CLASSICALLY UNSTABLE APPROXIMATIONS FOR LINEAR EVOLUTION EQUATIONS AND APPLICATIONS

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Abstract

Temporal discretization methods for evolutionary differential equations that factorize the resolvent into a product of easily computable operators have great numerical appeal. For instance, the alternating direction implicit (ADI) method of Peaceman-Rachford for 2-D parabolic problems greatly reduces the simulation time when compared with the Crank-Nicolson scheme. However, like many other factorized approximation methods that exhibit numerical stability, the ADI method is known to satisfy only the Von Neumann stability condition, a necessary condition that is usually surmised as sufficient in practical cases as pointed out by Lax and Richtmyer. Intensive efforts have been directed to understand the Von Neumann condition, e.g. by John, Lax and Richtmyer, Lax, Lax and Wendroof, and Strang. Their way of investigation is to find conditions under which the Von Neumann condition becomes sufficient for stability. In this dissertation, we found a factorized temporal approximation method and a well-posed problem for which the method is unstable but satisfies the Von Neumann stability condition. However, the method still exhibits excellent numerical stability even for large time step sizes.

Thus, to better understand the Von Neumann condition, we investigate the relation between stability and convergence in directions not covered by the Lax equivalence theorem which equates the stability with convergence for all initial values under some uniform consistency condition. To do that, we extend the Trotter-Kato theorem and the Chernoff product formula to possibly unstable "spatial" and "temporal" approximations and indicate how our results can be used for some unstable factorized approximation methods.

1 Introduction

In this dissertation we consider single step temporal approximation methods for the evolutionary system

$$\begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = f \in D(A), \end{cases} \tag{1.1}$$

where A is a spatial (i.e. independent of time), closed linear operator in a Banach space E, and $D(A)$ denotes the domain of A. Single step temporal approximation methods of the form

$$u^{n+1} = V(\Delta t)u^n \tag{1.2}$$

approximate the solution at time $(n+1)\Delta t$ by applying an operator $V(\Delta t)$ to the approximate solution u^n at time $n\,\Delta t$, where the **solution** of equation (1.1) refers to a continuously differentiable function $t \mapsto u(t) \in D(A)$ which satisfies (1.1). For all approximation procedures, the final goal is "fast" convergence of the approximate solutions to the true solution.

The convergence of the approximate solutions computed by the method (1.2) depends not only on the operators $V(\Delta t)$, but clearly on the operator A and the regularity of the initial values of the solutions as well. In this dissertation, we restrict ourselves to linear problems which are well-posed (see [9, 17, 28]) in the sense that

- the spatial operator A is densely defined in the Banach space E,
- for each initial value $f \in D(A)$, there exists a solution u of (1.1), and
- for each solution u of (1.1), for each $t > 0$, $u(t)$ continuously depends on $u(0)$.

The three conditions above implies (see also [9, 17, 28]) that there exists a strongly continuous family of bounded linear operators $S := \{S(t) : t \in [0, \infty)\}$ which, for each $t \geq 0$, maps the initial value f to $u(t)$, the value of the solution at time t. We call this operator family S the C_0-semigroup generated by A. The emphases of this dissertation are

- to investigate approximation methods which are stable in senses weaker than the classical one (see below), but still strong enough to imply convergence for a large set of initial values, and
- to investigate "benefits" of weakened stability requirements with regard to the final goal of approximations — "fast" convergence.

Since these "benefits" are the major motivation of this dissertation research, we describe some of them in this introductory chapter.

Before we proceed, we would like to list some additional notations used in this dissertation. The range of an operator A is denoted by $R(A)$, and $D(A^\infty) := \bigcap_{n=1}^{\infty} D(A^n)$. We denote by $\mathbf{L}(E)$ the set of bounded linear operators on a Banach space E. Here we call an operator L **bounded** if it is bounded on a dense subset of E, and we identify it with its unique bounded extension to E. We denote the everywhere defined (and hence bounded) resolvent of a closed linear operator A by $R(\lambda,A) := (\lambda - A)^{-1}$, and we denote by $\rho(A) := \{\lambda \in \mathbf{C} : (\lambda - A)^{-1} \in \mathbf{L}(E)\}$ the resolvent set of A. The letters $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$, respectively, denote the set of positive integers, the set of real numbers, and the set of complex numbers. Moreover, $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$, and $\mathbf{R}^+ := [0, \infty)$.

1.1 Notions of Stability

It is well known that the convergence of the approximate solutions computed by the method (1.2) depends very much on the stability of the method, that is, the uniform norm-boundedness of the powers of the operators $V(\Delta t)$. This dependence was first recognized in 1928 by Courant, Friedrichs and Lewy in their important paper [6], and later characterized (first by Lax in a seminar talk at New York University in January 1954) in the following theorem reformulated from a result in an 1956 paper of Lax and Richtmyer [17].

Theorem (Lax Equivalence Theorem) *Let S be the C_0-semigroup generated by A. Assume that a strongly continuous one-parameter family of bounded linear operators $V := \{V(t) : t \in [0, \delta]\}$ with $V(0) = I$ satisfies the* **uniform consistency condition**

$$\lim_{\Delta t \to 0} \tfrac{V(\Delta t) - I}{\Delta t} S(t)f = AS(t)f \tag{1.3}$$

uniformly for t in compact intervals for each $f \in D(A)$. Then the following statements are equivalent.

(i) $\lim_{n\to\infty} V(t/n)^n f = S(t)f$ *uniformly for $t \in [0, T]$ for each $T > 0$ and $f \in E$.*

(ii) *For any $T > 0$ and compact subset $E_C \subset E$, $\lim_{n\to\infty} V(t/n)^n f = S(t)f$ uniformly for $(t, f) \in [0, T] \times E_C$.*

(iii) *The family of bounded operators V is* **stable** *in the sense that there exists an increasing positive function $M(\cdot)$ such that for any $T > 0$*

$$\|V(t)^n\| \le M(T) \tag{1.4}$$

for those $(n, t) \in \mathbf{N} \times [0, \delta]$ with $nt \in [0, T]$.

