathcal{E}_c$, then $: e^{\langle \cdot, h \rangle} : \in (\mathcal{E})_\beta$ for all $0 \leq \beta < 1$. If $y \in \mathcal{E}_c'$, then $: e^{\langle \cdot, h \rangle} : \in (\mathcal{E})_\beta^*$. For such a y and ξ we have

$$\langle\langle : e^{\langle \cdot, y \rangle}, : e^{\langle \cdot, \xi \rangle} : \rangle\rangle = e^{\langle y, \xi \rangle}. \tag{2.7}$$

The renormalized exponential functions $\{ : e^{\langle \cdot, \xi \rangle} : \ \big| \ \xi \in \mathcal{E}_c \}$ are linearly independent and span a dense subspace of $(\mathcal{E})_\beta$.

For all $\Phi \in (\mathcal{E})_\beta^*$, the S-transform $S\Phi$ of Φ is defined to be the function on $\mathcal{E}_c$ given by

$$S\Phi(\xi) = \langle\langle \Phi, : e^{\langle \cdot, \xi \rangle} : \rangle\rangle, \qquad \xi \in \mathcal{E}_c \tag{2.8}$$

The S-transform is an injective function because the exponential functions span a dense subspace of $(\mathcal{E})_\beta$. By using the S-transform, we can extend the operator ∂_t from $(\mathcal{S})_\beta$ to the space $(\mathcal{S})_\beta^*$ as follows.

For $\Phi \in (\mathcal{S})_\beta^*$, let $F(\xi) = S\Phi(\xi)$ and let $F'(\xi; t)$ be the functional derivative of F at ξ. The function F is said to have the first variation if it satisfies the condition

$$F(\xi + \delta\xi) = F(\xi) + \delta F(\xi) + o(\delta\xi), \qquad \xi, \delta\xi \in \mathcal{S}(\mathbb{R}),$$

where $\delta F(\xi)$ is given by

$$\delta F(\xi) = \int_{\mathbb{R}} F'(\xi;t)\delta\xi(t)dt.$$

Definition 2.1. (see [20]). *Suppose that $F'(xi;t)$ where $\xi \in \mathcal{S}(\mathbb{R})$ is the S-transform of some generalized function. Then we define $\partial_t \Phi$ to be the generalized function with S-transform $F'(\xi;t)$, i.e.,*

$$S(\partial_t \Phi)(\xi) = F'(\xi;t), \qquad \xi \in \mathcal{S}(\mathbb{R}).$$

Example. The Hermite Brownian functional gives us the following

$$\begin{aligned}
\Phi &= \langle : \cdot^{\otimes n} : , 1_{[0,t]}^{\otimes n} \rangle, \\
F(\xi) &= S\Phi(\xi) = \xi(t)^n, \\
\delta F(\xi) &= n\xi(t)^{n-1}\delta\xi(t) = \int_{\mathbb{R}} n 1_{[0,t]}(u)\xi(u)^{n-1}\delta\xi(u)\, du, \\
F'(\xi;t) &= n 1_{[0,t]}(t)\xi(t)^{n-1} = n\xi(t)^{n-1}, \\
\partial_t \Phi &= n\langle : \cdot^{n-1} : , 1_{[0,t]}^{\otimes(n-1)} \rangle.
\end{aligned}$$

The main purpose of the S-transform is to change an otherwise infinite-dimensional object into a finite-dimensional one. By so doing, ordinary calculus can be applied to the finite-dimensional case before finally reconverting it back (if possible) into the infinite-dimensional setup. A good example is when solving white noise stochastic diffrential equations. As it was done in chapter 13 of [21], the S-transform was first applied to the white noise diffrential equation thereby obtaining an ordinary differential equation, before solving it and then reconverting the result back to the white noise setup. It is therefore necessary to have unique characterization theorems for both test and generalized functions which we will state without proof. The interested reader is referred to [21] chapter 8. We will start by characterizing generalized functions in the following theorem. The next theorem will then characterize test functions; and in both cases, the S-transform plays a very important role.

Theorem 2.2. Let $\Phi \in (\mathcal{E}_\beta)^*$. Then its S-transform $F = S\Phi$ satisfies the condition:

(a) For any ξ and η in $\mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.

(b) There exists nonnegative constants K, a, and p such that

$$|F(\xi)| \leq K \exp\left[a|\xi|_p^{\frac{2}{1-\beta}}\right], \qquad \forall \xi \in \mathcal{E}_c$$

Conversely, suppose a function F defined on $\mathcal{E}_c$ satisfies the above two conditions. Then there exists a unique $\Phi \in (\mathcal{E})^*_\beta$ such that $F = S\Phi$ and for any q satisfying the condition that $e^2\left(\frac{2a}{1-\beta}\right)^{1-\beta}\|A^{-(q-p)}\|^2_{HS} < 1$, the following inequality holds:

$$\|\Phi\|_{-q,-\beta} \leq K\left(1 - e^2\left(\frac{2a}{1-\beta}\right)^{1-\beta}\|A^{-(q-p)}\|^2_{HS}\right)^{-1/2}$$

Theorem 2.3. *Let F be a function on $\mathcal{E}_c$ satisfying the conditions:*

(a) For any ξ and η in $\mathcal{E}_c$, the function $F(z\xi + \eta)$ is an entire function of $z \in \mathbb{C}$.

(b) There exists positive constants K, a, and p such that

$$|F(\xi)| \leq K \exp\left[a|\xi|_{-p}^{\frac{2}{1+\beta}}\right], \qquad \forall \xi \in \mathcal{E}_c.$$

Then there exists a unique $\varphi \in (\mathcal{E}_\beta)^*$ such that $F = S\varphi$. In fact, $\varphi \in (\mathcal{E}_q)_\beta$ for any $q \in [0, \infty)$ satisfying the condition $e^2\left(\frac{2a}{1+\beta}\right)^{1+\beta}\|A^{-(p-q)}\|^2_{HS} < 1$ and

$$\|\varphi\|_{q,\beta} \leq K\left(1 - e^2\left(\frac{2a}{1+\beta}\right)^{1+\beta}\|A^{-(p-q)}\|^2_{HS}\right)^{-1/2}$$

In the next theorem, we state the necessary and sufficient condition for determining the convergence of generalized functions. This is done in terms of the S-transform. For a proof, see Theorem 8.6 in [21].

Theorem 2.4. *Let $\Phi_n \in (\mathcal{E})^*_\beta$ and $F_n = S\Phi_n$. Then Φ_n converges strongly in $(\mathcal{E})^*_\beta$ if and only if the following conditions are satisfied:*

(a) $\lim_{n\to\infty} F_n(\xi)$ exists for each $\xi \in \mathcal{E}_c$.

(b) There exists nonnegative constants $K, a,$ and p, independent of n, such that

$$|F_n(\xi)| \leq K \exp\left[a|\xi|_p^{\frac{2}{1-\beta}}\right], \qquad \forall n \in \mathbb{N},\ \xi \in \mathcal{E}_c.$$

.

2.5 Pettis and Bochner Integrals

Let $(M, \mathcal{B}, m)$ be a sigma-finite measure space. Let $\mathbb{X}$ be a Banach space with norm $\|\cdot\|$. A function $f : M \to \mathbb{X}$ is called weakly measurable if $\langle w, f(\cdot)\rangle$ is measurable for each $w \in \mathbb{X}^*$ (with the understanding that $\langle \cdot, \cdot \rangle$ denotes the bilinear pairing of $\mathbb{X}^*$ and $\mathbb{X}$). A weakly measurable $\mathbb{X}$-valued function f on M is called Pettis integrable if it satisfies the following conditions:

(1) For any $w \in \mathbb{X}^*$, $\langle w, f(\cdot)\rangle \in L^1(M)$.

(2) For any $E \in \mathcal{B}$, there exists an element $J_E \in \mathbb{X}$ such that

$$\langle w, J_E \rangle = \int_E \langle w, f(u)\rangle \, dm(u), \qquad \forall w \in \mathbb{X}^*.$$

This J_E is unique. It is denoted by $(P)\int_E f(u)\, dm(u)$ and is called the Pettis integral of f on E. It is noted that if $\mathbb{X}$ is reflexive, then f is Pettis integrable if and only if f is weakly measurable and $\langle w, f(\cdot)\rangle \in L^1(M)$ for each $w \in \mathbb{X}^*$.

Let $f : M \to \mathbb{X}$ be a countably-valued function given by $f = \sum_{k=1}^\infty x_k 1_{E_k}$, $E_k \in \mathcal{B}$ disjoint. Then f is called Bochner integrable if

$$\sum_{k=1}^\infty \|x_k\| m(E_k) < \infty$$

For any $E \in \mathcal{B}$, the Bochner integral of F on E is defined by

$$(B)\int_E f(u)\, dm(u) = \sum_{k=1}^{\infty} x_k m(E \cap E_k).$$

A function $f : M \to \mathbb{X}$ is said to be almost separably-valued if there exists a set E_0 such that $m(E_0) = 0$ and the set $f(M \backslash E_0)$ is separable. Let $f : M \to \mathbb{X}$ be a weakly measurable and an almost separably-valued function. Then the function $\|f(\cdot)\|$ is measurable. Consider a sequence $\{f_n\}$ of weakly measurable functions such that f_n converges almost everywhere. Then the limit function $f = \lim_{n\to\infty} f_n$ is also weakly measurable. Moreover, if the sequence $\{f_n\}$ is also countably-valued, then both functions f and $f - f_n$ are weakly measurable and almost separably-valued. Hence $\|f\|$ and $\|f - f_n\|$ are measurable. It therefore follows from the above paragraph that a function $f : M \to \mathbb{X}$ is called Bochner integrable if there exists a sequence $\{f_n\}$ of Bochner integrable functions such that

1. $\lim_{n\to\infty} f_n = f$ almost everywhere

2. $\lim_{n\to\infty} \int_M \|f(u) - f_n(u)\|\, dm(u) = 0$.

From condition 2, it follows that for any $E \in \mathcal{B}$, $\int_E f_n(u)\, dm(u), n \geq 1$, is Cauchy in $\mathbb{X}$. The Bochner integral of f on E is then defined as

$$(B)\int_E f(u)\, dm(u) = \lim_{n\to\infty} \int_E f_n(u)\, dm(u).$$

Condition (2) also allows us to state the following conditions for determining Bochner integrability.

A function $f : M \to \mathbb{X}$ is Bochner integrable if and only if the following conditions are satisfied:

1. f is weakly measurable,

2. f is almost separably-valued, and

3. $\int_M \|f(u)\|\, dm(u) < \infty$.

It is observed that if f is Bochner integrable, then it is also Pettis integrable and we have equality of the two integrals:

$$(P)\int_E f(u)\, dm(u) = (B)\int_E f(u)\, dm(u), \qquad \forall E \in \mathcal{B}.$$

The converse is not true. See examples in [21] Section 13.2.

2.6 White Noise Integrals

A white noise integral is a type of integral for which the integrand takes values in the space $(\mathcal{E})^*_\beta$ of generalized functions. As an example consider the integral $\int_0^t e^{-c(t-s)} : \dot{B}(s)^2 :\ ds$ where $\dot{B}$ is white noise. In this case the integrand is an $(\mathcal{S})^*_\beta$-valued measurable function on $[0,t]$. A second example is the integral $\int_0^t \partial_s^* \Phi(s)\, ds$ where Φ is also $(\mathcal{S})^*_\beta$-valued. If $\int_0^t \partial_s^* \Phi(s)\, ds$ is a random variable in (L^2) then the white noise integral is called a Hitsuda-Skorokhod integral. This integral will be defined in detail in the next section and will be our center of focus. So in general, if $(\mathcal{E})_\beta \subset (L^2) \subset (\mathcal{E})^*_\beta$ is a Gel'fand triple, and Φ is an $(\mathcal{E})^*_\beta$-valued function on a measurable space $(M, \mathcal{B}, m)$, a white noise integral is of the type

$$\int_E \Phi(u)\, dm(u), E \in \mathcal{B}.$$

Despite the fact that $(\mathcal{E})^*_\beta$ is not a Banach space, these integrals can be defined in the Pettis or Bochner sense by the use of the S-transform.

- *White noise integrals in the Pettis sense:*

We need to define $\int_E \Phi(u)\, dm(u)$ as the generalized function in $(\mathcal{E})^*_\beta$ that satisfies the following:

$$S\left(\int_E \Phi(u)\,dm(u)\right)(\xi) = \int_E S(\Phi(u))(\xi)\,dm(u), \qquad \xi \in \mathcal{E}_c.$$

In particular, if $\Phi(u)$ is replaced by $\partial_u^*\Phi(u)$ we have

$$S\left(\int_E \partial_u^*\Phi(u)\,dm(u)\right)(\xi) = \int_E \xi(u)S(\Phi(u))(\xi)\,dm(u), \qquad \xi \in \mathcal{E}_c. \tag{2.9}$$

The above two equations then call for the following conditions on the function Φ to be satisfied:

(a) $S(\Phi(u))(\xi)$ is measurable for any $\xi \in \mathcal{E}_c$.

(b) $S(\Phi(\cdot))(\xi) \in L^1(M)$ for any $\xi \in \mathcal{E}_c$.

(c) For any $E \in \mathcal{B}$, the function $\int_E S(\Phi(u))(\cdot)\,dm(u)$ is a generalized function in $(\mathcal{E})_\beta^*$. (This can be verified by using the characterization theorem for generalized functions).

The statement in (c) can be rewritten as

$$\langle\langle \int_E \Phi(u)\,dm(u), :e^{\langle\cdot,\xi\rangle}:\rangle\rangle = \int_E \langle\langle \Phi(u), :e^{\langle\cdot,\xi\rangle}:\rangle\rangle\,dm(u), \qquad \xi \in \mathcal{E}_c.$$

Since the linear span of the set $\{:e^{\langle\cdot,\xi\rangle}:, \xi \in \mathcal{E}_c\}$ is dense in $(\mathcal{E})_\beta^*$, the above equation implies that

$$\langle\langle \int_E \Phi(u)\,dm(u), \varphi\rangle\rangle = \int_E \langle\langle \Phi(u), \varphi\rangle\rangle\,dm(u), \qquad \varphi \in (\mathcal{E})_\beta^* \tag{2.10}$$

In terms of the S-transform, Pettis integrability can be characterized using the following theorem. For a proof, see section 13.4 [21].

Theorem 2.5. *Suppose a function $\Phi : M \to (\mathcal{E})_\beta^*$ satisfies the following conditions:*

(1) $S(\Phi(\cdot))(\xi)$ is measurable for any $\xi \in \mathcal{E}_c$.

(2) There exists nonnegative numbers K, a and p such that

$$\int_M S(\Phi(u))(\xi)\, dm(u) \leq K \exp\left[a|\xi|_p^{\frac{2}{1-\beta}}\right], \qquad \xi \in \mathcal{E}_c.$$

Then Φ is Pettis integrable and for any $E \in \mathcal{B}$,

$$S\left(\int_E \Phi(u)\, dm(u)\right)(\xi) = \int_E S(\Phi(u))(\xi)\, dm(u), \qquad \xi \in \mathcal{E}_c.$$

- *White noise integrals in the Bochner sense:*

Since the space $(\mathcal{E})^*_\beta$ is not a Banach space, the definition of the Bochner integral from the last section cannot be used to define the white noise integral $\int_E \Phi(u)\, dm(u)$. As it turns out, we will need much stronger conditions than those used in the defining white noise integrals in the Pettis sence. As earlier observed, we know that $(\mathcal{E})^*_\beta = \cup_{p\geq 0}(\mathcal{E}_p)^*_\beta$ and each of the spaces $(\mathcal{E}_p)^*_\beta$ is a separable Hilbert space. With is in mind, the white noise integral $\int_E \Phi(u)\, dm(u)$ can be defined in the Bochner sense in the following way.

Let $\Phi : M \to (\mathcal{E})^*_\beta$. Then Φ is Bochner integrable if it satisfies the following conditions:

1. Φ is weakly measurable

2. There exists $p \geq 0$ such that $\Phi(u) \in (\mathcal{E}_p)^*_\beta$ for almost all $u \in M$ and $\|\Phi(\cdot)\|_{-p,-\beta} \in L^1(M)$.

If Φ is Bochner integrable, then we have

$$\left\|\int_M \Phi(u)\, dm(u)\right\|_{-p,-\beta} \leq \int_M \|\Phi(u)\|_{-p,-\beta}\, dm(u)$$

The following theorem contains the conditions for Bochner integrability in terms of the S-transform and helps estimate the norm $\|\Phi(u)\|_{-p,-\beta}$ of Φ. See [21] section 13.5.

Theorem 2.6. *Let $\Phi : M \to (\mathcal{E})^*_\beta$ be a function satisfying the conditions*

1. *$S(\Phi(\cdot))(\xi)$ is measurable for any $\xi \in \mathcal{E}_c$*

2. *There exists $p \geq 0$ and nonnegative functions $L \in L^1(M), b \in L^\infty(M)$, and an m-null set E_0 such that*

$$|S(\Phi(u))(\xi)| \leq L(u) \exp\left[b(u)|\xi|_p^{\frac{2}{1-\beta}}\right], \qquad \forall \xi \in \mathcal{E}_c,\ u \in E_0^c.$$

Then Φ is Bochner integrable and $\int_M \Phi(u)\, dm(u) \in (\mathcal{E}_q)^*_\beta$ for any $q > p$ such that

$$e^2 \left(\frac{2\|b\|_\infty}{1-\beta}\right)^{1-\beta} \|A^{-(q-p)}\|^2_{HS} < 1 \tag{2.11}$$

where $\|b\|_\infty$ is the essential supremum of b. It turns out that for such q,

$$\begin{aligned} &\| \int_M \Phi(u)\, dm(u) \|_{-q,-\beta} \\ &\leq \|L\|_1 \left(1 - e^2 \left(\frac{2\|b\|_\infty}{1-\beta}\right) \|A^{-(q-p)}\|^2_{HS}\right)^{-1/2}. \end{aligned} \tag{2.12}$$

An example worth noting is the following. Let $F \in \mathcal{S}'(\mathbb{R})$ and $f \in L^2(\mathbb{R}), f \neq 0$. Then $F(\langle \cdot, f\rangle)$ is a generalized function and if the Fourier transform $\hat{F} \in L^\infty(\mathbb{R})$, then $F(\langle \cdot, f\rangle)$ is represented as a white noise integral by

$$F(\langle \cdot, f\rangle) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iu\langle \cdot, f\rangle} \hat{F}(u)\, du. \tag{2.13}$$

In this case $\Phi(u) = e^{iu\langle \cdot, f\rangle}\hat{F}(u), u \in \mathbb{R}$ and it satisfies the conditions in Theorem 2.4 for Pettis integrability.

2.7 Donsker's Delta Function

Consider the Gel'fand triple $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$. For $F \in \mathcal{S}'(\mathbb{R})$ and non-zero $f \in L^2(\mathbb{R})$, it is known that the composition $F(\langle \cdot, f\rangle)$ is a generalized Brownian functional [17], [21]. In particular, let $F = \delta_a$ be the Dirac delta function at a. Then, for $f = 1_{[0,t]}$, and $\langle \cdot, 1_{[0,t]}\rangle = B(t)$ a Brownian motion, the function $\delta_a(B(t))$ is a generalized Brownian functional. $\delta_a(B(t))$ is called Donsker's delta function. Sometimes $\delta_a(B(t))$ is written as $\delta(B(t)-a)$. Some of the applications of Donsker's delta function include among others:

(a) Solving partial differential equations

(b) White noise representation of Feynman integrals

(c) Local time $L(t,x) = \int_0^t \delta(B(s) - x)\, ds$.

Donsker's delta function $\delta(B(t) - a)$ can be expressed in two ways. First, as a white noise integral in the Pettis sense represented as

$$\delta(B(t) - a) = \int_{\mathbb{R}} e^{iu(B(t)-a)}\, du.$$

Note that this is a special case of equation (2.13) with $F = \delta_a$ and $f = 1_{[0,1]}$. Second, in terms of its Wiener-Itô decomposition also given by

$$\begin{aligned}\delta(B(t) - a) &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} \sum_{n=0}^{\infty} \frac{1}{n! t^n} : a^n :_t \langle : \cdot^{\otimes n} : , 1_{[0,t]}^{\otimes n}\rangle \\ &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} \sum_{n=0}^{\infty} \frac{1}{n! t^n} : a^n :_t : B(t)^n :_t\end{aligned}$$

This representation is using the fact that $\langle : \cdot^{\otimes n} : , 1_{[0,t]}^{\otimes n}\rangle =: B(t)^n :_t$. In general, for $F \in \mathcal{S}'(\mathbb{R})$, the Wiener-Itô decomposition for $F(B(t))$ is given by

$$F(B(t)) = \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^{\infty} \frac{1}{n! t^n} \langle F, \xi_{n,t}\rangle : B(t)^n :_t \tag{2.14}$$

where $\xi_{n,t}(x) =: x^n :_t e^{-\frac{x^2}{2t}}$ is a function in $\mathcal{S}(\mathbb{R})$ and $\langle , \rangle$ is the pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ (See [22]). Another representation for Donsker's delta function is the subject of discussion in the second part of this dissertation, which is called the Clark-Ocone Representation formula. The second representaion above will be very important in obtaining the result.

Chapter 3. A Generalization of the Itô Formula

This is the chapter that contains the first result of this dissertation. In the first two sections we define the Itô integral and some other related concepts. Section two contains an overview of the ordinary Itô formula. In section three we introduce the problem plus the machinery involved in solving it while the fourth section contains the major result.

3.1 The Itô Integral

Definition 3.1. Let $T \subset \mathbb{R}$ and let $(\Omega, \mathcal{F}, P)$ be a probability space. A *stochastic process* $X(t, \omega)$ is a measurable mapping $X: T \times \Omega \to \mathbb{R}$ such that:

(a) For each ω, $X(\cdot, \omega)$ is a sample path i.e. for every fixed ω (hence for every observation), $X(\cdot, \omega)$ is an $\mathbb{R}$-valued function defined on T.

(b) For each t, $X(t, \cdot)$ is a random variable.

For a given stochastic process X, we let $X(t)$ denote the random variable $X(t, \cdot)$.

Definition 3.2. A *Brownian motion* $B(t)$ is a stochastic process satisfying the following conditions:

1. $P\{B(0) = 0\} = 1$.

2. For any $0 < t < s$, $B(s) - B(t) \sim N(0, s-t)$; i.e.

$$P\{B(s) - B(t) \le x\} = \frac{1}{\sqrt{2\pi(s-t)}} \int_{-\infty}^{x} e^{-\frac{y^2}{2(s-t)}}\, dy.$$

3. $B(t)$ has independent increments; i.e. for any $0 < t_1 < \cdots < t_n$ the random variables $B(t_1), B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1})$ are independent.

4. $P\{\omega \mid B(\cdot, \omega) \text{ is continuous}\} = 1$.

Definition 3.3. A *filtration* on $T \subset \mathbb{R}$ is an increasing family $\{\mathcal{F}_t\}$ of σ-fields, $\mathcal{F}_t \subset \mathcal{F}$ with $\mathcal{F}_s \subset \mathcal{F}_t \qquad$ for all $s \leq t$; $s, t \in T$.

Let $B(t)$ be a Brownian motion. Take a filtration $\{\mathcal{F}_t;\ a \leq t \leq b\}$ such that:

(1) $B(t)$ is $\mathcal{F}_t$- measurable

(2) For any $s \leq t, B(t) - B(s)$ is independent of $\mathcal{F}_s$.

Definition 3.4. A stochastic process $f(t, \omega)$ with filtration $\{\mathcal{F}_t\}$ is called *non-anticipating* if for any $t \in [a, b]$, $f(t, \cdot)$ is measurable with respect to $\mathcal{F}_t$.

As examples, the functions $B(t)$, $B(t)^2$, and $B(t)^2 - t$ are non-anticipating.

The well-known Itô integral is a stochastic integral of the form $\int_a^b f(t, \omega)\, dB(t, \omega)$.

It is defined for any stochastic process $f(t, \omega)$ satisfying the following conditions:

(1) f is non-anticipating

(2) $\int_a^b |f(t, \cdot)|^2\, dt < \infty$ almost surely. (i.e. almost all sample paths are in $L^2(a, b)$.)

For the definition of a stochastic integral see [1], [25].

When f is a deterministic function and in $L^2(a, b)$, it is obviously non-anticipating and in this case $\int_a^b f(t)\, dB(t)$ is known as the Wiener integral.

Definition 3.5. A stochastic process $M(t)$ is called a *martingale* with respect to a filtration $\{\mathcal{F}_t\}$ if $E(M(t)|\mathcal{F}_s) = M(s)\ \forall s \leq t$.

Examples:

1. A Brownian motion $B(t)$ is a martingale with respect to the filtration $\mathcal{F}_t$ generated by $\{B(s); s \leq t\}$, because if $s \leq t$, then

$$\begin{aligned} E[B(t)|\mathcal{F}_s] &= E[B(t) - B(s) + B(s)|\mathcal{F}_s] \\ &= E[B(t) - B(s)|\mathcal{F}_s] + E[B(s)|\mathcal{F}_s] \\ &= 0 + B(s) \end{aligned}$$

Here we have used the facts that $E[B(t) - B(s)|\mathcal{F}_s] = 0$ since $B(t) - B(s)$ is independent of $\mathcal{F}_s$ and that $E[B(s)|\mathcal{F}_s] = B(s)$ since $B(s)$ is $\mathcal{F}_s$-measurable.

2. Let $M(t) = B(t)^2 - t$. Then $M(t)$ is a martingale. To see this, let $s \leq t$. Then by using the fact that $E[B(t)B(s)|\mathcal{F}_s] = B(s)E[B(t)|\mathcal{F}_s] = B(s)^2$ we have

$$\begin{aligned} E[M(t) - M(s)|\mathcal{F}_s] &= E[B(t)^2 - B(s)^2 - (t-s)|\mathcal{F}_s] \\ &= E[B(t)^2 - (2B(s)^2 - B(s)^2) - (t-s)|\mathcal{F}_s] \\ &= E[B(t)^2 - 2B(t)B(s) + B(s)^2 - (t-s)|\mathcal{F}_s] \\ &= E[(B(t) - B(s))^2 - (t-s)|\mathcal{F}_s] \\ &= (t-s) - (t-s) = 0. \end{aligned}$$

This then shows that $E(M(t)|\mathcal{F}_s) = M(s)$ and so $B(t)^2 - t$ is a martingale.

If the conditions above in the definition of the Itô integral are replaced with the following:

(a) f is non-anticipating

(b) $E(\int_a^b |f(t,\cdot)|^2\, dt) < \infty$ (i.e. $f \in L^2([a,b] \times \Omega)$),

then condition (b) is stronger than condition (2) above and the Itô integral defined using conditions (a) and (b) is a martingale.

3.2 The Itô Formula

In this section we briefly describe the well-known Itô formula [1], [25].

Definition 3.6. Let $[a,b]$ be a fixed interval. An *Itô process* is a stochastic process of the form

$$X(t) = X(a) + \int_a^t f(s)\, dB(s) + \int_a^t g(s)\, ds, \;\; t \in [a,b] \tag{3.1}$$

where $X(a)$ is $\mathcal{F}_a$-measurable, f is nonanticipating and $\int_a^b |f(s)|^2\, ds < \infty$ almost surely, g is nonanticipating, and $\int_a^b |g(s)|\, ds < \infty$ almost surely.

Take an Itô process $X(t)$ as defined by equation (3.1) and let $\theta(t,x)$ be a $\mathcal{C}^2$ function in x and $\mathcal{C}^1$ in t. Then the well-known Itô formula states that

$$\begin{aligned}\theta(t,X(t)) &= \theta(a,X(a)) + \int_a^t \frac{\partial\theta}{\partial x}\big(s,X(s)\big)f(s)\,dB(s)\\ &+ \int_a^t \left(\frac{\partial\theta}{\partial s}\big(s,X(s)\big) + \frac{\partial\theta}{\partial x}\big(s,X(s)\big)g(s) + \frac{1}{2}\frac{\partial^2\theta}{\partial x^2}\big(s,X(s)\big)f(s)^2\right)ds.\end{aligned} \tag{3.2}$$

for all $t \in [a,b]$. Observe that $\theta(t,X(t))$ is also an Itô process.

In comparison with ordinary calculus, the chain rule says that for such a function θ and G a deterministic $\mathcal{C}^1$ function

$$\theta(t,G(t)) = \theta(a,G(a)) + \int_a^t \frac{\partial\theta}{\partial s}\big(s,G(s)\big)\,ds + \int_a^t \frac{\partial\theta}{\partial x}\big(s,G(s)\big)G'(s)\,ds \tag{3.3}$$

If we compare equations (3.2) and (3.3) we see a significant difference. In equation (3.2) there is an extra term $\frac{1}{2}\int_a^t \frac{\partial^2\theta}{\partial x^2}(s,X(s))f(s)^2\,ds$ which is often called the *correction term* for stochastic calculus.

Examples:

1. Let $X(t) = B(t),\ \theta(t,x) = f(x)$ a $\mathcal{C}^2$-function. Itô's formula in this case is given by

$$f(B(t)) = f(B(a)) + \int_a^t f'(B(s))\,dB(s) + \frac{1}{2}\int_a^t f''(B(s))\,ds.$$

2. Let $X(t) = B(t)$ and let $\theta(x) = x^n$. Then

$$B(t)^n = B(a)^n + n\int_a^t B(s)^{n-1}\,dB(s) + \frac{n(n-1)}{2}\int_a^t B(s)^{n-2}\,ds.$$

In particular, put $t = b$ and $n = 3$ to obtain

$$\int_a^b B(t)^2\,dB(t) = \frac{1}{3}\left\{\left(B(b)^3 - B(a)^3\right) - 3\int_a^b B(t)\,dt\right\}.$$

For $n = 2$ we have

$$\int_a^b B(t)\, dB(t) = \frac{1}{2}\{B(b)^2 - B(a)^2 - (b-a)\}.$$

3.3 An Extension of the Itô Integral

The property that the integrands in the Itô processes be nonanticipating is needed in order to apply the Itô formula discussed in section 3.2 above. However, in many cases this essential condition is not satisfied. Take the example of the stochastic integral $\int_0^1 B(1)dB(t), t < 1$. Clearly, $B(1)$ is anticipating and $\int_0^1 B(1)dB(t)$ cannot be defined as an Itô integral.

A number of extensions of the Itô integral exist. One such extension is by Itô. In [12] he extended it to stochastic integrals for integrands which may be anticipating. In particular, he showed that $\int_0^1 B(1)dB(t) = B(1)^2$. In [9] a special type of integral called the *Hitsuda-Skorokhod integral* (check the definition 3.7 below) was introduced as a motivation to obtain an Itô type formula for such functions as $\theta(B(t), B(c)),\ t < c$, for a $\mathcal{C}^2$- function θ. (We note here that $B(c), t < c$ is not $\mathcal{F}_t$ measurable). As shown in Theorem 3.8 below, this integral is also an extension of the Itô integral.

Consider the Gel'fand triple $(\mathcal{S})_\beta \subset (L^2) \subset (\mathcal{S})^*_\beta$ which comes from the Gel'fand triple $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$ as described in section 2.1. Suppose $\Phi : [a, b] \to (\mathcal{S})^*_\beta$ is Pettis integrable. Then the function $t \to \partial_t^*\Phi(t)$ is also Pettis integrable and equation (2.9) holds. Also, if $\Phi : [a, b] \to (\mathcal{S})^*_\beta$ is Bochner integrable, then the function $t \to \partial_t^*\Phi(t)$ is also Bochner integrable; and because Bochner integrability implies Pettis integrability, equation (2.9) still holds. We now define a particular extension of the Itô integral which forms the basis of our result.

Definition 3.7. Let $\varphi : [a, b] \to (\mathcal{S})^*_\beta$ be Pettis integrable. The white noise integral $\int_a^b \partial_t^* \varphi(t)dt$ is called the *Hitsuda-Skorokhod integral* of φ if it is a random variable in (L^2).

Consider a stochastic process $\varphi(t)$ in the space $L^2([a, b] \times \mathcal{S}'(\mathbb{R}))$ which is nonanticipating. The Itô integral $\int_a^b \varphi(t)\, dB(t)$ for the process $\varphi(t)$ can be expressed as a white noise integral in the Pettis sense. The following theorem due to Kubo and Takenaka [15] (see also [21] Theorem 13.12 for a proof) implies that the Hitsuda-Skorokhod integral is an extension of the Itô integral to $\varphi(t)$ which might be anticipating. A look at the example following the next theorem will throw some light on the difference between Itô's extension and the Hitsuda-Skorokhod integral.

Theorem 3.8. *Let $\varphi(t)$ be nonanticipating and $\int_a^b \|\varphi(t)\|_0^2 dt < \infty$. Then the function $\partial_t^* \varphi(t), t \in [a, b]$, is Pettis integrable and*

$$\int_a^b \partial_t^* \varphi(t)dt = \int_a^b \varphi(t)\, dB(t). \tag{3.4}$$

where the right hand side is the Itô integral of φ.

Example 3.9. Let $\varphi(t) = B(1),\ t \leq 1$. It was observed earlier that by Itô's extension, $\int_0^1 B(1)\, dB(t) = B(1)^2$. However, for the Hitsuda-Skorokhod integral, $\int_0^1 \partial_t^* B(1)\, dt = B(1)^2 - 1$. This equality can be verified by using the S-transform in the following way:

Since $B(1) = \langle \cdot, 1_{[0,1)} \rangle$, we have $(SB(1))(\xi) = \int_0^1 \xi(s)\, ds$. Now, by the use of equation (2.9),

$$\begin{aligned} S\left(\int_0^1 \partial_t^* B(1)\, dt\right)(\xi) &= \int_0^1 \xi(t)(SB(1))(\xi)\, dt \\ &= \int_0^1 \int_0^1 \xi(t)\xi(s)\, dtds \\ &= S\langle : \cdot^{\otimes 2} :, 1_{[0,1)}^{\otimes 2} \rangle(\xi). \end{aligned}$$

Therefore, $\int_0^1 \partial_t^* B(1)\,dt = \langle :\cdot^{\otimes 2}:, 1_{[0,1)}^{\otimes 2}\rangle$. We can apply the results from section 2.2 concerning wick tensors to get

$$\begin{aligned}
\int_0^1 \partial_t^* B(1)\,dt &=: \langle \cdot, 1_{[0,1)}\rangle^2 :_1 \\
&= \langle \cdot, 1_{[0,1)}\rangle^2 - 1 \\
&= B(1)^2 - 1
\end{aligned}$$

We clearly see a difference between the integral $\int_0^1 B(1)\,dB(t)$ as defined by Itô and the Hitsuda-Skorokhod integral of $B(1)$.