Proof: (i)$\Rightarrow$(iii). Let $T > 0$. Since for each $f \in E$ the map $t \mapsto V(t)f$ is continuous, we have that for each $T > 0$ $\sup\{\|V(t)f\| : t \in [0, T] \cap [0, \delta]\} < \infty$. Then by the uniform boundedness principle (see [38], pp. 201),

$$\sup\{\|V(t)\| : t \in [0, T] \cap [0, \delta]\} < \infty. \tag{1.5}$$

Now statement (i) menas that for each $f \in E$, $\lim_{n\to\infty} V(t/n)^n f = S(t)f$ uniformly for $t \in [0, T]$, then we obtain that $\sup_{t\in[0,T],\, \frac{t}{n}\le\delta} \|V(t/n)^n f - S(t)f\| < 1$ for all $n > N_f$ for some f-dependent integer $N_f > 0$, which implies that

$$\sup_{n>N_f,\, t\in[0,T],\, \frac{t}{n}\le\delta} \|V(t/n)^n f\| < 1 + \sup_{t\in[0,T]} \|S(t)f\|.$$

Now for each $f\in E$,

$$\begin{aligned}
\sup_{n\in\mathbf{N},t\in[0,T],\frac{t}{n}\le\delta}\|V(t/n)^n f\| &\le \left(\sup_{n>N_f,t\in[0,T],\frac{t}{n}\le\delta}+\sup_{n\le N_f,t\in[0,T],\frac{t}{n}\le\delta}\right)\|V(t/n)^n f\| \\
&\le 1+\sup_{t\in[0,T],\frac{t}{n}\le\delta}\|S(t)f\|+\sup_{n\le N_f,t\in[0,T],\frac{t}{n}\le\delta}\|V(t/n)^n f\| \\
&\le 1+\sup_{t\in[0,T],\frac{t}{n}\le\delta}\|S(t)f\|+\sup_{n\le N_f,t\in[0,T],\frac{t}{n}\le\delta}\|V(t)\|^n\|f\|.
\end{aligned}$$

Then it follows from (1.5) and the C_0-semigroup properties of S that

$$\sup_{(n,t)\in\mathbf{N}\times[0,T]}\|V(t/n)^n f\|<\infty.$$

By the uniform boundedness principle, $\sup_{(n,t)\in\mathbf{N}\times[0,T]}\|V(t/n)^n\|<\infty$.

Now for each $T\ge 0$, let $S_T:=\{(n,t)\in\mathbf{N}\times[0,T]:t/n\le\delta\}$, and define a positive function $M:T\mapsto M(T)=\sup_{(n,t)\in S_T}\|V(t/n)^n\|$. Since $S_T\subset S_{T'}$ for $T<T'$, the function $M(\cdot)$ is increasing. Let $(n,t)\in\mathbf{N}\times[0,\delta]$ with $nt\in[0,T]$. Then $(n,nt)\in S_T$. Thus, by the definition of the function $M(\cdot)$,

$$\|V(t)^n\|=\|V(nt/n)^n\|\le M(T),$$

which is statement (ii).

(iii)$\Rightarrow$(ii). For $f\in D(A)$, $T>0$, and $(n,t)\in\mathbf{N}\times[0,\delta]$ with $t\le T$,

$$\begin{aligned}
\|V(t/n)^n f-S(t)f\| &= \textstyle\sum_{i=1}^n\|V(t/n)^{i-1}[V(t/n)-S(t/n)]S(\frac{n-i}{n}t)f\| \\
&\le \textstyle\sum_{i=1}^n M(T)\|[V(t/n)-S(t/n)]S(\frac{n-i}{n}t)f\|,
\end{aligned}$$

where the last inequality is due to the stability condition (1.4). Then, from the above inequality, we can further obtain

$$\begin{aligned}
\|V(\tfrac{t}{n})^n f-S(t)f\| &\le \tfrac{t}{n}\textstyle\sum_{i=1}^n M(T)\|[(\frac{V(\frac{t}{n})-I}{t/n}-A)-(\frac{S(\frac{t}{n})-I}{t/n}-A)]S(\frac{n-i}{n}t)f\| \\
&\le tM(T)\sup_{1\le i\le n}\|[(\tfrac{V(\frac{t}{n})-I}{t/n}-A)-(\tfrac{S(\frac{t}{n})-I}{t/n}-A)]S(\tfrac{n-i}{n}t)f\| \\
&\le TM(T)\sup_{1\le i\le n}\|[(\tfrac{V(\frac{t}{n})-I}{t/n}-A)-(\tfrac{S(\frac{t}{n})-I}{t/n}-A)]S(\tfrac{n-i}{n}t)f\|.
\end{aligned} \tag{1.6}$$

From the uniform consistency condition (1.3), it follows that

$$\lim_{n\to\infty}\sup_{1\le i\le n}\|\left(\tfrac{V(t/n)-I}{t/n}-A\right)S(\tfrac{n-i}{n}t)f\|=0. \tag{1.7}$$

The C_0-semigroup S also satisfies the uniform consistency condition, since for each $f\in D(A)$ and $T>0$,

$$\begin{aligned}
\|\tfrac{S(\Delta t)-I}{\Delta t}S(t)f-AS(t)f\| &= \|S(t)\left(\tfrac{S(\Delta t)-I}{\Delta t}f-Af\right)\| \\
&\le \sup_{t\in[0,T]}\|S(t)\|\cdot\|\tfrac{S(\Delta t)-I}{\Delta t}f-Af\|.
\end{aligned}$$

Then we have that $\lim_{n\to\infty} \sup_{1\le i\le n} \|\left(\frac{S(t/n)-I}{t/n}-A\right) S(\frac{n-i}{n}t)f\| = 0$, which, together with (1.7), implies that $\lim_{n\to\infty} \left\|\left[\left(\frac{V(t/n)-I}{t/n}-A\right) - \left(\frac{S(t/n)-I}{t/n}-A\right)\right] S(\frac{n-i}{n}t)f\right\| = 0$. We obtain from (1.6) that for $f\in D(A)$,

$$\lim_{n\to\infty} \|V(t/n)^n f - S(t)f\| = 0 \tag{1.8}$$

uniformly for $t\in[0,T]$.