From now on, the space $L^2([a,b];(L^2))$ will be identified with the Hilbert space $L^2([a,b]\times\mathcal{S}')$.

Example 3.10. It is not always true that for any $\varphi \in L^2([a,b];(L^2))$ the white noise integral $\int_a^b \partial_t^*\varphi(t)dt$ is a Hitsuda-Skorokhod integral. Consider the following example: Let

$$\varphi(t) = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n!}} \langle :\cdot^{\otimes n}:, 1_{[0,1)}^{\otimes n}\rangle.$$

(This is a case where φ is constant in t). Applying the S-transform to $\int_0^1 \partial_t^*\varphi(t)dt$ as in the above example gives us the following result

$$\begin{aligned}
S\left(\int_0^1 \partial_t^*\varphi(t)\,dt\right)(\xi) &= \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n!}} \int_0^1 \xi(t) S\left(\langle :\cdot^{\otimes n}:, 1_{[0,1)}^{\otimes n}\rangle\right)(\xi)\,dt \\
&= \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n!}} \langle 1_{[0,1)}, \xi\rangle \, \langle 1_{[0,1)}^{\otimes n}, \xi^{\otimes n}\rangle \\
&= \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n!}} \langle 1_{[0,1)}^{\otimes(n+1)}, \xi^{\otimes(n+1)}\rangle \\
&= S\left(\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n!}} \langle :\cdot^{\otimes(n+1)}:, 1_{[0,1)}^{\otimes(n+1)}\rangle\right)(\xi)
\end{aligned}$$

Therefore,

$$\int_0^1 \partial_t^*\varphi(t)\,dt = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n!}} \langle :\cdot^{\otimes(n+1)}, 1_{[0,1)}^{\otimes(n+1)}\rangle.$$

Now if we compute the L^2-norms of $\varphi(t)$ and $\int_0^1 \partial_t^* \varphi(t)dt$ we have

$$\int_0^1 \|\varphi(t)\|_0^2 dt = \int_0^1 \sum_{n=1}^{\infty} n! \frac{1}{n^2 n!} |1_{[0,1)}|_0^{2n} dt = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

while

$$\| \int_0^1 \partial_t^* \varphi(t) dt \|_0^2 = \sum_{n=1}^{\infty} (n+1)! \frac{1}{n^2 n!} |1_{[0,1)}|_0^{2(n+1)} = \sum_{n=1}^{\infty} \frac{n+1}{n^2} = \infty.$$

Thus even though $\varphi \in L^2([0,1];(L^2))$, $\int_0^1 \partial_t^* \varphi(t)dt$ is not a Hitsuda-Skorokhod integral. This then calls for some restrictions on the stochastic process $\varphi(t)$ in order for the white noise integral $\int_0^1 \partial_t^* \varphi(t)dt$ to be a Hitsuda-Skorokhod integral. First we define a very important operator and a subspace of (L^2) which will be very important in our main result.

Definition 3.11. For $\varphi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : , f_n \rangle$, we define

$$N\varphi = \sum_{n=1}^{\infty} n \langle : \cdot^{\otimes n} :, f_n \rangle.$$

The operator N is called the *number operator.* Moreover, the power N^r, $r \in \mathbb{R}$, of the number operator is defined in the following way: for $\varphi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} :, f_n \rangle$,

$$N^r \varphi = \sum_{n=1}^{\infty} n^r \langle : \cdot^{\otimes n} :, f_n \rangle.$$

For any $r \in \mathbb{R}$, N^r is a continuous linear operator from $(\mathcal{S})_\beta$ into itself and from $(\mathcal{S})_\beta^*$ into itself. Let $\mathcal{W}^{1/2}$ be the Sobolev space of order $\frac{1}{2}$ for the Gel'fand triple $(\mathcal{S})_\beta \subset (L^2) \subset (\mathcal{S})_\beta^*$. In other words, for $[a,b] \subset \mathbb{R}^+$, $\mathcal{W}^{1/2}$ will denote the set of $\varphi \in (L^2)$ such that $(\partial_t \varphi)_{t \in [a,b]} \in L^2([a,b];(L^2))$. The norm on $\mathcal{W}^{1/2}$ will be defined as

$$
\begin{aligned}
\|\varphi\|_{\frac{1}{2}}^2 &= \|(N+1)^{\frac{1}{2}}\varphi\|_0^2 \\
&= \left\| \sum_{n=0}^{\infty} (n+1)^{1/2} \langle : \cdot^{\otimes n} : , f_n \rangle \right\|_0^2 \\
&= \sum_{n=0}^{\infty} n!(n+1)|f_n|_0^2 \\
&= \sum_{n=0}^{\infty} n!|f_n|_0^2 + \sum_{n=0}^{\infty} n!n|f_n|_0^2 \\
&= \|\varphi\|_0^2 + \|N^{1/2}\varphi\|_0^2
\end{aligned}
$$

Now Theorem 9.27 in [21] gives us the fact that if φ is given with the property that $N^{1/2}\varphi \in (L^2)$, and $t \in [a,b]$, then $\|N^{1/2}\varphi\|_0^2 = \int_a^b \|\partial_t\varphi\|_0^2 \, dt$. Hence

$$\|\varphi\|_{\frac{1}{2}}^2 = \|\varphi\|_0^2 + \int_a^b \|\partial_t\varphi\|_0^2 \, dt.$$

and so

$$\mathcal{W}^{\frac{1}{2}} \equiv \{\varphi \in (L^2); \|\varphi\|_{\frac{1}{2}} < \infty\}.$$

For more information on Sobolev spaces and the Number operator, see [23], [21].

The following theorem gives the desired condition on the function $\varphi(t)$ in order for $\int_a^b \partial_t^*\varphi(t)dt$ to be a Hitsuda-Skorokhod integral. This condition is determined by the number operator N and the space $\mathcal{W}^{1/2}$ plays a major role. The proof for this condition can be found in [21] Theorem 13.16.

Theorem 3.12. *Let $\varphi \in L^2([a,b];\mathcal{W}^{\frac{1}{2}})$. Then $\int_a^b \partial_t^*\varphi(t)\,dt$ is a Hitsuda-Skorokhod integral and*

$$\left\| \int_a^b \partial_t^*\varphi(t)\,dt \right\|_0^2 = \int_a^b \|\varphi(t)\|_0^2 \, dt + \int_a^b \int_a^b \Big(\big(\partial_t\varphi(s), \partial_s\varphi(t) \big) \Big)_0 \, ds dt \qquad (3.5)$$

where $((\cdot,\cdot))_0$ *is the inner product on* (L^2)*. Moreover,*

$$\left|\int_a^b\int_a^b \Big(\big(\partial_t\varphi(s),\partial_s\varphi(t)\big)\Big)_0\, dsdt\right| \le \int_a^b \|N^{\frac{1}{2}}\varphi(t)\|_0^2\, dt. \tag{3.6}$$

As a remark, when $\varphi\in L^2([a,b];(L^2))$ is nonanticipating, then the inner product $\Big(\big(\partial_t\varphi(s),\partial_s\varphi(t)\big)\Big)_0 = 0$ for almost all $(s,t)\in[a,b]^2$. Therefore, when attempting to compute the L^2-norm for the integral by using equation (3.5), we obtain the following very useful result that relates the norms:

$$\left\|\int_a^b \partial_t^*\varphi(t)\, dt\right\|_0^2 = \int_a^b \|\varphi(t)\|_0^2\, dt = \left\|\int_a^b \varphi(t)\, dB(t)\right\|_0^2.$$

The Hitsuda-Skorokhod integral is related to two other extensions of the Itô integral: the forward and backward integrals. For $\varphi\in L^2([a,b];\mathcal{W}^{\frac{1}{2}})$ let $\partial_{t^+}\varphi(t)$ and $\partial_{t^-}\varphi(t)$ be the right-hand white noise derivative and left-hand white noise derivative of φ respectively (see [21] definition 13.25). The forward integral of φ is defined as

$$\int_a^b \varphi(t)dB(t^+) \equiv \int_a^b \partial_{t^+}\varphi(t)dt + \int_a^b \partial_t^*\varphi(t)dt,$$

provided both integrals on the right hand side are random variables in (L^2). In particular,

$$\int_0^1 B(1)dB(t^+) = B(1)^2$$

which agrees with $\int_0^1 B(1)dB(t) = B(1)^2$ as shown by Itô's in [12]. Similarly, the backward integral of φ is defined as

$$\int_a^b \varphi(t)dB(t^-) \equiv \int_a^b \partial_{t^-}\varphi(t)dt + \int_a^b \partial_t^*\varphi(t)dt.$$

3.4 An Anticipating Itô Formula

In this section we present our main result for this part of the dissertation. It is about a particular generalization of the ordinary Itô formula to a class of functions that are anticipating.

Let $B(t)$ be a Brownian motion given by $B(t) = \langle \cdot, 1_{[0,t)} \rangle$. For a $\mathcal{C}^2$-function θ, a simple case of the Itô formula is given by:

$$\theta(B(t)) = \theta(B(a)) + \int_a^t \theta'(B(s))\, dB(s) + \frac{1}{2}\int_a^t \theta''(B(s))\, ds,$$

where $0 \leq a \leq t$.

If we assume that $\theta(B(\cdot)),\ \theta'(B(\cdot)),\ \theta''(B(\cdot)) \in L^2([a,b];(L^2))$, then Theorem 3.8 enables us to write the above equality as

$$\theta(B(t)) = \theta(B(a)) + \int_a^t \partial_s^* \theta'(B(s))\, ds + \frac{1}{2}\int_a^t \theta''(B(s))\, ds \tag{3.7}$$

where $\int_0^t \partial_s^* \theta'(B(s)) ds$ is a Hitsuda-Skorokhod integral. In [21], the following two generalizations of the white noise version of the Itô formula in equation (3.7) were considered:

(a) $\theta(X(t), B(c))$ for a $\mathcal{C}^2$- function θ and a Wiener integral $X(t), t \leq c$.

(b) $\theta(B(t))$ with a generalized function θ in $\mathcal{S}'(\mathbb{R})$.

The main tool for the proofs of the formulas obtained for the above two generalizations is the S-transform. The result for the generalization of (a) is stated below as a theorem and as earlier explained, our new formula will be using this particular generalization. See [21] Theorem 13.21 for a complete proof.

Theorem 3.13. *Let $0 \leq a \leq c \leq b$. Let $X(t) = \int_a^t f(s)\, dB(s)$ be a Wiener integral with $f \in L^\infty([a,b])$ and let $\theta(x,y)$ be a $\mathcal{C}^2$-function on $\mathbb{R}^2$ such that*

$$\theta(X(\cdot), B(c)), \frac{\partial^2 \theta}{\partial x^2}(X(\cdot), B(c)), \frac{\partial^2 \theta}{\partial x \partial y}(X(\cdot), B(c))$$

are all in $L^2([a,b];(L^2))$. Then for any $a \leq t \leq b$, the integral

$$\int_a^t \partial_s^* \left(f(s) \frac{\partial \theta}{\partial x}(X(s), B(c)) \right) ds$$

is a Hitsuda-Skorokhod integral and the following equalities hold in (L^2)*:*

(1) for $a \leq t \leq c$,

$$\theta(X(t), B(c)) = \theta(X(a), B(c)) + \int_a^t \partial_s^* \left(f(s) \frac{\partial \theta}{\partial x}(X(s), B(c)) \right) ds$$
$$+ \int_a^t \left(\frac{1}{2} f(s)^2 \frac{\partial^2 \theta}{\partial x^2}(X(s), B(c)) + f(s) \frac{\partial^2 \theta}{\partial x \partial y}(X(s), B(c)) \right) ds,$$

(2) for $c < t \leq b$,

$$\theta(X(t), B(c)) = \theta(X(a), B(c)) + \int_a^t \partial_s^* \left(f(s) \frac{\partial \theta}{\partial x}(X(s), B(c)) \right) ds$$
$$+ \frac{1}{2} \int_a^t f(s)^2 \frac{\partial^2 \theta}{\partial x^2}(X(s), B(c))\, ds + \int_a^c f(s) \frac{\partial^2 \theta}{\partial x \partial y}(X(s), B(c))\, ds.$$

In [29], Nicholas Prevault developed a generalization similar to one in (a) above to processes of the form $Y(t) = \theta(X(t), F)$, where F is a smooth random variable depending on the whole trajectory of $(B(t))_{t \in \mathbb{R}^+}$, and where $(X(t))_{t \in \mathbb{R}^+}$ is an adapted Itô process. His proof relies on the expression of infinitesimal time changes on Brownian functionals using the Gross Laplacian. For our generalization in this section, $X(t)$ is taken to be a Wiener integral while the function F is chosen to be in the space $\mathcal{W}^{1/2}$. The resulting formula is similar to the one obtained in [29], except for the method of computation. We use a limiting process of Theorem 3.13 above in order to maintain the use of the S-transform as used in [21].

The following is the main result:

Theorem 3.14. *Let* $f \in L^\infty([a, b])$ *and* $X(t) = \int_a^t f(s)\, dB(s)$ *be a Wiener integral. Let* $F \in \mathcal{W}^{1/2}$ *and* $\theta \in \mathcal{C}_b^2(\mathbb{R}^2)$ *such that*

$$\theta(X((\cdot), F)), \frac{\partial^2 \theta}{\partial x^2}(X(\cdot), F), \frac{\partial^2 \theta}{\partial x \partial y}(X(\cdot), F)$$

are all in $L^2([a,b];(L^2))$. Then for any $a \le t \le b$, the integral

$$\int_a^t \partial_s^* \left(f(s) \frac{\partial \theta}{\partial x}(X(s), F) \right) ds$$

is a Hitsuda-Skorokhod integral and the following equality holds in (L^2):

$$\theta(X(t), F) = \theta(X(a), F) + \int_a^t \partial_s^* \left(f(s) \frac{\partial \theta}{\partial x}(X(s), F) \right) ds$$
$$+ \frac{1}{2} \int_a^t f(s)^2 \frac{\partial^2 \theta}{\partial x^2}(X(s), F)\, ds + \int_a^t f(s)(\partial_s F) \frac{\partial^2 \theta}{\partial x \partial y}(X(s), F)\, ds$$

The proof is presented in the following steps. First it will be shown that for $F = B(c),\ 0 \le a \le c \le b$ the above formula holds and that it coincides with the formula in Theorem 3.13. Secondly a special choice of F will be taken in the following way: We know that the span of the set $\{: e^{\langle \cdot, g \rangle} : \ ; g \in L^2(\mathbb{R})\}$ is dense in (L^2). If we let $g_1, g_2, \cdots, g_k \in L^2(\mathbb{R})$ and

$$\begin{aligned} F_N = {} & \lambda_1 \sum_{m=1}^{N} \frac{1}{m!} \langle : \cdot^{\otimes m} : , g_1^{\otimes m} \rangle \\ & + \lambda_2 \sum_{m=1}^{N} \frac{1}{m!} \langle : \cdot^{\otimes m} : , g_2^{\otimes m} \rangle \\ & \vdots \\ & + \lambda_k \sum_{m=1}^{N} \frac{1}{m!} \langle : \cdot^{\otimes m} : , g_k^{\otimes m} \rangle, \end{aligned}$$

then, as $N \longrightarrow \infty,\ F_N \longrightarrow F$ in (L^2) where $F = \lambda_1 : e^{\langle \cdot, g_1 \rangle} : + \lambda_2 : e^{\langle \cdot, g_2 \rangle} : + \cdots + \lambda_k : e^{\langle \cdot, g_k \rangle} :$. In our proof we shall assume that $F_N \longrightarrow F$ in $\mathcal{W}^{1/2}$. Also, g will be taking on the form $g = \sum_{j=1}^k \alpha_j 1_{[0, c_j)},\ k \in \mathbb{N},\ \alpha_j, c_j \in \mathbb{R}$. The formula will then be generalized to $\theta(X(t), F_N)$ with F_N chosen as above. An extension to $\theta(X(t), F)$ with general $F \ \in \mathcal{W}^{1/2}$ will be achieved via a limiting process. The following is the proof:

Proof. In the proof of Theorem 3.13 in [21] which uses the S-transform, there are two components that were treated separately: (1) when $a \le t \le c$, and (2) when $c < t \le b$.

Now suppose that $F = B(c)$, $0 \le a \le c \le b$. We claim that our new formula in the above Theorem is correct when we replace F with $B(c)$. To see that this is correct it is enough to show that $B(c) \in \mathcal{W}^{1/2}$. We proceed by computing the norm for $B(c)$ in the space $\mathcal{W}^{1/2}$. Indeed, since $B(c) = \langle \cdot, 1_{[0,c)} \rangle$, we have

$$\partial_s B(c) = 1_{[0,c)}(s)$$

and

$$\begin{aligned} \|B(c)\|_0^2 &= \|\langle \cdot, 1_{[0,c)} \rangle\|_0^2 = |1_{[0,c)}|_0^2 \\ &= \int_{\mathbb{R}} 1_{[0,c)}(s)\, ds = \int_0^c ds = c. \end{aligned}$$

Therefore,

$$\begin{aligned} \|B(c)\|_{\mathcal{W}^{1/2}} &= \|B(c)\|_0^2 + \int_a^b \|\partial_s B(c)\|_0^2\, ds \\ &= c + (b-a). \end{aligned}$$

Hence, $B(c) \in \mathcal{W}^{1/2}$. Now since $\partial_s B(c) = 1_{[0,c)}(s)$, we then have

$$\int_a^t f(s)(\partial_s B(c)) \frac{\partial^2 \theta}{\partial x \partial y}(X(s), B(c))\, ds = \begin{cases} \int_a^t f(s) \frac{\partial^2 \theta}{\partial x \partial y}(X(s), B(c))\, ds & \text{if } a \le t \le c, \\ \int_a^c f(s) \frac{\partial^2 \theta}{\partial x \partial y}(X(s), B(c))\, ds & \text{if } c < t \le b. \end{cases}$$

Thus, the two components (1) and (2) above in Theorem 3.13 are put together as one piece so as to satisfy the above theorem.

In general, suppose $a \le c_1 \le c_2 \le \cdots \le c_p \le b$. Let $\tilde{\theta}(x, y_1, \cdots, y_p)$ be a function defined on $\mathbb{R}^{p+1}$ and of class $\mathcal{C}^2$. Then we have the following formula which is simply a generalization of the one above

$$
\begin{aligned}
\tilde{\theta}(X(t), B(c_1), \cdots, B(c_p)) = \tilde{\theta}(X(a), B(c_1), \cdots, B(c_p)) \\
&+ \int_a^t \partial_s^* \left(f(s) \frac{\partial \tilde{\theta}}{\partial x} \Big(X(s), B(c_1), \cdots, B(c_p) \Big) \right) ds \\
&+ \frac{1}{2} \int_a^t f(s)^2 \frac{\partial^2 \tilde{\theta}}{\partial x^2} \Big(X(s), B(c_1), \cdots, B(c_p) \Big) ds \\
&+ \int_a^t f(s) \sum_{j=1}^p (\partial_s B(c_j)) \frac{\partial^2 \tilde{\theta}}{\partial x \partial y_j} \Big(X(s), B(c_1), \cdots, B(c_p) \Big) ds.
\end{aligned}
$$

For a suitable function $G : \mathbb{R}^p \longrightarrow \mathbb{R}$ and of class $\mathcal{C}^2$, we can transform the function $\tilde{\theta}$ to coincide with θ as a function defined on $\mathbb{R}^2$ in the following way: $\tilde{\theta}\big(X(t), B(c_1), \cdots, B(c_p)\big) = \theta\Big(X(t), G\big(B(c_1), \cdots, B(c_p)\big)\Big) = \theta(x, y)$. With this transformation and using the chain rule the above integral equation then takes on the following form:

$$
\begin{aligned}
\theta(X(t), &G(B(c_1), \cdots, B(c_p))) = \theta(X(a), G(B(c_1), \cdots, B(c_p))) \\
&+ \int_a^t \partial_s^* (f(s) \frac{\partial \theta}{\partial x} (X(s), G(B(c_1), \cdots, B(c_p)))) ds \\
&+ \frac{1}{2} \int_a^t f(s)^2 \frac{\partial^2 \theta}{\partial x^2} (X(s), G(B(c_1), \cdots, B(c_p))) \, ds \\
&+ \int_a^t f(s) \sum_{j=1}^p (\partial_s B(c_j)) \frac{\partial^2 \theta}{\partial x \partial y} (X(s), G(B(c_1), \cdots, B(c_p))) \\
&\times \frac{\partial}{\partial y_j} G(B(c_1), \cdots, B(c_p)) \, ds.
\end{aligned}
$$

Now let

$$
\begin{aligned}
F_N = {} & \lambda_1 \sum_{m=1}^{N} \frac{1}{m!} \langle :\cdot^{\otimes m}:, (\sum_{j=1}^{p^{(1)}} \alpha_j^{(1)} 1_{[0,c_j^{(1)})})^{\otimes m} \rangle \\
& + \lambda_2 \sum_{m=1}^{N} \frac{1}{m!} \langle :\cdot^{\otimes m}:, (\sum_{j=1}^{p^{(2)}} \alpha_j^{(2)} 1_{[0,c_j^{(2)})})^{\otimes m} \rangle \\
& \vdots \\
& + \lambda_{k-1} \sum_{m=1}^{N} \frac{1}{m!} \langle :\cdot^{\otimes m}:, (\sum_{j=1}^{p^{(k-1)}} \alpha_j^{(k-1)} 1_{[0,c_j^{(k-1)})})^{\otimes m} \rangle \\
& + \lambda_k \sum_{m=1}^{N} \frac{1}{m!} \langle :\cdot^{\otimes m}:, (\sum_{j=1}^{p^{(k)}} \alpha_j^{(k)} 1_{[0,c_j^{(k)})})^{\otimes m} \rangle.
\end{aligned}
$$

We know that for any $n \in \mathbb{N}$,

$$
\begin{aligned}
\langle :\cdot^{\otimes n}:, 1_{[0,c)}^{\otimes n} \rangle &= \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!!(-c)^k B(c)^{n-2k} \\
&= \sum_{i=1}^{m} a_i B(c)^i.
\end{aligned}
$$

for some constants a_i and m. Some summands could as well be zero. This then becomes a polynomial in $B(c)$. We can similarly write F_N as a polynomial in Brownian motion as follows

$$
\begin{aligned}
F_N = {} & \lambda_1 \sum_{m_1^{(1)} \cdots m_p^{(1)}} a_{m_1^{(1)} \cdots m_p^{(1)}} B(c_1^{(1)})^{m_1^{(1)}} \cdots B(c_p^{(1)})^{m_p^{(1)}} \\
& + \lambda_2 \sum_{m_1^{(2)} \cdots m_p^{(2)}} a_{m_1^{(2)} \cdots m_p^{(2)}} B(c_1^{(2)})^{m_1^{(2)}} \cdots B(c_p^{(2)})^{m_p^{(2)}} \\
& \vdots \\
& + \lambda_k \sum_{m_1^{(k)} \cdots m_p^{(k)}} a_{m_1^{(k)} \cdots m_p^{(k)}} B(c_1^{(k)})^{m_1^{(k)}} \cdots B(c_p^{(k)})^{m_p^{(k)}},
\end{aligned}
$$

where again some coefficients $a_{m_1^{(j)} \cdots m_p^{(j)}}$ could be zero for some j. Suppose F_N is restricted to only one summand out of the k summands above. i.e. let us suppose that

$$F_N = \lambda_1 \sum_{m_1^{(1)} \cdots m_p^{(1)}} a_{m_1^{(1)} \cdots m_p^{(1)}} B(c_1^{(1)})^{m_1^{(1)}} \cdots B(c_p^{(1)})^{m_p^{(1)}}. \tag{3.8}$$

Then we see that the function F_N is a good choice for our composition. That is, we let $F_N = G(B(c_1^{(1)}), \cdots, B(c_p^{(1)}))$ so that the following ensues

$$\theta\left(X(t), G(B(c_1^{(1)}), \cdots, B(c_p^{(1)}))\right) = \theta\Big(X(t), \lambda_1 \sum_{m_1^{(1)} \cdots m_p^{(1)}} a_{m_1^{(1)} \cdots m_p^{(1)}} B(c_1^{(1)})^{m_1^{(1)}} \cdots B(c_p^{(1)})^{m_p^{(1)}}\Big).$$

Moreover,

$$\frac{\partial}{\partial y_j} G\left(B(c_1^{(1)}), \cdots, B(c_p^{(1)})\right) = \lambda_1 \sum_{m_1^{(1)} \cdots m_p^{(1)}} a_{m_1^{(1)} \cdots m_p^{(1)}} B(c_1^{(1)})^{m_1^{(1)}} \cdots m_j^{(1)} B(c_j^{(1)})^{m_j^{(1)}-1} \cdots B(c_p^{(1)})^{m_p^{(1)}}.$$

Therefore,

$$\begin{aligned}
\theta(X(t), F_N) &= \theta(X(a), F_N) + \int_a^t \partial_s^* \left(f(s)\frac{\partial \theta}{\partial x}(X(s), F_N)\right) ds \\
&+ \frac{1}{2}\int_a^t f(s)^2 \frac{\partial^2 \theta}{\partial x^2}(X(s), F_N) ds + \int_a^t \sum_{j=1}^p f(s)(\partial_s B(c_j^{(1)})) \\
&\times \frac{\partial^2 \theta}{\partial x \partial y}(X(s), F_N)\lambda_1 \sum_{m_1^{(1)} \cdots m_p^{(1)}} a_{m_1^{(1)} \cdots m_p^{(1)}} B(c_1^{(1)})^{m_1^{(1)}} \cdots m_j^{(1)} \\
&\times B(c_j^{(1)})^{m_j^{(1)}-1} \cdots B(c_p^{(1)})^{m_p^{(1)}} ds.
\end{aligned}$$

Now, by the product rule,

$$\partial_s(\varphi\psi) = (\partial_s\varphi)\psi + \varphi(\partial_s\psi).$$

We then incorporate this rule into the last summand of the most recent equation above to obtain the following

$$\begin{aligned}\sum_{j=1}^{p} f(s)\frac{\partial^2\theta}{\partial x\partial y}(X(s),F_N)\lambda_1 \sum_{m_1^{(1)}\cdots m_p^{(1)}} & a_{m_1^{(1)}\cdots m_p^{(1)}} B(c_1^{(1)})^{m_1^{(1)}} \cdots m_j^{(1)} \\ & \times B(c_j^{(1)})^{m_j^{(1)}-1}\cdots B(c_p^{(1)})^{(1)}\partial_s B(c_j^{(1)}) \\ & = f(s)\frac{\partial^2\theta}{\partial x\partial y}(X(s),F_N)(\partial_s F_N).\end{aligned}$$

Then, by linearity, the above result can be extended to all the k summands in the original expression for F_N. Hence for $F_N \in \mathcal{W}^{1/2}$ the formula in our Theorem is true; i.e.,

$$\begin{aligned}\theta(X(t),F_N) &= \theta(X(a),F_N) + \int_a^t \partial_s^* \left(f(s)\frac{\partial\theta}{\partial x}(X(s),F_N)\right) ds \\ &+ \frac{1}{2}\int_a^t f(s)^2\frac{\partial^2\theta}{\partial x^2}(X(s),F_N)\,ds + \int_a^t f(s)(\partial_s F_N)\frac{\partial^2\theta}{\partial x\partial y}(X(s),F_N)\,ds.\end{aligned}$$

As stated earlier, by assumption, $F_N \longrightarrow F$ as $N \longrightarrow \infty$ in the $\mathcal{W}^{1/2}$-norm, where

$$F = \lambda_1 : e^{\langle\cdot,g_1\rangle} : + \lambda_2 : e^{\langle\cdot,g_2\rangle} : + \cdots + \lambda_k : e^{\langle\cdot,g_k\rangle} :$$

with

$$g_n = \left(\sum_{j=1}^{p^{(n)}} \alpha_j^{(n)} 1_{[0,c_j^{(n)})}\right)^{\otimes m}, \quad 1 \le n \le k.$$

Therefore, there exists a subsequence $\{F_{N_k}\}_{k\ge1} \subset \{F_N\}_{N\ge1}$ such that $F_{N_k} \longrightarrow F$ as $k \longrightarrow \infty$ almost surely. For such a subsequence, the following is true

$$\begin{aligned}\theta(X(t),F_{N_k}) &= \theta(X(a),F_{N_k}) + \int_a^t \partial_s^* \left(f(s)\frac{\partial\theta}{\partial x}(X(s),F_{N_k})\right) ds \\ &+ \frac{1}{2}\int_a^t f(s)^2\frac{\partial^2\theta}{\partial x^2}(X(s),F_{N_k})ds + \int_a^t f(s)(\partial_s F_{N_k})\frac{\partial^2\theta}{\partial x\partial y}(X(s),F_{N_k})ds.\end{aligned}$$

We claim that the following convergences hold almost surely as k:

$$(i)\theta(X(t), F_{N_k}) \longrightarrow \theta(X(t), F)$$

$$(ii)\theta(X(a), F_{N_k}) \longrightarrow \theta(X(a), F)$$

$$(iii)\frac{1}{2}\int_a^t f(s)^2 \frac{\partial^2\theta}{\partial x^2}\partial^2\theta\partial x^2(X(s), F)ds$$

$$(iv)\int_a^t f(s)(\partial_s F_{N_k})\frac{\partial^2\theta}{\partial x\partial y}(X(s), F_{N_k})ds \longrightarrow \int_a^t f(s)(\partial_s F)\frac{\partial^2\theta}{\partial x\partial y}(X(s), F)ds$$

$$(v)\int_a^t \partial_s^*\left(f(s)\frac{\partial\theta}{\partial x}(X(s), F_{N_k}\right)ds \longrightarrow \int_a^t \partial_s^*\left(f(s)\frac{\partial\theta}{\partial x}(X(s), F\right)ds$$

Proof of (i) and (ii):

Because of continuity of θ, since $F_{N_k} \longrightarrow F\ a.s$, those two convergences also hold almost surely.

Proof of (iii):

Let $\omega \in \mathcal{S}'(\mathbb{R})$ be fixed. Since F_{N_k} converges, it is a bounded sequence. i.e. there exists $M > 0$ such that $|F_{N_k}(\omega)| \leq M$ for all k. Therefore, the function given by $\frac{\partial^2\theta}{\partial x^2} : [a, b] \times [-M, M] \longrightarrow \mathbb{R}$ is continuous on compact sets and hence uniformly continuous. Thus given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x_1 - x_2|^2 + |y_1 - y_2|^2 < \delta$, we have

$$\left|\frac{\partial^2\theta}{\partial x^2}(x_1, y_1) - \frac{\partial^2\theta}{\partial x^2}(x_2, y_2)\right| < \frac{2\epsilon}{\|f\|_\infty^2(t-a)} \tag{3.9}$$

Also, by the convergence of $\{F_{N_k}\}_{k\geq 1}$, there exists a number $N(\epsilon)$ depending on ϵ such that

$$k \geq N(\epsilon) \Longrightarrow |F_{N_k}(\omega) - F(\omega)| < \delta,$$

which implies that

$$\begin{aligned}
\tfrac{1}{2} &\quad \left| \int_a^t f(s)^2 \left(\frac{\partial^2 \theta}{\partial x^2}(X(s), F_{N_k}(\omega)) - \frac{\partial^2 \theta}{\partial x^2}(X(s), F(\omega)) \right) ds \right| \\
\leq &\quad \frac{\|f\|_\infty^2}{2} \int_a^t \left| \frac{\partial^2 \theta}{\partial x^2}(X(s), F_{N_k}(\omega)) - \frac{\partial^2 \theta}{\partial x^2}(X(s), F(\omega)) \right| ds \\
< &\quad \frac{\|f\|_\infty^2}{2} \frac{2\epsilon}{\|f\|_\infty^2 (t-a)}(t-a) \\
= &\quad \epsilon
\end{aligned}$$

and so (iii) is proved.

Proof of (iv):

It is known that in a Hilbert space H, if $x_n \longrightarrow x$ and $y_n \longrightarrow y$, then $\langle x_n, y_n \rangle \longrightarrow \langle x, y \rangle$ where $\langle \cdot, \cdot \rangle$ is the inner product on H. By taking $H = L^2([a,t])$ with the Lebesgue measure it follows then that

$$\int_a^t f(s)(\partial_s F_{N_k}(\omega)) \frac{\partial^2 \theta}{\partial x \partial y}(X(s), F_{N_k}(\omega)) ds \equiv \left\langle \partial . F_{N_k}(\omega), f(\cdot) \frac{\partial^2 \theta}{\partial x \partial y}(X(\cdot), F_{N_k}(\omega)) \right\rangle$$

and

$$\int_a^t f(s)(\partial_s F(\omega)) \frac{\partial^2 \theta}{\partial x \partial y}(X(s), F(\omega)) ds \equiv \left\langle \partial . F(\omega), f(\cdot) \frac{\partial^2 \theta}{\partial x \partial y}(X(\cdot), F(\omega)) \right\rangle .$$

By the same uniform continuity argument used above in proving (iv), as $k \longrightarrow \infty$, we see that

$$f(\cdot) \frac{\partial^2 \theta}{\partial x \partial y}(X(\cdot), F_{N_K}(\omega)) \longrightarrow f(\cdot) \frac{\partial^2 \theta}{\partial x \partial y}(X(\cdot), F(\omega))$$

in $L^2([a,t])$.

Now $F_{N_k} \longrightarrow F$ in $\mathcal{W}^{1/2}$. Hence $F_{N_k} \longrightarrow F$ in (L^2) and $\int_a^t \|\partial_s F_{N_k} - \partial_s F\|_0^2 \, ds \longrightarrow 0$ as $k \longrightarrow \infty$. Therefore a change of integrals is justified and the following is true:

$$\begin{aligned}\int_{\mathcal{S}'(\mathbb{R})} \int_a^t |\partial_s F_N(\omega) - \partial_s F(\omega)|^2 ds d\mu(\omega) &= \int_a^t \int_{\mathcal{S}'(\mathbb{R})} |\partial_s F_N(\omega) - \partial_s F(\omega)|^2 d\mu(\omega) ds \\ &= \int_a^t \|\partial_s F_N - \partial_s F\|^2_{(L^2)} ds \longrightarrow 0.\end{aligned}$$

Let

$$\int_a^t |\partial_s F_{N_k}(\omega) - \partial_s F(\omega)|^2 ds = H_{N_k}(\omega).$$

Then, $\{H_{N_k}(\omega)\}_{k \geq 1}$ is a sequence of positive functions and by the above result, $H_{N_k}(\omega) \longrightarrow 0$ in $L^1(\mu)$. Therefore,

$$\partial . F_{N_k}(\omega) \longrightarrow \partial . F(\omega)$$

in $L^2([a,t])$ and so (iv) is true.