For $n\in\mathbf{N}$ and $t\in[0,\delta]$, let $L_n(t) = V(t/n)^n - S(t)$. Then, the uniform convergence in (1.8) implies that $\lim_{n\to\infty} L_n(t)f = 0$ uniformly for $t\in[0,T]$ for each $f\in D(A)$. And the stability condition (1.4) provides us the inequality $\|L_n(t)\| \le 2M(T)$. Now, for a compact subset E_C, and $\epsilon > 0$, there exist $k = k(\epsilon)$ elements $f_1, f_2, \cdots, f_k \in E_C$ such that for any $f\in E_C$, there exists $f_{i_f} \in \{f_1, f_2, \cdots, f_k\}$ with $\|f - f_{i_f}\| \le \epsilon/8M(T)$. Since $D(A)$ is dense in E, there exist $g_1, g_2, \cdots, g_k \in D(A)$ such that $\sup_{1\le i\le k} \|g_i - f_i\| \le \epsilon/8M(T)$. Since $\lim_{n\to\infty} L_n(t)f = 0$ uniformly for $t\in[0,T]$ for each $f\in D(A)$, it follows that $\lim_{n\to\infty} \sup_{1\le i\le k} \|L_n(t)g_i\| = 0$ uniformly for $t\in[0,T]$. Then there exists $N = N(\epsilon)$ such that for $n \ge N$, $\sup_{t\in[0,T]} \sup_{1\le i\le k} \|L_n(t)g_i\| < \epsilon/2$. Now, for $f\in E_C$, $t\in[0,T]$, and $n \ge N(\epsilon)$,

$$\begin{aligned}
\sup_{t\in[0,T]} \|L_n(t)f\| &\le \sup_{t\in[0,T]} \left[\|L_n(t)(f-f_{i_f})\| + \|L_n(t)(f_{i_f}-g_{i_f})\| + \|L_n(t)g_{i_f}\|\right] \\
&\le \sup_{t\in[0,T]} \left[\|L_n(t)\| \cdot \left(\|f-f_{i_f}\| + \|f_{i_f}-g_{i_f}\|\right) + \sup_{1\le i\le k} \|L_n(t)g_i\|\right] \\
&\le 2M(T)[\epsilon/8M(T) + \epsilon/8M(T)] + \epsilon/2 \\
&= \epsilon,
\end{aligned}$$

which proves statement (ii).

(ii)$\Rightarrow$(i). The implication is obvious. $\square$

Statement (iii) of the Lax equivalence theorem is the classical definition of stability for general time dependent problems. Now we give two characterizations of it.

Lemma 1.1 *Let $\{V(t) : t\in[0,\delta]\}$ be a one-parameter family of bounded linear operators with $V(0) = I$. The following statements are equivalent.*

(i) *There exists an increasing positive function $M(t)$ such that $\|V(t)^n\| \le M(nt)$ for all $(n,t)\in\mathbf{N}\times[0,\delta]$.*

(i i) *There exists an increasing positive function $M(\cdot)$ such that $\|V(t)^n\| \le M(T)$ for any $T > 0$ and $(n,t)\in\mathbf{N}\times[0,\delta]$ with $nt\in[0,T]$.*

(iii) *There exist δ-dependent positive constants G and ω such that for all $t\in[0,\delta]$,*

$$\|V(t)^n\| \le G\, e^{n\omega t}. \tag{1.9}$$

Proof: (i)$\Rightarrow$(ii): Let $T>0$. For $(n,t)\in\mathbf{N}\times[0,\delta]$ with $nt\in[0,T]$, statement (i) implies that $\|V(t)^n\|\le M(nt)$. Since $M(t)$ is an increasing function of t, it follows that $M(nt)\le M(T)$.

(ii)$\Rightarrow$(iii): The inequality (1.9) obviously holds for $t=0$. So, we proceed to prove (1.9) for the case $t>0$.

Let $T=2\delta$, and let $\omega=\frac{2}{T}\ln M(T)$. Then $e^{0.5\omega T}=M(T)$. It follows from statement (ii) that $\|V(t)^{[\frac{T}{t}]}\|\le M(T)$ since $[\frac{T}{t}]t\le T$, where $[\frac{T}{t}]$ denotes the largest integer less than $\frac{T}{t}$. But $e^{0.5\omega T}=M(T)$, so we have

$$\|V(t)^{[\frac{T}{t}]}\|\le e^{0.5\omega T}\le e^{[\frac{T}{t}]\cdot\frac{0.5\omega T}{T/t-1}}=e^{[\frac{T}{t}]\cdot\frac{0.5\omega Tt}{T-t}}.$$

Since $T=2\delta$, we obtain from the above inequality that

$$\|V(t)^{[\frac{2\delta}{t}]}\|\le e^{[\frac{2\delta}{t}]\cdot\frac{\omega\delta t}{2\delta-t}}\le e^{[\frac{2\delta}{t}]\cdot\frac{\omega\delta t}{2\delta-\delta}}=e^{[\frac{2\delta}{t}]\omega t},$$

where the second inequality holds because $t\le\delta$. For notational simplicity, let $N=N(\delta,t):=[\frac{2\delta}{t}]$. Then the above inequality becomes

$$\|V(t)^N\|\le e^{N\omega t}. \tag{1.10}$$

Now, for any $n\in\mathbf{N}$, there exist $k,m\in\mathbf{N}_0$ such that $n=kN+m$ and $m<N$. Thus,

$$\|V(t)^n\|=\|V(t)^{kN+m}\|\le\|V(t)^N\|^k\cdot\|V(t)^m\|. \tag{1.11}$$

Since $m<N=[\frac{2\delta}{t}]<\frac{2\delta}{t}$, we obtain that $mt\le 2\delta$. Then it follows from statement (ii) that $\|V(t)^m\|\le M(2\delta)$, which, together with (1.11), implies that

$$\|V(t)^n\|\le M(2\delta)\|V(t)^N\|^k.$$

Now, it follows from (1.10) that $\|V(t)^n\|\le M(2\delta)e^{kN\omega t}\le M(2\delta)e^{n\omega t}$. Setting $G=M(2\delta)$, we obtain statement (iii) from the above inequality.

(iii)$\Rightarrow$(i): Statement (i) follows by setting $M(t)=G\,e^{\omega t}$. $\square$

Now with this lemma, we obtain the following theorm concerning the equivalence of the convergence and stability under a consistency condition weaker and much easier to check than the uniform consistency condition in the Lax equivalence theorem.

Theorem 1.1 *Let S be the C_0-semigroup generated by A. Assume that a strongly continuous one-parameter family of bounded linear operators $V:=\{V(t):t\in[0,\delta]\}$ with $V(0)=I$ satisfies the* **consistency condition**

$$\lim_{t\to0}\frac{V(t)-I}{t}f=Af \tag{1.12}$$

for each $f\in D(A)$. Then the following statements are equivalent.