Proof of (v):

Let us approximate the (L^2)-norm of the difference of the two integrals. We claim the norm of this difference goes to zero. For convenience let us use the following notation. Let

$$f(s)\frac{\partial \theta}{\partial x}(X(s), F_{N_k}(\omega)) = \varphi_{N_k}(s)$$

and

$$f(s)\frac{\partial \theta}{\partial x}(X(s), F(\omega)) = \varphi(s).$$

Then by Theorem 3.12, we have the following

$$\begin{aligned}\left\| \int_a^t \partial_s^* \varphi_{N_k} \, ds - \int_a^t \partial_s^* \varphi \, ds \right\|_0^2 &= \int_a^t \|\varphi_{N_k} - \varphi\|_0^2 \, ds \\ &+ \int_a^t \int_a^t \big((\partial_s \varphi_{N_k}(u) - \partial_s \varphi(u), \partial_u \varphi_{N_k}(s) - \partial_u \varphi(s)) \big)_0 \, du \, ds\end{aligned} \tag{3.10}$$

where $((\cdot,\cdot))_0$ is the inner product on (L^2).

Since $F_{N_k}(\omega) \longrightarrow F(\omega)$ in (L^2), we can choose a subsequence so that the convergence is almost surely. Thus for such a subsequence, there exists Ω_0 such that $P\{\Omega_0\} = 1$ and for ω fixed, we have $F_{N_k}(\omega) \longrightarrow F(\omega)$. Then, by the Mean Value Theorem, for each s,

$$\left|\frac{\partial\theta}{\partial x}\big(X(s), F_{N_k}(\omega)\big) - \frac{\partial\theta}{\partial x}\big(X(s), F(\omega)\big)\right| = \left|\frac{\partial^2\theta}{\partial x\partial y}\big(X(s), \xi(\omega)\big)\right| \cdot \left|F_{N_k}(\omega) - F(\omega)\right|$$

where $\xi(\omega)$ is between $F_{N_k}(\omega)$ and $F(\omega)$. Since $\theta \in \mathcal{C}_b^2(\mathbb{R}^2)$, $\left|\frac{\partial^2\theta}{\partial x\partial y}(X(s), \xi(\omega))\right| < C$ for some constant C. So,

$$\left\|\frac{\partial\theta}{\partial x}(X(s), F_{N_k}(\omega)) - \frac{\partial\theta}{\partial x}(X(s), F(\omega))\right\|_0 \leq C\,\|F_{N_k}(\omega) - F(\omega)\|_0.$$

As earlier noted, the convergence of the subsequence $\{F_{N_k}\}_{k\geq 1}$ of numbers implies that for some fixed constant $M > 0$, the quantity $\|F_{N_k}(\omega) - F(\omega)\|_0 \leq M$, $\forall k$. Therefore, using the fact that $F_{N_k}(\omega) \longrightarrow F(\omega)$ in (L^2), we have that

$$\|\varphi_{N_k} - \varphi\|_0 \leq \|f\|_\infty C\|F_{N_k} - F\|_0 \longrightarrow 0,$$

and also that

$$\begin{aligned}\|\varphi_{N_k} - \varphi\|_0 &\leq \|f\|_\infty C\|F_{N_k} - F\|_0 \\ &\leq \|f\|_\infty CM.\end{aligned}$$

Since this bound is independent of both s and k, by the Bounded Convergence Theorem, as $k \longrightarrow \infty$,

$$\int_a^t \|\varphi_{N_k} - \varphi\|_0^2\, ds \longrightarrow 0.$$

The convergence of the second summand of the right hand side of equation (3.10) goes as follows

$$\begin{aligned}
&\left|\int_a^t \int_a^t \big((\partial_s\varphi_{N_k}(u) - \partial_s\varphi(u), \partial_u\varphi_{N_k}(s) - \partial_u\varphi(s))\big)_0 \, du\, ds\right| \\
&\qquad \leq \int_a^t \int_a^t \|\partial_s\varphi_{N_k}(u) - \partial_s\varphi(u)\|_0 \, \|\partial_u\varphi_{N_k}(s) - \partial_u\varphi(s)\|_0 \, du\, ds \\
&\qquad \leq \frac{1}{2}\int_a^t \int_a^t \left(\|\partial_s\varphi_{N_k}(u) - \partial_s\varphi(u)\|_0^2 + \|\partial_u\varphi_{N_k}(s) - \partial_u\varphi(s)\|_0^2\right) du\, ds \\
&\qquad = \int_a^t \int_a^t \|\partial_u\varphi_{N_k}(s) - \partial_u\varphi(s)\|_0^2 \, du\, ds \\
&\qquad \leq \int_a^t \left(\int_{\mathbb{R}} \|\partial_u\varphi_{N_k}(s) - \partial_u\varphi(s)\|_0^2 \, du\right) ds.
\end{aligned}$$

The Number operator, N can also be expressed as $N = \int_{\mathbb{R}} \partial_s^*\partial_s \, ds$. (see [21]). Hence, for any $\varphi \in L^2([a,b];\mathcal{W}^{1/2})$ we have

$$\begin{aligned}
\|N^{1/2}\varphi\|_0^2 &= ((N\varphi, \varphi))_0 \\
&= \int_{\mathbb{R}} ((\partial_s^*\partial_s\varphi, \varphi))_0 \, ds \\
&= \int_{\mathbb{R}} ((\partial_s\varphi, \partial_s\varphi))_0 \, ds \\
&= \int_{\mathbb{R}} \|\partial_s\varphi\|_0^2 \, ds.
\end{aligned}$$

Therefore, by this result,

$$\int_a^t \left(\int_{\mathbb{R}} \|\partial_u\varphi_{N_k}(s) - \partial_u\varphi(s)\|_0^2 \, du\right) ds = \int_a^t \|N^{1/2}\partial_u\varphi_{N_k}(s) - \partial_u\varphi(s)\|_0^2 \, ds.$$

But $N^{1/2}$ is a bounded operator from $\mathcal{W}^{1/2}$ into (L^2). Therefore, by the same Mean Value Theorem argument, since $F_{N_k} \longrightarrow F$ also in $\mathcal{W}^{1/2}$ by our original assumption, we have that

$$\left|\int_a^t \int_a^t \big((\partial_s\varphi_{N_k}(u) - \partial_s\varphi(u), \partial_u\varphi_{N_k}(s) - \partial_u\varphi(s))\big)_0 \, du\, ds\right| \longrightarrow 0$$

as $k \longrightarrow \infty$.

Therefore, it has been shown that the convergence in (iii) is true in the (L^2)-norm. We then pick a further subsequence of the subsquence $\{F_{N_k}\}_{k\geq 1}$ also denoted by $\{F_{N_k}\}_{k\geq 1}$ such that convergence hold almost surely. This then proves (v) and completes the proof of the Theorem. $\square$

Chapter 4. A Generalization of the Clark-Ocone Formula

In this second part of the dissertation we show that the well-known Clark-Ocone formula [4], [27] can be extended to generalized Brownian (white noise) functionals that meet certain criteria and that the formula takes on the same form as the classical one. We look at a specific example of Donsker's delta function which is a generalized Brownian functional.

In section 4.1 we first state the Clark-Ocone formula as it is applied to Brownian functionals in [4], [27] that meet certain conditions. We then state the white noise version of the formula and verify it for functionals in the space $\mathcal{W}^{1/2}$. In section 4.2, we verify the formula for a generalized Brownian functional of the form $\Phi = \sum_{n=0}^{\infty}\langle : \cdot^{\otimes n} : , F_n\rangle$, $F_n \in L^2(\mathbb{R}^{\hat{\otimes} n})$. In section 4.3, we use th e Itô formula to verify the Clark-Ocone formula for the Hermite Brownian functional $: B(t)^n :_t$. This is important in view of the fact that Donsker's delta function has a representation in terms of the Hermite Brownian functional as shown in equation (2.14). In section 4.4, we compute the formula for Donsker's delta function as a specific example of the formula obtained in section 4.2 using the result in section 4.3. It should be pointed out that in [6] the same result was obtained by using a totally different approach. Our result is purely computational and much simpler. In section 4.5 we extend the formula to generalized Brownian functionals of the form $F = f(B(t))$ where f is a tempered distribution; i.e. $f \in \mathcal{S}'(\mathbb{R})$.

4.1 Representation of Brownian Functionals by Stochastic Integrals

Consider a probability space $(\Omega, \mathcal{F}, P)$. Let $B(t)$ be the Brownian motion $B(t) = \langle \cdot, 1_{[0,t)}\rangle$, $0 \le t \le 1$ and $\mathcal{F}_t = \sigma\{B(s) | 0 \le s \le t\}$ the filtration it generates. In [4],

[27] the following results are considered. If $F(B(t))$ is any finite functional of Brownian motion, then it can be represented as a stochastic integral. This representation is not unique. However, if $E(F^2(B(\cdot))) < \infty$, then according to martingale representation theory, F does have a unique representation as the sum of a constant and an Itô stochastic integral

$$F = E(F) + \int_0^1 \phi(t)\, dB(t), \tag{4.1}$$

where the process ϕ belongs to the space $L^2([0,1] \times \Omega; \mathbb{R})$ and is $\mathcal{F}_t$-measurable.

The following specific class of Brownian functionals was considered. If F is Frećhet-differentiable and satisfies certain technical regularity conditions, then F has an explicit expression as a stochastic integral in which the integrand consists of the conditional expectations of the Fréchet differential. It is this explicit representation for the integrand that gives rise to the Clark-Ocone formula. In the white noise setup, if we replace Fréchet differentiability with white noise differentiation, then we need the condition that $F \in \mathcal{W}^{1/2}$ for the result to have meaning. The formula then can be represented in two ways as shown below and we will verify each case separately. In the satement of the theorem, we shall assume that $T \subset \mathbb{R}$. The second form of the formula will be easier to verify using the S-transform.

Theorem 4.1 (The Clark-Ocone Formula). *Let $\mathcal{W}^{1/2}$ be the sobolev space from the Gel'fand triple $(\mathcal{S})_\beta \subset (L^2) \subset (\mathcal{S})^*_\beta$. Suppose $B(t)$ is the Brownian motion given by $B(t) = \langle \cdot, 1_{[0,t)} \rangle$, $t \in T$ with $\mathcal{F}_t = \sigma\{B(s) | 0 \leq s \leq t\}$ the filtration it generates. Let $F \in \mathcal{W}^{1/2}$ be a Brownian functional. Then, the Clark-Ocone representation formula for such an F is given by*

$$F = E(F) + \int_T E(\partial_t F | \mathcal{F}_t)\, dB(t). \tag{4.2}$$

We can rewrite the stochastic integral in the above equation as

$$\int_T E(\partial_t F|\mathcal{F}_t)\,dB(t) = \int_T \partial_t E(\partial_t F|\mathcal{F}_t)\,dt + \int_T \partial_t^* E(\partial_t F|\mathcal{F}_t)\,dt.$$

But $\partial_t E(\partial_t F|\mathcal{F}_t) = 0$ since $(E(\partial_t F|\mathcal{F}_t))$ is non-anticipating. (See [21] Lemma 13.11). Therefore, another formulation for equation (4.2) is

$$F = E(F) + \int_T \partial_t^* E(\partial_t F|\mathcal{F}_t)\,dt, \tag{4.3}$$

where ∂_t is the white noise differential operator, ∂_t^* is its adjoint, and the integral in equation (4.3) is regarded as a white noise integral in the Pettis sense.

Such formulas are very useful in the determination of hedging portfolios. Another application is in the context of determining the quadratic variation process of Brownian martingales.

We start with verifying the representation (4.2). The space $\mathcal{W}^{1/2}$ can be represented as the following

$$\mathcal{W}^{1/2} = \left\{ F \in (L^2) | F = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : , f_n \rangle,\ f_n \in L^2(\mathbb{R})^{\hat{\otimes} n},\ \sum_{n=1}^{\infty} n n! |f_n|_0^2 < \infty \right\}.$$

This is so because by Theorem 9.27 in [21], $\int_T \|\partial_t F\|_0^2\,dt = \sum_{n=1}^{\infty} n n! |f_n|_0^2$.

As an observation first, we need to make sure that the integral in equation (4.2) is well defined by computing its (L^2) norm. We will assume that F is real valued. Otherwise, $F = F_1 + iF_2$ if F is complex valued so that the two pieces are treated separately. By using Jensen's inequality we can have

$$\begin{aligned} E\left(\int_T E(\partial_t F|\mathcal{F}_t)\,dB(t)\right)^2 &= \int_T E\left(E(\partial_t F|\mathcal{F}_t)\right)^2\,dt \\ &\le \int_T E\left(E(\partial_t F)^2|\mathcal{F}_t\right)\,dt \\ &= \int_T E(\partial_t F)^2\,dt \end{aligned}$$

$$\leq \int_{\mathbb{R}} E(\partial_t F)^2 \, dt$$
$$= \int_{\mathbb{R}} \|\partial_t F\|_0^2 \, dt < \infty.$$

So the integral is well defined as an (L^2) function.

F will take on the form $F = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : , f_n \rangle$ with $f_n \in L^2(\mathbb{R}^{\hat{\otimes} n})$. The Wick tensor $\varphi = \langle : \cdot^{\otimes n} : , f_n \rangle$ is referred to as the nth order homogeneous chaos. It is simply another name for the multiple Wiener integral $I_n(f_n)$ of degree n for the function f_n. We first need to verify the formula for φ. The following Lemma will be useful in the verification. We will first recall that in integral form, $I_n(f_n)$ can be written as $I_n(f_n) = \int_{\mathbb{R}^n} f(t_1, \cdots, t_n) \, dB(t_1) \cdots dB(t_n)$. We will use second form to verify the first version of the formula.

Lemma 4.2. *Let $\varphi_n = I_n(f)$ be the multiple Wiener integral of degree n with $f \in L^2(\mathbb{R}^{\hat{\otimes} n})$. Let $B(t)$ be the Brownian motion given by $B(t) = \langle \cdot, 1_{[0,t)} \rangle$, $t \in \mathbb{R}$ with $\mathcal{F}_t = \sigma\{B(s) | 0 \leq s \leq t\}$, the filtration it generates. Then*

$$E(\varphi_n | \mathcal{F}_t) = I_n(f 1_{(-\infty,t]}^{\otimes n}).$$

Using another notation, for $\varphi_n = \int_{\mathbb{R}^n} f(t_1, \cdots, t_n) \, dB(t_1) \cdots dB(t_n)$, we have

$$E(\varphi_n | \mathcal{F}_t) = \int_{(-\infty,t]^n} f(t_1, \cdots, t_n) \, dB(t_1) \cdots dB(t_n).$$

Proof. Let $n \geq 1$ be fixed. Let $A_1, A_2, \cdots A_k$ be pairwise disjoint intervals in $\mathbb{R}$. Denote by ε_n the set of special step functions of the form

$$f_n(t_1, t_2, \cdots, t_n) = \sum_{i_1, \cdots, i_n = 1}^{k} a_{i_1 \cdots i_n} 1_{A_{i_1} \times \cdots \times A_{i_n}}(t_1, \cdots, t_n)$$

with the property that the coefficients $a_{i_1, \cdots i_n}$ are zero if any two of the indices $i_1, \cdots, i_m$ are equal. In other words f should vanish on the rectangles that intersect any diagonal subspace $\{t_i = t_j, \, i \neq j\}$. Assume A_{i_j} is of the form $(t_{i_j}, t_{i_{j+1}}]$. Then

$I_n(f_n)$ can be expressed in the form

$$\begin{aligned} I_n(f_n) &= \sum_{i_1,\cdots,i_n=1}^{k} a_{i_1\cdots i_n}(B(t_{i_2})-B(t_{i_1}))\cdots(B(t_{i_n})-B(t_{i_{n-1}})) \\ &= \sum_{i_1,\cdots,i_n=1}^{k} a_{i_1\cdots i_n}\langle\cdot, 1_{(t_{i_1},t_{i_2}]}\rangle\cdots\langle\cdot, 1_{(t_{i_{n-1}},t_{i_n}]}\rangle \end{aligned}$$

By linearity, it suffices to take $f_n = 1_{A_1\times\cdots\times A_n}$ where the $A_i's$ are mutually disjoint intervals in $\mathbb{R}$ of the form $A_i = (t_i, t_{i+1}]$. In this case, we obtain the following representation

$$\begin{aligned} E(I_n(f_n) \mid \mathcal{F}_t) &= E(\langle\cdot, 1_{(t_1,t_2]}\rangle\cdots\langle\cdot, 1_{(t_{n-1},t_n]}\rangle \mid \mathcal{F}_t) \\ &= \prod_{i=1}^{n} E(\langle\cdot, 1_{(t_i,t_{i+1}]\cap(-\infty,t]}\rangle + \langle\cdot, 1_{(t_i,t_{i+1}]\cap(t,\infty)}\rangle \mid \mathcal{F}_t) \\ &= \prod_{i=1}^{n}(\langle\cdot, 1_{(t_i,t_{i+1}]\cap(-\infty,t]}\rangle) \\ &= I_n(1_{(A_1\cap(-\infty,t])\times\cdots\times(A_n\cap(-\infty,t])}) \\ &= I_n(1_{A_1\times\cdots\times A_n}1_{(-\infty,t]}^{\otimes n}) \\ &= I_n(f_n 1_{(-\infty,t]}^{\otimes n}) \\ &= \int_{(-\infty,t]^n} 1_{A_1\times\cdots\times A_n} dB(t_1)\cdots dB(t_n) \end{aligned}$$

We then conclude that the formula is also true for general $f \in L^2(\mathbb{R}^{\hat{\otimes} n})$ because the set ε_n is dense in $L^2(\mathbb{R}^{\hat{\otimes} n})$. This then completes the proof for the lemma. $\square$

We are now ready to verify the formula for the nth order homogeneous chaos. Let $\varphi = \langle : \cdot^{\otimes n} : , f_n\rangle, f_n \in L^2(\mathbb{R}^{\otimes n})$, $n \geq 1$ be the nth order homogeneous chaos. Then in integral form,

$$\begin{aligned} \varphi &= \int_{\mathbb{R}^n} f_n(t_1,\cdots,t_n)\, dB(t_1)\cdots dB(t_n) \\ &= n! \int_{\mathbb{R}}\int_{-\infty}^{t_n}\cdots\int_{-\infty}^{t_3}\int_{-\infty}^{t_2} f_n(t_1,\cdots,t_n) dB(t_1)\cdots dB(t_n) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \left(n! \int_{-\infty}^{t_n} \cdots \int_{-\infty}^{t_3} \int_{-\infty}^{t_2} f_n(t_1, \cdots, t_n) dB(t_1) \cdots dB(t_{n-1}) \right) dB(t_n) \\
&= \int_{\mathbb{R}} n! \frac{1}{(n-1)!} \left(\int_{(-\infty,t]^{n-1}} f_n(t_1, \cdots, t_{n-1} dB(t_1) \cdots dB(t_{n-1}) \right) dB(t) \\
&= \int_{\mathbb{R}} \left(n \int_{(-\infty,t]^{n-1}} f_n(t_1, \cdots, t_{n-1} dB(t_1) \cdots dB(t_{n-1}) \right) dB(t)
\end{aligned}
$$

Now since $\partial_t \varphi = n\langle : \cdot^{\otimes(n-1)} :, f_n(t, \cdot)\rangle$, by the lemma above, $E(\partial_t \varphi \mid \mathcal{F}_t) = n \int_{(-\infty,t]^{n-1}} f_n(t_1, \cdots, t_{n-1}, t) dB(t_1) \cdots dB(t_{n-1})$ and so

$$
\begin{aligned}
\varphi &= \int_{\mathbb{R}} \left(n \int_{(-\infty,t]^{n-1}} f_n(t_1, \cdots, t_{n-1}, t) dB(t_1) \cdots dB(t_{n-1}) \right) dB(t) \\
&= \int_{\mathbb{R}} E(\partial_t \varphi \mid \mathcal{F}_t) dB(t).
\end{aligned}
$$

If we have a finite sum $F_N = \sum_{n=0}^{N} \langle : \cdot^{\otimes n} : , f_n\rangle$, then

$$
\begin{aligned}
F_N &= f_0 + \sum_{n=1}^{N} \langle : \cdot^{\otimes n} : , f_n\rangle \\
&= E(F_N) + \int_{\mathbb{R}} E(\partial_t F_N \mid \mathcal{F}_t) dB(t).
\end{aligned}
$$

Now for general $F = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : , f_n\rangle \in \mathcal{W}^{1/2}$, we have the following.

$$
\begin{aligned}
F &= \sum_{n=0}^{N} \langle : \cdot^{\otimes n} : , f_n\rangle + \sum_{n=N+1}^{\infty} \langle : \cdot^{\otimes n} : , f_n\rangle \\
&= F_N + F_{N+1} \\
&= E(F_N) + \int_{\mathbb{R}} E(\partial_t F_N \mid \mathcal{F}_t) dB(t) + F_{N+1}
\end{aligned}
$$

We note that $E(F_N) = E(F)$ and $F_N \longrightarrow F$ in (L^2) because $\|F - F_N\|_0^2 = \sum_{n=N+1}^{\infty} n! |f_n|_0^2 \longrightarrow 0$ as $n \longrightarrow \infty$. We claim that $\int_{\mathbb{R}} E\left(\partial_t F_N | \mathcal{F}_t\right) dB(t) \longrightarrow \int_{\mathbb{R}} E\left(\partial_t F | \mathcal{F}_t\right) dB(t)$ in (L^2) as $N \longrightarrow \infty$. In fact all the convergences also hold if we work in the space $\mathcal{W}^{1/2}$, which indeed is our main concern. However, for purposes of ease of computation we will work in the space (L^2) because of the way the two norms are related to each other. Now if we start by estimating the (L^2)-norm of the difference between these two quantities we obtain the following

$$
\begin{aligned}
\left\| \int_{\mathbb{R}} E(\partial_t F|\mathcal{F}_t)\, dB(t) - \int_{\mathbb{R}} E(\partial_t F_N|\mathcal{F}_t)\, dB(t) \right\|_0^2 & \\
&= E\left\{ \left(\int_{\mathbb{R}} E(\partial_t (F - F_N)|\mathcal{F}_t)\, dB(t) \right)^2 \right\} \\
&= E\left\{ \left(\int_{\mathbb{R}} E(\partial_t F_{N+1}|\mathcal{F}_t)\, dB(t) \right)^2 \right\} \\
&= \int_{\mathbb{R}} E\left(E(\partial_t F_{N+1}|\mathcal{F}_t) \right)^2\, dt \\
&\leq \int_{\mathbb{R}} E(E(\partial_t F_{N+1})^2|\mathcal{F}_t)\, dt \\
&= \int_{\mathbb{R}} E(\partial_t F_{N+1})^2\, dt \\
&= \int_{\mathbb{R}} \|\partial_t F_{N+1}\|_0^2\, dt \\
&= \sum_{n=N+1}^{\infty} n n! |F_n|_0^2 \longrightarrow 0.
\end{aligned}
$$

Hence, for general $F \in \mathcal{W}^{1/2}$ we can decompose F as in equation (4.2).

We now verify the representation in equation (4.3). The main tool for the verification is the S-transform. First we prove that the integral $\int_T \partial_t^* E(\partial_t F|\mathcal{F}_t)\, dt$ is a white noise integral defined in the Pettis sense. We need to show that the three conditions in section (2.9) are satisfied. We will take $\Phi : \mathbb{R} \longrightarrow (\mathcal{S})_\beta^*$ where $\Phi(t) = \partial_t^* E(\partial_t F|\mathcal{F}_t)$. We note that $t \mapsto S(\Phi(t))(\xi)$ is measurable because

$$
\begin{aligned}
S(\Phi(t))(\xi) &= S(\partial_t^* E(\partial_t F|\mathcal{F}_t)(\xi) \\
&= \xi(t) S(E(\partial_t F|\mathcal{F}_t))(\xi) \\
&= \xi(t) \sum_{n=1}^{\infty} n \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t, s_1, \cdots, s_{n-1}) \xi(s_1) \cdots \xi(s_{n-1})\, ds_1 \cdots ds_{n-1},
\end{aligned}
$$

which is a product of measurable functions. Hence the function $\partial_t^* E(\partial_t F|\mathcal{F}_t)$ is weakly measurable to settle the first of the three necessary conditions.

The second condition is to show that $t \mapsto \langle\langle \Phi, \varphi \rangle\rangle \in L^1(\mathbb{R})\ \forall \varphi \in (\mathcal{S})_\beta$. Let $\varphi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : , g_n \rangle \in (\mathcal{S})_\beta$. Then

$$\begin{aligned}
\int_{\mathbb{R}} \big| \langle\langle \partial_t^* E(\partial_t F | \mathcal{F}_t), \varphi \rangle\rangle \big| \, dt &= \int_{\mathbb{R}} \big| \langle\langle E(\partial_t F | \mathcal{F}_t), \partial_t \varphi \rangle\rangle \big| \, dt \\
&\leq \int_{\mathbb{R}} \big\| E(\partial_t F | \mathcal{F}_t) \big\|_0 \big\| \partial_t \varphi \big\|_0 \, dt \\
&\leq \int_{\mathbb{R}} \big\| \partial_t F \big\|_0 \big\| \partial_t \varphi \big\|_0 \, dt \\
&\leq \sqrt{\int_{\mathbb{R}} \big\| \partial_t F \big\|_0^2 \, dt \int_{\mathbb{R}} \big\| \partial_t \varphi \big\|_0^2 \, dt}.
\end{aligned}$$

We know that $\int_{\mathbb{R}} \big\| \partial_t F \big\|_0^2 \, dt < \infty$ by assumption. To show that $\int_{\mathbb{R}} \big\| \partial_t \varphi \big\|_0^2 \, dt < \infty$, it is enough to show that $\sum_{n=1}^{\infty} n n! |g_n|_0^2 < \infty$. We see that for any $p \geq 0$,

$$\sum_{n=1}^{\infty} n n! |g_n|_0^2 \leq \sum_{n=1}^{\infty} n! 2^n \lambda_1^{-2pn} |g_n|_p^2$$

If we choose p large enough such that $2\lambda_1^{-2p} < 1$, we see that $\sum_{n=1}^{\infty} 2^n \lambda_1^{-2pn} n! |g_n|_p^2 \leq \sum_{n=1}^{\infty} n! |g_n|_p^2 = \|\varphi\|_p^2 < \infty$. Therefore the second condition has been satisfied. The third condition in section (2.9) is implied by the above two conditions as shown in [21]. Hence the integral $\int_T \partial_t^* E(\partial_t F | \mathcal{F}_t) \, dt$ is defined in the Pettis sense.

We now claim that $F = E(F) + \int_T \partial_t^* E(\partial_t F | \mathcal{F}_t) \, dt \ \forall F \in \mathcal{W}^{1/2}$. We prove this first for exponential functions $F = F_\eta = \sum_{n=0}^{\infty} \frac{1}{n!} \langle : \cdot^{\otimes n} : , \eta^{\otimes n} \rangle, \qquad \eta \in L^2(\mathbb{R})$. First we check that $F_\eta \in \mathcal{W}^{1/2}$.

$$\begin{aligned}
\sum_{n=1}^{\infty} n n! \left| \frac{\eta^{\otimes n}}{n!} \right|_0^2 &= \sum_{n=1}^{\infty} \frac{n}{n!} |\eta|_0^{2n} \\
&= |\eta|_0^2 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} |\eta|_0^{2(n-1)} \\
&= |\eta|_0^2 e^{|\eta|_0^2} < \infty.
\end{aligned}$$

Now to show that $F_\eta = E(F_\eta) + \int_{\mathbb{R}} \partial_t^* E(\partial_t F_\eta|\mathcal{F}_t)\, dt$ we will have to check if the S-transforms of the two sides are equal.

$$\begin{aligned}
S\Big(E(F_\eta) + \int_{\mathbb{R}} \partial_t^* E(\partial_t F_\eta|\mathcal{F}_t)\, dt\Big)(\xi) &= S(E(F_\eta))(\xi) + S\Big(\int_{\mathbb{R}} \partial_t^* E(\partial_t F_\eta|\mathcal{F}_t)\, dt\Big)(\xi) \\
&= 1 + \int_{\mathbb{R}} \xi(t) S(E(\partial_t F_\eta|\mathcal{F}_t))(\xi)\, dt \\
&= 1 + \int_{\mathbb{R}} \xi(t) S(E\eta(t) F_\eta|\mathcal{F}_t))(\xi)\, dt \\
&= 1 + \int_{\mathbb{R}} \xi(t)\eta(t) S(E(F_\eta|\mathcal{F}_t))(\xi)\, dt \\
&= 1 + \int_{\mathbb{R}} \xi(t)\eta(t) S\Big(\sum_{n=0}^{\infty} \langle :\cdot^{\otimes n}: , \frac{1}{n!}\eta 1_{(-\infty,t]}^{\otimes n}\rangle\Big)(\xi)\, dt \\
&= 1 + \int_{\mathbb{R}} \xi(t)\eta(t) S(F_{\eta 1_{(-\infty,t]}})(\xi)\, dt \\
&= 1 + \int_{\mathbb{R}} \xi(t)\eta(t) \langle\langle F_{\eta 1_{(-\infty,t]}}, F_\xi\rangle\rangle\, dt \\
&= 1 + \int_{\mathbb{R}} \xi(t)\eta(t) e^{\langle \eta 1_{(-\infty,t]}, \xi\rangle}\, dt \\
&= 1 + \int_{\mathbb{R}} \xi(t)\eta(t) e^{\int_{-\infty}^t \eta(s)\xi(s)\, ds} dt \\
&= 1 + \int_{-\infty}^{\infty} \frac{d}{dt}\Big(e^{\int_{-\infty}^t \eta(s)\xi(s)\, ds}\Big) dt \\
&= 1 + e^{\int_{-\infty}^t \eta(s)\xi(s)\, ds}\Big|_{-\infty}^{\infty} \\
&= 1 + e^{\int_{-\infty}^{\infty} \eta(s)\xi(s)\, ds} - 1 \\
&= e^{\int_{-\infty}^t \eta(s)\xi(s)\, ds} \\
&= e^{\langle \eta, \xi\rangle} \\
&= \langle\langle F_\eta, F_\xi\rangle\rangle \\
&= S(F_\eta)(\xi)
\end{aligned}$$

This then verifies that $F_\eta = E(F_\eta) + \int_{\mathbb{R}} \partial_t^* E(\partial_t F_\eta|\mathcal{F}_t)\, dt$. We now have to show that the span of the set $\{F_\eta,\, \eta \in L^2(\mathbb{R})\}$ is dense in $\mathcal{W}^{1/2}$. Let $F = \langle :\cdot^{\otimes n}: , f_n\rangle \in \mathcal{W}^{1/2}$, $f_n \in L^2(\mathbb{R}^{\hat{\otimes} n})$. It is enough to show that each $\langle :\cdot^{\otimes n}: , f_n\rangle$ can be approximated in the $\mathcal{W}^{1/2}$-norm by exponential functions. Now each f_n can be approximated by $\eta^{\otimes n}$, where $\eta \in L^2(\mathbb{R})$ by the polarization identity. Thus $\langle : \cdot^{\otimes n} : , f_n \rangle$ is approximated by $\langle : \cdot^{\otimes n} : , \eta^{\otimes n} \rangle$. But we have the equation

$$\langle : \cdot^{\otimes n} : , \eta^{\otimes n} \rangle = \frac{d^n}{dt^n} F_{t\eta} \Big|_{t=0}$$

and $\frac{d^n}{dt^n} F_{t\eta}\big|_{t=0}$ is in the closure of the span of exponential functions (it is a limit of a limit of finite linear combinations of exponential functions). Therefore, there exists a sequence $\{F_n\}_{n\geq 1}$ in the span of $\{F_\eta \mid \eta \in L^2(\mathbb{R})\}$ such that $F_n \longrightarrow F$ in $\mathcal{W}^{1/2}$. Since $E(F_n) \longrightarrow E(F)$ it remains to show that $\int_{\mathbb{R}} \partial_t^* E(\partial_t F_n \mid \mathcal{F}_t)\, dt \longrightarrow \int_{\mathbb{R}} \partial_t^* E(\partial_t F \mid \mathcal{F}_t)\, dt$. This is equivalent to showing that the convergence exists weakly. Let $\varphi \in (\mathcal{S})_\beta$. Then we have

$$\begin{aligned}
\Big|\langle\langle \int_{\mathbb{R}} \partial_t^* E(\partial_t(F_n - F) \mid \mathcal{F}_t)\, dt, \varphi \rangle\rangle\Big| &= \Big|\int_{\mathbb{R}} \langle\langle \partial_t^* E(\partial_t(F_n - F) \mid \mathcal{F}_t), \varphi \rangle\rangle\, dt\Big| \\
&\leq \int_{\mathbb{R}} \big|\langle\langle E(\partial_t(F_n - F) \mid \mathcal{F}_t), \partial_t \varphi \rangle\rangle\big|\, dt \\
&\leq \int_{\mathbb{R}} \big\|E(\partial_t(F_n - F) \mid \mathcal{F}_t)\big\|_0 \big\|\partial_t \varphi\big\|_0\, dt \\
&\leq \sqrt{\int_{\mathbb{R}} \big\|\partial_t(F_n - F)\big\|_0^2\, dt \int_{\mathbb{R}} \big\|\partial_t \varphi\big\|_0^2\, dt} \\
&\longrightarrow 0.
\end{aligned}$$

This then completes the verification of formula (4.3).

4.2 A Generalized Clark-Ocone Formula

This is the main result of our second part of the dissertation. As pointed out earlier on, extending the Clark-Ocone formula to generalized Brownian functionals that meet some restrictions yields a formula of the same form as equation (4.2), but for ease of computation we will verify the form as given in equation (4.3). For our result, we start with laying down the kind of restrictions that the generalized Brownian functional has to satisfy.

Let $\Phi \in (\mathcal{S})^*_\beta$ with the property that $\Phi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : , f_n \rangle$, $f_n \in L^2(\mathbb{R}^{\hat{\otimes} n})$, $\partial_t \Phi$ exists in $(\mathcal{S})^*_\beta$ and $\int_{\mathbb{R}} \left\| \partial_t \Phi \right\|^2_{-p,-\beta} dt < \infty$. We first need to extend the definition of the conditional expectation $E(\partial_t \Phi \mid \mathcal{F}_t)$.