(i) $\lim_{n\to\infty}V(t/n)^nf=S(t)f$ *uniformly for $t\in[0,T]$ for each $T>0$ and $f\in E$.*

(ii) *The family of bounded operators V is stable in the sense of (1.4).*

Proof: (i)$\Rightarrow$(ii). In part (i)$\Rightarrow$(iii) of the proof of the Lax equivalence theorem, we did not use the uniform consistency condition at all, which means that the implication (i)$\Rightarrow$(ii) of this theorem also holds.

(ii)$\Rightarrow$(i). By Lemma 1.1, the stability of V implies its exponential stability. Then under the consistency condition (1.12), the implication in this direction follows from the Chernoff product formula [7] (also see Chapter 3). □

In view of the equivalence of the stability of the method (1.2) and the convergence of the method for all initial values in the space E under the consistency condition (1.12), there are two possible ways to proceed without risking losing convergence while weakening stability. They are

- strengthening the consistency condition (1.12), and/or
- restricting the set of initial values for which the approximation methods will converge.

In this dissertation, we choose to restrict the set of initial values instead of trying to strengthen the consistency. Let us explain why such a restriction is both practically acceptable and mathematically appropriate.

It is usually the case in actual numerical computations or in the mathematical study of Cauchy problems modeled from evolutionary systems in sciences or engineering that the initial values of the problem are in the domain of A. It is well-known that when the initial value f is not in the domain of A, the meaning of the solution of the Cauchy problem (1.1) has to be interpreted in the temporally non-differential sense of mild solutions (see [9, 28]). Thus, if one is interested only in classical solutions, then one needs convergence only for initial values in $D(A)$. This allows one to consider stability requirements which are possibly weaker than the one in the Lax equivalence theorem. For example, Butzer and Weis [4], and Butzer, Dickmeis and Nessel [5] have considered stability and convergence in this direction. They introduced a weakened notion of stability which they called stability-with-orders.

Definition 1.1 *An approximation method $\{V(t) : t \in [0, \delta]\}$ of bounded linear operators with $V(0) = I$ is said to be* **stable of order** $\beta \geq 0$ *(or β-stable for short) if there exists an increasing positive function $M(\cdot)$ such that for all $t \in (0, \delta]$ and $n \in \mathbf{N}$,*

$$\|V^n(t)\| \leq t^{-\beta} M(nt). \tag{1.13}$$

With the introduction of this "stability with orders" condition, they showed that a β-stable approximation method $V(t)$ converges for all initial values in a subset $S \subset D(A)$ if, for some $\alpha > \beta$, $V(t)$ is **uniformly consistent of order** α on the set $S \subset E$ in the sense that for each $T > 0$ and $f \in S$, there exists a constant $C_{T,f} > 0$ such that for all $t \in [0, T]$ and Δt in a neighborhood of of 0, $\|V(\Delta t)S(t)f - S(t+\Delta t)f\| \leq C_{T,f} \cdot (\Delta t)^{\alpha+1}$.

With the introduction of this notion of weaker stability, Butzer, Dickmeis, Nessel and Weis furthered the understanding of the relation between stability and convergence established in the Lax equivalence theorem. Besides the mathematical significance, weakening stability requirements also has practical computation benefits. A weaker stability requirement allows one to design more flexible approximation methods which might be advantageous in other aspects. One example is the alternating direction implicit (ADI) method

$$V_{ADI}(t) := (I-\tfrac{t}{2}A_1)^{-1}(I-\tfrac{t}{2}A_2)^{-1}(I+\tfrac{t}{2}A_2)(I+\tfrac{t}{2}A_1) \tag{1.14}$$

proposed by Peaceman and Rachford [29] for two dimensional parabolic problems where A is split into directional components $A = A_1 + A_2$ with $D(A) \subset D(A_i)$ for $i = 1, 2$. The ADI method is an approximate factorization of the second order Crank-Nicolson method

$$V_{CN}(t) := (I-\tfrac{t}{2}A)^{-1}(I+\tfrac{t}{2}A), \tag{1.15}$$

where an approximation method $V := \{V(t) : t \in [0, \delta]\}$ is called to be of **p-th order** accurate if there exists a set $D(A^p) \subset E$ dense in E such that for each $f \in D$, there exists a constant $C > 0$ such that for t in a neighborhood of of 0,

$$\|V(t)f - S(t)f\| \le C \cdot t^{p+1}. \tag{1.16}$$

The Crank-Nicolson method is well known to be of second order. This can be seen by expanded the solution $S(t)f$ and the Crank-Nicolson approximation $V_{CN}(t)f$ into Taylor series upto the third derivatives. The ADI method is an approximation to the Crank-Nicolson method, but it attains the same order of accuracy. The significance of the order of accuracy can be seen from the following proposition for methods that commute with the operator A. For more general results, see Lax's [16] and Butzer and Weis' [4].

Proposition 1.1 *Let S be the C_0-semigroup generated by A satisfying the stability condition $\|S(t)\| \le Me^{\omega t}$ for some positive constants M, ω independent of t. Let $V := \{V(t) : t \in [0, \delta]\}$ with $V(0) = I$ be a strongly continuous one-parameter family of bounded linear operators satisfying the stability condition $\|V(t)^n\| \le Me^{nt\omega}$ for all $n \in \mathbf{N}$ and $t \in [0, \delta]$. Suppose that V is of p-th order for the evolutionary system (1.1) on $D(A^{p+1})$. Then, for each $f \in D(A^{p+1})$, there exists a positive constant C such that*

$$\overline{\lim}_{n\to\infty} \frac{\|V(t/n)^n f - S(t)f\|}{(t/n)^p} \le C$$

if for each $t \ge 0$, $V(t)Af = AV(t)f$ for all $f \in D(A^{p+1})$.