Definition 4.3. Let $\Phi \in (\mathcal{S})^*_\beta$ that satisfies the above property. For each t, we define the *conditional expectation* of $\partial_t \Phi$ with respect to $\mathcal{F}_t$ to be the function whose S-transform is given by

$$\begin{aligned} &S(E(\partial_t \Phi \mid \mathcal{F}_t))(\xi) \\ &\qquad := \sum_{n=1}^{\infty} n \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t, s_1, \cdots, s_{n-1}) \xi(s_1) \cdots \xi(s_{n-1}) \, ds_1 \cdots ds_{n-1}. \end{aligned}$$

Of course for such a definition to be valid, $S(E(\partial_t \Phi \mid \mathcal{F}_t))(\xi)$ must satisfy the conditions in the characterization Theorem for generalized functions as introduced in chapter 2. That is, we need to check that for $\xi, \eta \in \mathcal{S}_c(\mathbb{R})$ and $z \in \mathbb{C}$ the function $S(E(\partial_t \Phi \mid \mathcal{F}_t))(\xi + \eta)$ is an entire function and that it satisfies the growth condition. Let us start by showing that the function is question is an entire function.

(a) Let $\xi, \eta \in \mathcal{S}_c(\mathbb{R})$ and any $z \in \mathbb{C}$. Then we have

$$\begin{aligned} S(E(\partial_t \Phi \mid \mathcal{F}_t))(\xi + \eta) &= \sum_{n=1}^{\infty} n \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t, s_1, \cdots, s_{n-1}) (z\xi + \eta)(s_1) \cdots \\ &\quad \cdots (z\xi + \eta)(s_{n-1}) \, ds_1 \cdots ds_{n-1} \\ &= z \sum_{n=1}^{\infty} n \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t, s_1, \cdots, s_{n-1}) \xi(s_1) \cdots \xi(s_{n-1}) \, ds_1 \cdots ds_{n-1} \\ &\quad + \sum_{n=1}^{\infty} n \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t, s_1, \cdots, s_{n-1}) \eta(s_1) \cdots \eta(s_{n-1}) \, ds_1 \cdots ds_{n-1}. \end{aligned}$$

Now by the Fundamental Theorem of Calculus, the two series in the last equality are complex differentiable. Therefore, the first condition has been satisfied.

(b) The growth condition:

$$
\begin{aligned}
&\left|S(E(\partial_t\Phi \mid \mathcal{F}_t))(\xi)\right| \\
&\quad = \left|\sum_{n=1}^{\infty} n \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t, s_1, \cdots, s_{n-1})\xi(s_1)\cdots\xi(s_{n-1})\, ds_1 \cdots ds_{n-1}\right| \\
&\quad \leq \left|\sum_{n=1}^{\infty} n\langle f_n, \xi^{\otimes n}\rangle\right| = \left|\langle\langle \partial_t\Phi, :e^{\langle\cdot,\xi\rangle}:\rangle\rangle\right| \\
&\quad \leq \|\partial_t\Phi\|_{-p,-\beta} \left\|:e^{\langle\cdot,\xi\rangle}:\right\|_{p,\beta} \\
&\quad \leq \|\partial_t\Phi\|_{-p,-\beta}\, 2^{\beta/2} \exp(1-\beta) 2^{\frac{2\beta-1}{1-\beta}} |\xi|_p^{2/1-\beta} \\
&\quad = K \exp[a|\xi|_p^{2/1-\beta}], \qquad \forall \xi \in \mathcal{E}_c,
\end{aligned}
$$

where $K = 2^{\beta/2}\|\partial_t\Phi\|_{-p,-\beta}$ and $a = (1-\beta)2^{\frac{2\beta-1}{1-\beta}}$. See Theorem 5.7 [21].

Therefore, a generalized function in the space $(\mathcal{S})_\beta^*$ exists for which the given expression is its S-transform; and that function is given by $E(\partial_t\Phi \mid \mathcal{F}_t)$.

We now claim that the integral $\int_{\mathbb{R}} \partial_t^* E(\partial_t\Phi \mid \mathcal{F}_t)\, dt$ is defined as a white noise integral in the Pettis sense. The argument will be similar to the $\mathcal{W}^{1/2}$ case considered earlier. It only suffices to show that $t \mapsto \langle\langle \partial_t^* E(\partial_t\Phi \mid \mathcal{F}_t), \varphi\rangle\rangle \in L^1(\mathbb{R})$, $\forall \varphi \in (\mathcal{S})_\beta$ since the measurability condition is obvious by the previous argument.

Let $\varphi = \sum_{n=1}^{\infty} \langle :\cdot^{\otimes n}:, g_n\rangle \in (\mathcal{S})_\beta$. Then

$$
\begin{aligned}
\int_{\mathbb{R}} \left|\langle\langle \partial_t^* E(\partial_t\Phi \mid \mathcal{F}_t), \varphi\rangle\rangle\right| dt &= \int_{\mathbb{R}} \left|\langle\langle E(\partial_t\Phi \mid \mathcal{F}_t), \partial_t\varphi\rangle\rangle\right| dt \\
&\leq \int_{\mathbb{R}} \left\|E(\partial_t\Phi \mid \mathcal{F}_t)\right\|_{-p,-\beta} \|\partial_t\varphi\|_{p,\beta}\, dt \\
&\leq \int_{\mathbb{R}} \|\partial_t\Phi\|_{-p,-\beta} \|\partial_t\varphi\|_{p,\beta}\, dt \\
&\leq \sqrt{\int_{\mathbb{R}} \|\partial_t\Phi\|_{-p,-\beta}^2\, dt \int_{\mathbb{R}} \|\partial_t\varphi\|_{p,\beta}^2\, dt}.
\end{aligned}
$$

By assumption, $\int_{\mathbb{R}} \|\partial_t\Phi\|_{-p,-\beta}^2\, dt < \infty$. Let us choose any $q > p \geq 0$ such that $4\lambda_1^{-2(q-p)} < 1$. We will the have the following for the computation.

$$
\begin{aligned}
\int_{\mathbb{R}} \|\partial_t \varphi\|_{p,\beta}^2 \, dt &= \int_{\mathbb{R}} \sum_{n=1}^{\infty} n^2((n-1)!)^{1+\beta} |g_n(t,\cdot)|_p^2 \, dt \\
&\leq \int_{\mathbb{R}} \sum_{n=1}^{\infty} n^2(n!)^{1+\beta} |g_n(t,\cdot)|_p^2 \, dt \\
&= \sum_{n=1}^{\infty} n^2(n!)^{1+\beta} |g_n|_p^2 \\
&\leq \sum_{n=1}^{\infty} 2^{2n}(n!)^{1+\beta} \lambda_1^{-2(q-p)n} |g_n|_q^2 \\
&\leq \sum_{n=1}^{\infty} (n!)^{1+\beta} |g_n|_q^2 = \|\varphi\|_q^2 < \infty.
\end{aligned}
$$

Hence, $\int_{\mathbb{R}} \partial_t^* E(\partial_t \Phi \mid \mathcal{F}_t) \, dt$ is defined as a white noise integral in the Pettis sense. The next Theorem then establishes the second version of the Clark-Ocone formula is given in equation (4.3) above.

Theorem 4.4. *Let $\Phi \in (\mathcal{S})_\beta^*$ with $\Phi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : , f_n \rangle$, $f_n \in L^2(\mathbb{R}^{\hat{\otimes} n})$. Assume that $\partial_t \Phi$ exists in $(\mathcal{S})_\beta^*$ and $\int_{\mathbb{R}} \|\partial_t \Phi\|_{-p,-\beta}^2 \, dt < \infty$ for some $p \geq 0$. Then, Φ can be represented as*

$$
\Phi = E(\Phi) + \int_{\mathbb{R}} \partial_t^* E(\partial_t \Phi \mid \mathcal{F}_t) \, dt. \tag{4.4}
$$

Proof. First let $\Phi_N = \sum_{n=0}^{N} \langle : \cdot^{\otimes n} : , f_n \rangle$. Then calculating its S-transform yields the following

$$
\begin{aligned}
S(\Phi_N)(\xi) &= \langle\langle \Phi_N, : e^{\langle \cdot, \xi \rangle} : \rangle\rangle \\
&= \sum_{n=0}^{N} \langle f_n, \xi^{\otimes n} \rangle \\
&= f_0 + \sum_{n=1}^{N} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(s_1, \cdots, s_n) \xi(s_1) \cdots \xi(s_n) ds_1 \cdots ds_n \\
&= f_0 + \sum_{n=1}^{N} \int_{\mathbb{R}} n \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t, s_1, \cdots, s_{n-1}) \xi(t) \xi(s_1) \cdots \\
&\quad \cdots \xi(s_{n-1}) ds_1 \cdots ds_{n-1} dt
\end{aligned}
$$

$$
\begin{aligned}
&= f_0 + \int_{\mathbb{R}} \xi(t) \Bigg(\sum_{n=1}^{N} n \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t, s_1, \cdots, s_{n-1}) \xi(s_1) \cdots \\
&\cdots \xi(s_{n-1}) ds_1 \cdots ds_{n-1} \Bigg) dt \\
&= f_0 + \int_{\mathbb{R}} \xi(t) S(E(\partial_t \Phi_N \mid \mathcal{F}_t))(\xi)\, dt \\
&= S(E(\Phi_N))(\xi) + S\Bigg(\int_{\mathbb{R}} \partial_t^*(E\partial_t \Phi_N \mid \mathcal{F}_t)\, dt \Bigg)(\xi) \\
&= S\Bigg(E(\Phi_N) + \int_{\mathbb{R}} \partial_t^*(E\partial_t \Phi_N \mid \mathcal{F}_t)\, dt \Bigg)(\xi).
\end{aligned}
$$

This then verifies the decomposition $\Phi_N = E(\Phi_N) + \int_{\mathbb{R}} \partial_t^* E(\partial_t \Phi_N \mid \mathcal{F}_t)\, dt$. Now for $\Phi = \sum_{n=0}^{\infty} \langle : \cdot^{\otimes n} : , f_n \rangle$, with $f_n \in L^2(\mathbb{R}^{\hat{\otimes} n})$, we have that $\Phi_N \longrightarrow \Phi$ in $(\mathcal{S})_\beta^*$ and $E(\Phi_N) \longrightarrow E(\Phi)$ as $N \longrightarrow \infty$. We only need to check that

$$
\int_{\mathbb{R}} \partial_t^*(E\partial_t \Phi_N \mid \mathcal{F}_t)\, dt \longrightarrow \int_{\mathbb{R}} \partial_t^*(E\partial_t \Phi \mid \mathcal{F}_t)\, dt.
$$

We will use Theorem 2.3 to show this convergence. We verify the two conditions which are as follows

(a)Let $F_N = S\Phi_N$. We claim that $\lim_{N \longrightarrow \infty} F_N(\xi)$ exists.

$$
\begin{aligned}
F_N(\xi) &= S(E(\Phi_N))(\xi) + S\left(\int_{\mathbb{R}} \partial_t^* E(\partial_t \Phi_N \mid \mathcal{F}_t)\, dt \right)(\xi) \\
&= f_0 + \int_{\mathbb{R}} \xi(t) S(E(\partial_t \Phi_N \mid \mathcal{F}_t))(\xi)\, dt.
\end{aligned}
$$

Now, we note that as $N \longrightarrow \infty$, we obtain the pointwise limit of the integrand in the above integral because

$$
\begin{aligned}
&S(E(\partial_t \Phi_N \mid \mathcal{F}_t))(\xi) \\
&\quad = \sum_{n=1}^{N} n \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t, s_1, \cdots, s_{n-1}) \xi(s_1) \cdots \xi(s_{n-1})\, ds_1 \cdots ds_{n-1} \\
&\quad \longrightarrow \sum_{n=1}^{\infty} n \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} f_n(t, s_1, \cdots, s_{n-1}) \xi(s_1) \cdots \xi(s_{n-1})\, ds_1 \cdots ds_{n-1}.
\end{aligned}
$$

Moreover, an attepmt to obtain a bound for the quantity $\left|S(E(\partial_t\Phi_N \mid \mathcal{F}_t))(\xi)\right|$ yields the following estimate

$$\begin{aligned}
&\left|SE(\partial_t\Phi_N \mid \mathcal{F}_t)(\xi)\right| \\
&\quad \leq \left|\sum_{n=1}^{N} n\int_{-\infty}^{t}\cdots\int_{-\infty}^{t} f_n(t,s_1,\cdots,s_{n-1})\xi(s_1)\cdots\xi(s_{n-1})\,ds_1\cdots ds_{n-1}\right| \\
&\quad \leq \left|\langle\langle\partial_t\Phi, :e^{\langle\cdot,\xi\rangle}:\rangle\rangle\right| \\
&\quad \leq \left\|\partial_t\Phi\right\|_{-p,-\beta}\left\|:e^{\langle\cdot,\xi\rangle}:\right\|_{p,\beta}.
\end{aligned}$$

This bound obtained above is integrable because

$$\int_{\mathbb{R}}\left\|\partial_t\Phi\right\|_{-p,-\beta}\left\|:e^{\langle\cdot,\xi\rangle}:\right\|_{p,\beta}dt = \left\|:e^{\langle\cdot,\xi\rangle}:\right\|_{p,\beta}\int_{\mathbb{R}}\left\|\partial_t\Phi\right\|_{-p,-\beta}dt < \infty,$$

since $\int_{\mathbb{R}}\left\|\partial_t\Phi\right\|^2_{-p,-\beta}dt < \infty$ by assumption in the Theorem.

Hence, by the Lebesgue Dominated Convergence Theorem, $\lim_{N\longrightarrow\infty} S(\Phi_N)(\xi) = f_0 + \int_{\mathbb{R}}\xi(t)\lim_{N\longrightarrow\infty} SE(\partial_t\Phi_N \mid \mathcal{F}_t)(\xi)\,dt$ and so $\lim_{N\longrightarrow\infty} F_N(\xi)$ exists.

(b) We claim that $F_N(\xi)$ satisfies the growth condition.

$$\begin{aligned}
\left|F_N(\xi)\right| &= \left|S(\Phi_n)(\xi)\right| \\
&\leq |f_0| + \int_{\mathbb{R}}\left|\xi(t)\right|\left|SE(\partial_t\Phi_N \mid \mathcal{F}_t)(\xi)\right|dt \\
&\leq |f_0| + \int_{\mathbb{R}}\left|\xi(t)\right|\left\|\partial_t\Phi\right\|_{-p,-\beta}\exp[a|\xi|_p^{\frac{2}{1-\beta}}]dt \\
&\leq |f_0| + \exp[a|\xi|_p^{\frac{2}{1-\beta}}]\int_{\mathbb{R}}|\xi(t)|^2\,dt\int_{\mathbb{R}}\left\|\partial_t\Phi\right\|^2_{-p,-\beta}dt.
\end{aligned}$$

But $\int_{\mathbb{R}}\|\partial_t\Phi\|^2_{-p.-\beta}dt < \infty$ and $\int_{\mathbb{R}}|\xi(t)|^2dt < \infty$ since $\xi \in \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. If we let $K = \int_{\mathbb{R}}|\xi(t)|^2\,dt\int_{\mathbb{R}}\left\|\partial_t\Phi\right\|^2_{-p,-\beta}dt$, we get the desired bound for the growth condition which is independent on N for every $\xi \in \mathcal{S}_c(\mathbb{R})$. We now conclude that $\int_{\mathbb{R}}\partial_t^*E(\partial_t\Phi_N \mid \mathcal{F}_t)\,dt \longrightarrow \int_{\mathbb{R}}\partial_t^*E(\partial_t\Phi \mid \mathcal{F}_t)\,dt$ since $SE(\partial_t\Phi_N \mid \mathcal{F}_t)(\xi) \longrightarrow SE(\partial_t\Phi \mid \mathcal{F}_t)(\xi)$ from the above argument. This then completes the verification for the extended formula (4.4). ☐

4.3 The Formula for the Hermite Brownian Functional

In this section we verify the Clark-Ocone formula for the Hermite Brownian functional using the Itô formula.

Lemma 4.5. Let $B(t)$, $t > 0$ be a Brownian motion. Then the Hermite Brownian functional $: B(t)^n :_t$ is a martingale.

Proof. Let $\lambda \in \mathbb{R}$ and set $Y_\lambda(t) = e^{\lambda B(t) - \frac{1}{2}\lambda t^2}$. We first show that the process $\{Y_\lambda(t),\ t \geq 0\}$ is a martingale (See [8]). First we show integrability. We observe that $\lambda(t)$ is $\mathcal{F}_t$-measurable. Moreover,

$$
\begin{aligned}
E[Y_\lambda(t)] &= e^{-\frac{1}{2}\lambda^2 t} \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{\lambda y - \frac{y^2}{2t}} dy \\
&= e^{-\frac{1}{2}\lambda^2 t} \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}(y^2 - 2t\lambda y)} dy \\
&= e^{-\frac{1}{2}\lambda^2 t} \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{1}{2t}(y - t\lambda)^2} e^{\frac{t^2\lambda^2}{2t}} dy \\
&= e^{-\frac{1}{2}\lambda^2 t} e^{\frac{1}{2}\lambda^2 t} = 1
\end{aligned}
$$

which shows integrability.

Now let $t > s$. We can write

$$
\begin{aligned}
Y_\lambda(t) &= e^{\lambda B(s) - \frac{1}{2}\lambda^2 s} e^{\lambda(B(t) - B(s)) - \frac{1}{2}\lambda^2 (t-s)} \\
&= Y_\lambda(s) e^{\lambda(B(t) - B(s)) - \frac{1}{2}\lambda^2 (t-s)}
\end{aligned}
$$

Since $B(t) - B(s)$ is independent of any $\mathcal{F}_s$-measurable function, we have

$$
\begin{aligned}
E[Y_\lambda(t) \mid \mathcal{F}_s] &= Y_\lambda(s) E[e^{\lambda B(t-s) - \frac{1}{2}\lambda^2 (t-s)}] \\
&= Y_\lambda(s) \text{ almost surely by the integrability result above.}
\end{aligned}
$$

We have the generating function for $: B(t)^n :_t$ given by

$$
\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} : B(t)^n :_t = e^{\lambda B(t) - \frac{1}{2}\lambda^2 t} \tag{4.5}
$$

Taking the conditional expectation on both sides of (4.5) we obtain

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} E(: B(t)^n :_t \mid \mathcal{F}_t) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} : B(s)^n :_s.$$

By comparing coefficients of λ^n, we get

$$E(: B(t)^n :_t \mid \mathcal{F}_t) =: B(s)^n :_s,$$

which completes the proof for the lemma. □

Next we verify the formula for the Hermite Brownian functional $: B(t)^n :_t$.