Proof: Let $f \in D(A^{p+1})$. Since $V(t)$ and A commute for all $t \ge 0$, it can be easily verified that $V(t)$ and $S(t)$ also commute. Then,

$$\begin{aligned}\|V(t/n)^n f - S(t)f\| &= \| \textstyle\sum_{i=1}^n V(t/n)^{n-i}[V(t/n) - S(t/n)]S(t/n)^{i-1} f \\ &= \| \textstyle\sum_{i=1}^n V(t/n)^{n-i} S(t/n)^{i-1}[V(t/n) - S(t/n)]f\|,\end{aligned}$$

where the last equality is due to the commutativity of $V(t)$ and $S(t)$. Then the stability conditions of S and V imply that

$$
\begin{aligned}
\|V(t/n)^n f - S(t)f\| &\leq \|\sum_{i=1}^n M e^{(n-i)\omega t/n} M e^{(i-1)\omega t/n} \|[V(t/n) - S(t/n)]f\| \\
&\leq M^2 e^{\omega t} \sum_{i=1}^n \|[V(t/n) - S(t/n)]f\| \\
&\leq M^2 e^{\omega t} \sum_{i=1}^n C_f (t/n)^{p+1},
\end{aligned}
\tag{1.17}
$$

where the last inequality is due to the p-th order accuracy of V, and C_f is a positive constant dependent upon f. Since $\sum_{i=1}^n C_f(t/n)^{p+1} = tC_f(t/n)^p$, it follows from (1.17) that

$$\|V(t/n)^n f - S(t)f\| \leq t\, C_f M^2 e^{\omega t} (t/n)^p,$$

from which the conclusion of this proposition follows immediately. □

While it retains the same order as that of the Crank-Nicolson method on $D(A^3)$, the ADI method (1.14) reduces the computation of the 2-D operator $(I - \frac{t}{2}A)^{-1}$ to that of two 1-D operators $(I - \frac{t}{2}A_1)^{-1}$ and $(I - \frac{t}{2}A_2)^{-1}$, resulting in great computation reduction especially when A is not self-adjoint. And this computation reduction does not come at the expense of numerical instability, as exhibited by numerical errors as small as those of the stable (see Chapter 4) Crank-Nicolson method. However, when the two components A_1 and A_2 do not commute, it is unknown if the ADI method is stable or β-stable, even though in this case it still exhibits perfect numerical stability. It will be proven in Chapter 4 that when each component A_i with $D(A) \subset D(A_i)$ satisfies the **quasi-dissipative** condition $\langle A_i f, f \rangle \leq \omega \|f\|^2$ for some constant ω and for all $f \in D(A_i)$, the ADI method satisfies the Von Neumann stability condition

$$\lim_{n\to\infty} \sqrt[n]{\|V_{ADI}(t)^n f\|} \leq e^{\omega' t}, \quad \text{for } t \in [0, \delta] \text{ and } f \in D(A) \tag{1.18}$$

for some constant $\omega' > \omega$ and an ω-dependent constant $\delta > 0$. A more detailed definition and historical background of this here newly introduced notion of stability are given below.

Another example of approximation methods with weakened stability but significant computation benefits is a domain decomposition based factorized temporal approximation method introduced by the author in his computer science dissertation proposal [40]. This method is both computationally and communicationally efficient for large scale parallel computing on distributed memory architecture machines. It exhibits excellent numerical stability, and satisfies the Von Neumann stability under certain assumptions (see Chapter 4). However, it is unknown if this method is stable or β-stable.

The Von Neumann stability condition (1.18) is usually stated only for methods for spatially discretized problems (see [17, 16, 18, 32]) in finite dimensional spaces E_h; i.e. for

$$\begin{cases} u_h'(t) = A_h u_h(t), \ t \geq 0 \\ u_h(0) = f_h \in E_h, \end{cases} \qquad h \in (0, \epsilon) \tag{1.19}$$

where $\{E_h : h \in (0, \epsilon)\}$ is a family of finite dimensional spaces approximating E in some sense, and $\{A_h : E_h \to E_h, h \in (0, \epsilon)\}$ is a family of bounded linear operators approximating

A is some sense. A temporal approximation method $\{V(t;A_h):(t,h)\in[0,\delta]\times(0,\epsilon)\}$ for the spatially discrete problem (1.19) is said to satisfy the **Von Neumann stability condition** if the spectral radii $\mathbf{r}(V(t;A_h)) := \overline{\lim}_{n\to\infty}\|V(t;A_h)^n\|^{1/n}$ of $V(t;A_h)$ satisfy

$$\mathbf{r}(V(t;A_h)) \le e^{\omega t} \quad \text{for all } (t,h)\in[0,\delta']\times(0,\epsilon) \tag{1.20}$$

for some constants $\omega \ge 0$ and $\delta' \in (0,\delta]$.

This spectral-radius based definition of Von Neumann stability is obviously extendible to an approximation method $V(t;A)$ in a Banach space when the operators $V(t;A)^n$ are bounded for all n large enough. However, the requirement that $V(t;A)^n$ are bounded operators for sufficiently large n is too restrictive. Take the ADI method (1.14) for example. The ADI method $V_{ADI}(t;A_h)$ for the spatially discrete problem (1.19) satisfies the Von Neumann stability condition (1.20) when $A_{h,1}$ and $A_{h,2}$ retain the quasi-dissipativity of A_1 and A_2. But in general, the operators $V_{ADI}(t;A)^n$ are not necessary bounded for n large enough (see Section 1.2). Thus, in order to deal with approximation methods in Banach spaces without recourse to spatial discretization, but still able to accommodate numerically perfectly stable methods like the ADI, we are not going to impose the boundedness restriction on the operator $V(t;A)^n$ for sufficiently large n. Hence, we generalize the usual Von Neumann stability condition (1.20) to approximation methods in general Banach spaces in the following sense.

Definition 1.2 *An approximation method $\{V(t):t\in[0,\delta]\}$ of linear operators $V(t)$ defined on a set $D\subset E$ is said to satisfy the* **Von Neumann stability condition on** D *if there exist constants $\omega\ge 0$ and $\delta'\in(0,\delta]$ such that for all $t\in[0,\delta']$ and $f\in D$,*

$$\lim_{n\to\infty}\sqrt[n]{\|V(t)^n f\|} \le e^{\omega\, t}. \tag{1.21}$$

An approximation method V is said to satisfy the **Von Neumann stability condition** *if V satisfies the Von Neumann stability condition on some* **dense** *subset of E.*

It is obvious that this definition of the Von Neumann stability condition coincides with the spectral radius based definition when $\{V(t):t\in[0,\delta]\}$ is a family of bounded operators. It is well known that stable methods consisting of bounded operators satisfy the Von Neumann stability condition in the sense that $\mathbf{r}(V(t)) \le e^{\omega t}$ for some constant $\omega\ge 0$ for t in a **neighborhood** of 0 [17]. Now, with the characterizations given in Lemma 1.1, we have the following stronger result. The proof is obvious and thus omitted.

Corollary 1.1 *Suppose that a temporal approximation method $\{V(t):t\in[0,\delta]\}$ is stable in the classically sense of Lemma 1.1. Then there exists a constant $\omega\ge 0$ such that $\mathbf{r}(V(t)) \le e^{\omega t}$ for all $t\in[0,\delta]$.*

This corollary states that the Von Neumann stability condition is a necessary condition for stability. Since there has been no observed numerical “misfortune” of any Von Neumann stable approximation method in practical computations, it is surmised, as pointed out by

Lax and Richtmyer in [17], that the Von Neumann stability condition (1.20) is also sufficient for stability. Intensive research has been carried out to find conditions under which the Von Neumann condition becomes sufficient for stability, e.g. by F. John [12], Lax and Richtmyer [17], Lax [16], Lax and Wendroff [18], and Strang [32]. However, to our knowledge the question if the Von Neumann stability condition (1.20) itself is sufficient for stability has remained open, until now.

1.2 An Unstable but Von-Neumann Stable Method

In this section we present an example in which a consistent factorized temporal approximation method satisfies the Von Neumann stability condition in both the usual sense (1.20) and the newly introduced sense (1.21), but is not stable in the classical sense (1.4) or in the stability-with-orders' sense (1.13). This shows that the Von Neumann condition is not sufficient for stability. However, the method still exhibits excellent numerical stability as indicated by measured errors in Tables 1 and 2 (see Section 1.2.3), and the measured errors are even smaller than those of approximated solutions computed using the stable implicit method.

1.2.1 The Problem and Its Spatial Approximation

Let $\Omega = [0, \pi]$. Let E denote the real Hilbert space $E := \{f \in L^2(\Omega) : f(0) = f(\pi) = 0\}$ with the L^2 inner product. It is known that every function f in this space has a Fourier series expansion $f = \sum_{k\in\mathbf{N}} \langle f, e_k\rangle\, e_k$, where $\{e_k\}_{k\in\mathbf{N}}$ is an orthonormal basis consisting of the functions $e_k(x) := \sqrt{\frac{2}{\pi}}\sin(kx)$. Let A_1 denote the differential operator $A_1 f = f''(x)$ on the domain $D(A_1) := \{f \in E : \sum_{k\in\mathbf{N}} \left|k^2 \langle f, e_k\rangle\right|^2 < \infty\}$. It is easily verifiable that

$$A_1 e_k = -k^2 e_k \quad \text{for all } k \in \mathbf{N}, \tag{1.22}$$

which implies that that A_1 is dissipative and hence generates a contraction semigroup. Then the Hille-Yosida theorem implies that

$$\|(I - tA_1)^{-1}\| \le 1 \qquad \text{for all } t \ge 0. \tag{1.23}$$

From the dissipativity of A_1, we also have that

$$\|(I + tA_1)(I - tA_1)^{-1}\| \le 1 \qquad \text{for all } t \ge 0, \tag{1.24}$$

since for all $f \in D(A)$,

$$\begin{aligned}
\|(I + tA_1)f\|^2 &= \|f\|^2 + 2t\mathrm{Re}\,\langle A_1 f, f\rangle + t^2\|A_1 f\|^2 \\
&\le \|f\|^2 - 2t\mathrm{Re}\,\langle A_1 f, f\rangle + t^2\|A_1 f\|^2 \\
&= \|(I - tA_1)f\|^2.
\end{aligned}$$

Define an operator A_2 by

$$A_2 f := \sum_{k=1}^{\infty} \langle f, e_{k^2}\rangle\, e_k. \tag{1.25}$$

By the Parseval equation $\|\sum a_k e_k\|^2 = \sum |a_k|^2$ for all $f = \sum_k a_k e_k \in E$. Then, we have that $\|A_2 f\|^2 = \sum_{k=1}^{\infty} |\langle f, e_{k^2}\rangle|^2 \le \sum_{k=1}^{\infty} |\langle f, e_k\rangle|^2 = \|f\|^2$, which shows that $\|A_2\| \le 1$ and hence

$$\|I + tA_2\| \le 1 + t. \tag{1.26}$$

Define an operator A on $D(A_1)$ by $Af := A_1 f + A_2 f$. Then, A generates a C_0-semigroup since A_1 generates a contraction semigroup and A_2 is bounded (see Theorem I.6.4, [9]).

For $n \in \mathbf{N}$, we choose the discrete space $\Omega_n := \left\{\frac{k\pi}{n} : k = 0, 1, \cdots, n\right\}$. Define, as a counterpart of E, a finite dimensional function space E_n on this discretized domain Ω_n by $E_n := \{f \in L^2(\Omega_n) : f(0) = f(\pi) = 0\}$, with the inner product defined as $\langle f, g\rangle_n = \frac{\pi}{n}\sum_{k=1}^{n-1} f(\frac{k\pi}{n})\, g(\frac{k\pi}{n})$. Define a projection operator $P_n : E \to E_n$ by $(P_n f)(x) = f(x)$ for $f \in E$ and $x \in \Omega_n$. With the discrete domain and function space in place, we will now discretize the two operators A_1 and A_2. We first choose an orthonormal basis of E_n. For $k = 1, 2, \cdots, n-1$, let $e_{n,k} := P_n e_k = \sqrt{\frac{2}{\pi}}\sin(kx)$ for $x \in \Omega_n$. Then, $\|e_{n,k}\| = 1$ for $k \in \{1, 2, \cdots, n-1\}$ since

$$\begin{aligned}
\|e_{n,k}\|^2 &= \tfrac{\pi}{n}\textstyle\sum_{x=1}^{n-1} \frac{2}{\pi}\sin^2(kx\pi/n) \\
&= \tfrac{2}{n}\textstyle\sum_{x=0}^{n-1} \frac{1-\cos(2kx\pi/n)}{2} \\
&= 1 - \tfrac{1}{n}\textstyle\sum_{x=0}^{n-1} \cos(2kx\pi/n) \\
&= 1 - \tfrac{1}{n}\ \mathrm{Re}\textstyle\sum_{x=0}^{n-1} e^{i\frac{2kx\pi}{n}} \\
&= 1 - \tfrac{1}{n}\ \mathrm{Re}\frac{1-e^{i\frac{2k\pi}{n}\cdot n}}{1-e^{i\frac{2k\pi}{n}}} \;=\; 1.
\end{aligned}$$