Theorem 4.6. Let $t \in [0,1]$ and $B(t)$ a Brownian motion. Then, $E(\partial_s : B(t)^n :_t \mid \mathcal{F}_s) = n : B(s)^{n-1} :_s 1_{[0,t]}(s)$ and the Clark-Ocone representation formula for $: B(t)^n :_t$ is given by:

$$: B(t)^n :_t = \int_0^t n : B(s)^{n-1} :_s dB(s). \tag{4.6}$$

Proof. Consider the function $\theta(t,x) =: x^n :_t$. Then by using the generating function

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} : x^n :_t = e^{\lambda x - \frac{1}{2}\lambda^2 t},$$

the following formulas can easily be verified:

$$\frac{\partial}{\partial t} \theta(t,x) = -\frac{1}{2} n(n-1) : x^{n-2} :_t .$$

$$\frac{\partial}{\partial x} \theta(t,x) = n : x^{n-1} :_t .$$

$$\frac{\partial^2}{\partial x^2} \theta(t,x) = \frac{1}{2} n(n-1) : x^{n-2} :_t .$$

By the Itô formula,

$$d\theta(t, B(t)) = \frac{\partial}{\partial x}\theta(t, B(t)) dB(t) + \left(\frac{1}{2} \frac{\partial^2}{\partial x^2} \theta(t, B(t)) + \frac{\partial}{\partial t} \theta(t, B(t)) \right) dt.$$

Putting it in integral form,

$$
\begin{aligned}
: B(t)^n :_t &= \int_0^t \frac{\partial}{\partial x}\theta(s, B(s))dB(s) + \int_0^t \left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\theta(s, B(s)) + \frac{\partial}{\partial s}\theta(s, B(s))\right) ds \\
&= \int_0^t n : B(s)^{n-1} :_s dB(s) + \int_0^t \frac{1}{2}n(n-1) : B(s)^{n-2} :_s ds \\
&- \int_0^t \frac{1}{2}n(n-1) : B(s)^{n-2} :_s ds \\
&= \int_0^t n : B(s)^{n-1} :_s dB(s).
\end{aligned}
$$

Since $E[: B(t)^n :_t] = 0$, it remains to be shown that

$$
\int_0^1 E(\partial_s : B(t)^n :_t | \mathcal{F}_s)dB(s) = \int_0^t n : B(s)^{n-1} :_s dB(s). \tag{4.7}
$$

We can first write $: B(t)^n :_t$ in terms of Wick tensors as

$$
: B(t)^n :_t = \langle : \cdot^{\otimes n} :, 1_{[0,t]}^{\otimes n} \rangle.
$$

We then obtain

$$
\begin{aligned}
\partial_s : B(t)^n :_t &= n \langle : \cdot^{\otimes (n-1)} :, 1_{[0,t]}^{\otimes (n-1)} 1_{[0,t]}(s) \\
&= n : B(t)^{n-1} :_t 1_{[0,t]}(s).
\end{aligned}
$$

Since $: B(t)^n :_t$ is a martingale as earlier noted, it follows that

$$
E(\partial_s : B(t)^n :_t | \mathcal{F}_s) = n : B(s)^{n-1} :_s 1_{[0,t]}(s).
$$

and so

$$
\begin{aligned}
\int_0^1 E(\partial_s : B(t)^n :_t | \mathcal{F}_s)dB(s) &= \int_0^1 n : B(s)^{n-1} :_s 1_{[0,t]}(s)dB(s) \\
&= \int_0^t n : B(s)^{n-1} :_s dB(s).
\end{aligned}
$$

which then verifies the Clark-Ocone formula for $: B(t)^n :_t$. $\square$

4.4 The Formula for Donsker's Delta Function

As a specific example of the results obtained in section 4.2 above, we consider Donsker's delta function. For δ_a the Dirac delta function at a and $B(t)$ a Brownian motion, the Donsker's delta function $\delta(B(t)-a)$ is a generalized Brownian functional with the following Wiener-Itô decomposition ([18]) given by:

$$\delta(B(t)-a) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} \sum_{n=0}^{\infty} \frac{1}{n! t^n} : a^n :_t \langle : \cdot^{\otimes n} :, 1_{[0,t]}^{\otimes n} \rangle. \tag{4.8}$$

Theorem 4.7. Let $\delta(B(t)-a)$ be Donsker's delta function. Then the Clark-Ocone representation formula for $\delta(B(t)-a)$ is given by:

$$\delta(B(t)-a) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} + \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \frac{(a-B(s))}{(t-s)^{\frac{3}{2}}} e^{-\frac{(a-B(s))^2}{2(t-s)}} dB(s). \tag{4.9}$$

Proof. Since $\langle : \cdot^{\otimes n} : , 1_{[0,t]}^{\otimes n} \rangle =: B(t)^n :_t$, equation (4.8) can also be written as

$$\delta(B(t)-a) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} \sum_{n=0}^{\infty} \frac{1}{n! t^n} : a^n :_t : B(t)^n :_t. \tag{4.10}$$

We now know from Theorem 4.6 that $: B(t)^n :_t = \int_{-\infty}^{t} n : B(s)^{n-1} :_s dB(s)$. Hence,

$$\begin{aligned}
\delta(B(t)-a) &= \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} + \frac{1}{\sqrt{2\pi t}} \sum_{n=1}^{\infty} \frac{1}{n! t^n} : a^n :_t : B(t)^n :_t \\
&= \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} + \frac{1}{\sqrt{2\pi t}} \sum_{n=1}^{\infty} \frac{1}{n! t^n} : a^n :_t \int_{-\infty}^{t} n : B(s)^{n-1} :_s dB(s) \\
&= \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} + \int_{-\infty}^{t} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} \frac{1}{(n-1)! t^n} : a^n :_t : B(s)^{n-1} :_s dB(s).
\end{aligned}$$

The following 2 formulas can be used to simplify the integrand in the last equality:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} : x^n :_1 : y^n :_1 = \frac{1}{\sqrt{1-t^2}} \exp\left[-\frac{t^2x^2 - 2txy + t^2y^2}{2(1-t^2)}\right], \qquad | t | < 1. \tag{4.11}$$

$$: (\sigma u)^n :_{\sigma^2} = \sigma^n : u^n :_1 . \tag{4.12}$$

From the second formula above, we get

$$: u^n :_1 = \frac{1}{\sigma^n} : (\sigma u)^n :_{\sigma^2} .$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{\sigma^n} : (\sigma x)^n :_{\sigma^2} \frac{1}{\lambda^n} : (\lambda y)^n :_{\lambda^2} = \frac{1}{\sqrt{1-t^2}} \exp\left[-\frac{t^2x^2 - 2txy + t^2y^2}{2(1-t^2)} \right].$$

Put $\sigma x = u$, $\lambda y = v$. Then

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{\sigma^n} : u^n :_{\sigma^2} \frac{1}{\lambda^n} : v^n :_{\lambda^2} = \frac{1}{\sqrt{1-t^2}} \exp\left[-\frac{\frac{t^2}{\sigma^2}u^2 - 2t\frac{u}{\sigma}\frac{v}{\lambda} + \frac{t^2}{\lambda^2}v^2}{2(1-t^2)} \right].$$

Let $t = \tau$. Then we obtain

$$\sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{1}{\sigma^n} : u^n :_{\sigma^2} \frac{1}{\lambda^n} : v^n :_{\lambda^2} = \frac{1}{\sqrt{1-\tau^2}} \exp\left[-\frac{\frac{\tau^2}{\sigma^2}u^2 - 2\tau\frac{1}{\sigma\lambda}uv + \frac{\tau^2}{\lambda^2}v^2}{2(1-\tau^2)} \right].$$

Put $\sigma^2 = t$, $\lambda^2 = s$. to obtain

$$\sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{1}{(\sqrt{t})^n} \frac{1}{(\sqrt{s})^n} : u^n :_t : v^n :_s = \frac{1}{\sqrt{1-\tau^2}} \exp\left[-\frac{\frac{\tau^2}{t}u^2 - 2\tau\frac{1}{\sqrt{ts}}uv + \frac{\tau^2}{s}v^2}{2(1-\tau^2)} \right]. \tag{4.13}$$

Differentiating both sides of equation (4.13) in v we obtain

$$\sum_{n=1}^{\infty} \frac{\tau^n}{(n-1)!} \frac{1}{(\sqrt{ts})^n} : u^n :_t : v^n :_s = \frac{1}{\sqrt{1-\tau^2}} \frac{\frac{2u\tau}{\sqrt{ts}} - \frac{2v\tau^2}{s}}{2(1-\tau^2)} \exp\left[-\frac{\frac{\tau^2u^2}{t} - \frac{2\tau uv}{\sqrt{ts}} + \frac{\tau^2v^2}{s}}{2(1-\tau^2} \right]. \tag{4.14}$$

Substitute a for u, $B(s)$ for v, and $\frac{\sqrt{s}}{\sqrt{t}}$ for τ in equation(4.14) to get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(n-1)!t^n} : a^n :_t : B(s)^n :_s \\
&= \frac{1}{\sqrt{1-\frac{s}{t}}} \frac{\frac{\sqrt{s}}{\sqrt{t}}\frac{1}{\sqrt{ts}}a - \frac{s}{t}\frac{1}{s}2B(s)}{2(1-\frac{s}{t})} \exp\left[-\frac{\frac{s}{t}\frac{1}{t}a^2 - 2\frac{\sqrt{s}}{\sqrt{t}}\frac{1}{\sqrt{ts}}aB(s) + \frac{s}{t}\frac{1}{s}B(s)^2}{2(1-\frac{s}{t})} \right] \\
&= \frac{\sqrt{t}a - \sqrt{t}B(s)}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{\frac{s}{t}a^2 - 2aB(s) + B(s)^2}{2(t-s)} \right].
\end{aligned}$$

We then get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{(n-1)!t^n} e^{-\frac{a^2}{2t}} : a^n :_t: B(s)^n :_s \\
&= \frac{\sqrt{t}(a-B(s))}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{a^2}{2t} - \frac{\frac{s}{t}a^2 - 2aB(s) + B(s)^2}{2(t-s)}\right].
\end{aligned}$$

The power for the exponential expression in the above equation reduces to

$$\begin{aligned}
-\frac{a^2}{2t} - \frac{\frac{s}{t}a^2 - 2aB(s) + B(s)^2}{2(t-s)} &= \frac{-a^2}{2t} - \frac{sa^2}{2t(t-s)} - \frac{-2aB(s) + B(s)^2}{2(t-s)} \\
&= \frac{-(t-s)a^2 + sa^2}{2t(t-s)} - \frac{-2aB(s) + B(s)^2}{2(t-s)} \\
&= -\frac{a^2}{2(t-s)} - \frac{-2aB(s) + B(s)^2}{2(t-s)} \\
&= -\frac{(a-B(s))^2}{2(t-s)}.
\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)!t^n} e^{-\frac{a^2}{2t}} : a^n :_t: B(s)^n :_s = \frac{\sqrt{t}(a-B(s))}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{(a-B(s))^2}{2(t-s)}\right].$$

Hence

$$\begin{aligned}
&\delta(B(t)-a) \\
&= \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} + \int_{-\infty}^{t} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} \frac{1}{(n-1)!t^n} : a^n :_t: B(s)^{n-1} :_s dB(s) \\
&= \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} + \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi t}} \frac{\sqrt{t}(a-B(s))}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{(a-B(s))^2}{2(t-s)}\right] dB(s) \\
&= \frac{1}{\sqrt{2\pi t}} e^{-\frac{a^2}{2t}} + \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \frac{\sqrt{t}(a-B(s))}{(t-s)^{\frac{3}{2}}} \exp\left[-\frac{(a-B(s))^2}{2(t-s)}\right] dB(s).
\end{aligned}$$

which gives us the desired result, and completes the proof. □

4.5 Generalization to Compositions with Tempered Distributions

In this section we generalize the Clark-Ocone formula to Brownian functionals of the form $f(B(t))$ where $f \in \mathcal{S}'(\mathbb{R})$. We first state a Theorem wich verifies that indeed $f(B(t))$ is a generalized function. For the proof see [21], [22].

Theorem 4.8. Let $f \in \mathcal{S}'(\mathbb{R})$. Then $f(B(t)) \equiv \langle f, \delta(B(t) - (\cdot)) \rangle$ is a generalized Brownian functional with the following series representation:

$$f(B(t)) = \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^{\infty} \frac{1}{n! t^n} \langle f, \xi_{n,t} \rangle :B(t)^n:_t. \tag{4.15}$$

where $\xi_{n,t}(x) =: x^n :_t e^{-\frac{x^2}{2t}}$ is a function in $\mathcal{S}(\mathbb{R})$ and $\langle \cdot, \cdot \rangle$ is the pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$.

Theorem 4.9. Let $f \in \mathcal{S}'(\mathbb{R})$. Then, the Brownian functional $f(B(t))$ has the following Clark-Ocone representation formula:

$$f(B(t)) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x) e^{-\frac{x^2}{2t}} dx + \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi(t-s)}} \left(\int_{\mathbb{R}} f'(x) e^{-\frac{(x-B(s))^2}{2(t-s)}} dx \right) dB(s). \tag{4.16}$$

Proof. For $f \in \mathcal{S}'(\mathbb{R})$ and $\xi_{n,t}(x) =: x^n :_t e^{-\frac{x^2}{2t}} \in \mathcal{S}(\mathbb{R})$ the pairing $\langle f, \xi_{n,t} \rangle$ is the integral $\int_{\mathbb{R}} f(x) :x^n:_t e^{-\frac{x^2}{2t}} dx$. Therefore,

$$\begin{aligned} f(B(t)) &= \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^{\infty} \frac{1}{n! t^n} \langle f, \xi_{n,t} \rangle :B(t)^n:_t \\ &= \frac{1}{\sqrt{2\pi t}} \sum_{n=0}^{\infty} \frac{1}{n! t^n} \left(\int_{\mathbb{R}} f(x) :x^n:_t e^{-\frac{x^2}{2t}} dx \right) :B(t)^n:_t \\ &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x) e^{-\frac{x^2}{2t}} dx + \sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi t}} \frac{1}{n! t^n} \left(\int_{\mathbb{R}} f(x) :x^n:_t e^{-\frac{x^2}{2t}} dx \right) :B(t)^n:_t. \end{aligned}$$

Let K denote the second summand in the last equality above. Then we obtain

$$\begin{aligned}
K &= \sum_{n=1}^{\infty} \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi t}} \frac{1}{n! t^n} \left(\int_{\mathbb{R}} f(x) : x^n :_t n : B(s)^{n-1} :_s e^{-\frac{x^2}{2t}} dx \right) dB(s) \\
&= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi t}} \left(\int_{\mathbb{R}} f(x) \sum_{n=1}^{\infty} \frac{1}{(n-1)! t^n} : x^n :_t : B(s)^{n-1} :_s e^{-\frac{x^2}{2t}} dx \right) dB(s).
\end{aligned}$$

But

$$\sum_{n=1}^{\infty} \frac{1}{(n-1)! t^n} : x^n :_t : B(s)^{n-1} :_s e^{-\frac{x^2}{2t}} = \frac{\sqrt{t}(x - B(s))}{(t-s)^{\frac{3}{2}}} e^{-\frac{(x-B(s))^2}{2(t-s)}}.$$

Therefore,

$$K = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} f(x) \frac{(x - B(s))}{(t-s)^{\frac{3}{2}}} e^{-\frac{(x-B(s))^2}{2(t-s)}} dx \right) dB(s).$$

Applying integration by parts to K yields the following

$$\begin{aligned}
K &= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{R}} f(x) \frac{1}{\sqrt{(t-s)}} \frac{(x - B(s))}{(t-s)} e^{-\frac{(x-B(s))^2}{2(t-s)}} dx \right) dB(s) \\
&= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \frac{-1}{\sqrt{(t-s)}} \left(\int_{\mathbb{R}} f(x) \frac{-(x - B(s))}{(t-s)} e^{-\frac{(x-B(s))^2}{2(t-s)}} dx \right) dB(s) \\
&= \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{(t-s)}} \left(\int_{\mathbb{R}} f'(x) e^{-\frac{(x-B(s))^2}{2(t-s)}} dx \right) dB(s).
\end{aligned}$$

Hence

$$f(B(t)) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x) e^{-\frac{x^2}{2t}} dx + \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi(t-s)}} \left(\int_{\mathbb{R}} f'(x) e^{-\frac{(x-B(s))^2}{2(t-s)}} dx \right) dB(s).$$

which is the Clark- Ocone formula for $f(B(t))$ for which

$$E[f(B(t))] = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x) e^{-\frac{x^2}{2t}} dx.$$

and

$$E(\partial_s f(B(t)) \mid \mathcal{F}_s) = \frac{1}{\sqrt{2\pi(t-s)}} \int_{\mathbb{R}} f'(x) e^{-\frac{(x-B(s))^2}{2(t-s)}} dx.$$

This finishes the proof. □

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