To see that

$$\langle e_{n,j}, e_{n,k}\rangle = 0 \quad \text{for } j, k \in \{1, 2, \cdots, n-1\}, j \neq k, \tag{1.27}$$

we first examine the case when $j-k$ is an odd number. In this case $j+k$ is also an odd number. Since $2\sin(\alpha)\cdot\sin(\beta) = \cos(\alpha-\beta) + \cos(\alpha-\beta)$,

$$\begin{aligned}
\langle e_{n,j}, e_{n,k}\rangle &= \tfrac{2}{n}\textstyle\sum_{x=1}^{n-1} \sin(jx\pi/n)\sin(kx\pi/n) \\
&= \tfrac{1}{n}\textstyle\sum_{x=1}^{n-1}\left[\cos\frac{(j-k)x\pi}{n} - \cos\frac{(j+k)x\pi}{n}\right] \\
&= \tfrac{1}{2n}\textstyle\sum_{x=1}^{n-1}\left[\cos\frac{(j-k)x\pi}{n} + \cos\frac{(j-k)(n-x)\pi}{n} - \cos\frac{(j+k)x\pi}{n} - \cos\frac{(j+k)(n-x)\pi}{n}\right] \\
&= \tfrac{1}{2n}\textstyle\sum_{x=1}^{n-1}\left[\cos\frac{(j-k)x\pi}{n} - \cos\frac{(j-k)x\pi}{n} - \cos\frac{(j+k)x\pi}{n} + \cos\frac{(j+k)x\pi}{n}\right] \\
&= 0.
\end{aligned}$$

Next, we examine the case when $j-k$ is a non-zero even number. In this case $j+k$ is a positive even number. So,

$$\begin{aligned}
\langle e_{n,j}, e_{n,k}\rangle &= \tfrac{2}{n}\textstyle\sum_{x=0}^{n-1} \sin(jx\pi/n)\sin(kx\pi/n) \\
&= \tfrac{1}{n}\textstyle\sum_{x=0}^{n-1}\left[\cos\frac{(j-k)x\pi}{n} - \cos\frac{(j+k)x\pi}{n}\right] \\
&= \tfrac{1}{n}\ \mathrm{Re}\textstyle\sum_{x=0}^{n-1}\left[e^{i(j-k)x\pi/n} - e^{i(j+k)x\pi/n}\right] \\
&= \tfrac{1}{n}\ \mathrm{Re}\left[\frac{1-e^{i(j-k)\pi}}{1-e^{i\frac{(j-k)\pi}{n}}} - \frac{1-e^{i(j+k)\pi}}{1-e^{i\frac{(j+k)\pi}{n}}}\right] \\
&= 0.
\end{aligned}$$

Now on E_n, we use the commonly used central finite difference to discretize A_1, yielding

$$A_{n,1} := \left(\frac{n}{\pi}\right)^2 \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

It was observed in Hockney's famous Fast Poisson Solver paper [10] that

$$A_{n,1}e_{n,k} = -a_{n,k}\ e_{n,k}, \tag{1.28}$$

where $a_{n,k} := \frac{2-2\cos\frac{k\pi}{n}}{(\pi/n)^2} > 0$. To see this equality, let $v_i (i = 0, 1, \cdots, n-1)$ denote the i-th entry in the vector $v := A_{n,1}e_{n,k}$. By definition,

$$v_i = (n/\pi)^2\sqrt{2/\pi}\left[\sin\tfrac{k(i-1)\pi}{n} - 2\sin\tfrac{ki\pi}{n} + \sin k(i+1)\pi n\right].$$

Notice that this equality also holds for $i=1$ and $i=n$–1 since $\sin\frac{k(1-1)\pi}{n} = \sin\frac{k(n-1+1)\pi}{n} = 0$. Then, we have that $v_i = \sin\frac{ki\pi}{n}[2\cos(k\pi/n)-2] = -a_{n,k}\,e_{n,k}$. Since $a_{n,k} > 0$, $A_{n,1}$ retains the dissipativity of A_1. From the dissipativity of $A_{n,1}$, we have that

$$\|(I - tA_{n,1})^{-1}\| \ \le\ 1 \quad \text{and} \quad \|(I-tA_{n,1})^{-1}(I+tA_{n,1})\| \ \le\ 1 \tag{1.29}$$

for all $t \ge 0$. For the operator A_2, we use the discretization

$$A_{n,2}\, f := \sum_{k=1}^{k^2<n} \left\langle f, e_{n,k^2}\right\rangle e_{n,k} \text{ for } f \in E_n. \tag{1.30}$$

Since $\{e_{n,k}\}_{k=1}^{n-1}$ is an orthonormal basis of E_n, it follows that $A_{n,2}$ is bounded by 1 and hence

$$\|I + tA_2\| \le 1 + t. \tag{1.31}$$

1.2.2 The Temporal Approximation Method

Define temporal approximation methods $V(t; A)$ and $V(t; A_n)$ for $n\in\mathbf{N}$ by

$$\begin{cases} V(t; A)\, f & := \quad (I - \frac{t}{2}A_1)^{-1}(I + tA_2)(I + \frac{t}{2}A_1)f, \\ V(t; A_n)P_n f & := \quad (I-\frac{t}{2}A_{n,1})^{-1}(I+tA_{n,2})(I+\frac{t}{2}A_{n,1})P_n f. \end{cases} \tag{1.32}$$

for $f\in D(A)$ and $t\in[0, 1]$. We shall show that

(a) $V(t; A)$ is consistent for $f\in D(A)$ in the sense of (1.12);

(b) $V(t; A)^n$ is unbounded for all $(n, t)\in\mathbf{N}\times(0, 1]$;

(c) $\{V(t; A) : t\in[0, 1]\}$ satisfies the Von Neumann stability condition (1.20) on $D(A)$ for all $t\in[0, 1]$;

(d) $V(t;A_n)$ is not unconditionally stable, where the **unconditional stability** of the temporal approximation method $\{V(t;A_n) : (n,t)\in \mathbf{N}\times[0,\infty)\}$ means the existence of a constant $\delta\in[0,1]$ and an increasing function $M(\cdot)$ on $[0,\infty)$ such that for all $n,k\in\mathbf{N}$ and $t\in[0,\delta]$,

$$\|V(t;A_n)^k\|_n \le M(kt);$$

(e) $\{V(t;A_n) : t\in[0,1], n\in\mathbf{N}\}$ satisfies the Von Neumann stability condition (1.20).

Proof: (a) For $f\in D(A)$,

$$\begin{aligned} V(t;A)f-f &= (I-\tfrac{t}{2}A_1)^{-1}(I+tA_2)(I+\tfrac{t}{2}A_1)f-f \\ &= (I-\tfrac{t}{2}A_1)^{-1}[(I+tA_2)(I+\tfrac{t}{2}A_1)-(I-\tfrac{t}{2}A_1)]f \\ &= t(I-\tfrac{t}{2}A_1)^{-1}[A+\tfrac{t}{2}A_2A_1]f. \end{aligned}$$

Thus we have that

$$\tfrac{V(t;A)f-f}{t}-Af = [(I-\tfrac{t}{2}A_1)^{-1}Af-Af]+\tfrac{t}{2}(I-\tfrac{t}{2}A_1)^{-1}A_2A_1f.$$

Since $A=A_1+A_2$ and A_2 is bounded, it follows that $D(A_1)=D(A)$. Then, for $f\in D(A)$, we have that $A_2A_1f\in E$. Since A_1 generates a contraction semigroup, the Hille-Yosida theorem implies that $\|(I-\frac{t}{2}A_1)^{-1}\|\le 1$ for all $t\ge 0$. Thus,

$$\|\tfrac{t}{2}(I-\tfrac{t}{2}A_1)^{-1}A_2A_1f\| \le \tfrac{t}{2}\|A_2A_1f\|\to 0$$

as $t\to 0$. And again since $\|(I-\frac{t}{2}A_1)^{-1}\|\le 1$, we obtain that

$$\begin{aligned} \|(I-\tfrac{t}{2}A_1)^{-1}g-g\| &= \|(I-\tfrac{t}{2}A_1)^{-1}[g-(I-\tfrac{t}{2}A_1)g]\| \\ &\le \|(I-\tfrac{t}{2}A_1)^{-1}\|\cdot\|\tfrac{t}{2}A_1g\| \\ &\le \tfrac{t}{2}\|A_1g\|\to 0 \end{aligned}$$

as $t\to 0$ for all $g\in D(A)$. The Banach's Convergence Theorem (see H. H. Schaefer [30], Theorem III.4.5) implies that $\lim\limits_{t\to 0}(I-\frac{t}{2}A_1)^{-1}g=g$ for all $g\in E$ since $D(A)$ is dense in E and $(I-\frac{t}{2}A_1)^{-1}$ is uniformly bounded. Replacing g by Af we obtain that $\lim_{t\to 0}(I-\frac{t}{2}A_1)^{-1}Af-Af=0$. Therefore, $\lim\limits_{t\to 0}\frac{V(t;A)f-f}{t}-Af=0$, which means that $V(t;A)$ is consistent.

(b) For $k\in\mathbf{N}$, $V(t;A)e_{k^2}=(I-\frac{t}{2}A_1)^{-1}(I+tA_2)(I+\frac{t}{2}A_1)e_{k^2}$ by definition (1.32) of the approximation method V. The equality (1.22) implies that

$$\left\{\begin{aligned} (I+\tfrac{t}{2}A_1)e_k &= (1-\tfrac{t}{2}k^2)e_k, \\ (I-\tfrac{t}{2}A_1)^{-1}e_k &= (1+\tfrac{t}{2}k^2)^{-1}e_k. \end{aligned}\right. \tag{1.33}$$

Thus we obtain that

$$V(t;A)e_{k^2} = (1-\tfrac{t}{2}k^4)(I-\tfrac{t}{2}A_1)^{-1}(I+tA_2)e_{k^2}. \tag{1.34}$$

By the definition of A_2, $(I+tA_2)e_{k^2}=e_{k^2}+te_k$. Hence it follows from (1.34) that

$$V(t;A)e_{k^2} = (1-\tfrac{t}{2}k^4)(I-\tfrac{t}{2}A_1)^{-1}(e_{k^2}+te_k).$$

Then, with (1.33) we obtain from the above equality that

$$\begin{aligned} V(t;A)e_{k^2} &= (1-\tfrac{t}{2}k^4)[\tfrac{1}{1+\frac{t}{2}k^4}e_{k^2} + \tfrac{t}{1+\frac{t}{2}k^2}e_k] \\ &= \tfrac{1-\frac{t}{2}k^4}{1+\frac{t}{2}k^4}e_{k^2} + \tfrac{t(1-\frac{t}{2}k^4)}{1+\frac{t}{2}k^2}e_k. \end{aligned} \tag{1.35}$$

Therefore,

$$\|V(t;A)e_{k^2}\| \geq \|\tfrac{t(1-\frac{t}{2}k^4)}{1+\frac{t}{2}k^2}e_k\| = |\tfrac{t(1-\frac{t}{2}k^4)}{1+\frac{t}{2}k^2}| \to \infty$$

as $k \to \infty$ for any $t > 0$. This shows that $V(t;A)$ is unbounded for any $t \in (0,\infty)$.

Now, let k be an integer that is not a square of an integer. Then,

$$\begin{aligned} V(t;A)e_k &= (I - \tfrac{t}{2}A_1)^{-1}(I + tA_2)(I + \tfrac{t}{2}A_1)e_k \\ &= (1 - \tfrac{t}{2}k^2)(I - \tfrac{t}{2}A_1)^{-1}(I + tA_2)e_k, \end{aligned} \tag{1.36}$$

where the last "=" sign is due to (1.33). By definition, $A_2\, e_k = \sum_i \langle e_k, e_{i^2}\rangle\, e_i = 0$. Then, it follows from (1.36) that $V(t;A)e_k = (1 - \frac{t}{2}k^2)(I - \frac{t}{2}A_1)^{-1}e_k$. With (1.33) we arrive at

$$V(t;A)e_k = \tfrac{1-\frac{t}{2}k^2}{1+\frac{t}{2}k^2}e_k. \tag{1.37}$$

For each $t \in (0,1]$, let $c_{1,k} := \frac{1-\frac{t}{2}k^2}{1+\frac{t}{2}k^2}$, $c_{2,k} := \frac{1-\frac{t}{2}k^4}{1+\frac{t}{2}k^4}$, and $b_k := \frac{t(1-\frac{t}{2}k^4)}{1+\frac{t}{2}k^2}$. When k is not a square of an integer, it follows from (1.37) and (1.35) that

$$\begin{cases} V(t;A)^n e_k = c_{1,k}^n e_k, & \text{for } n \in \mathbf{N}, \\ V(t;A)e_{k^2} = c_{2,k}e_{k^2} + b_k e_k. \end{cases} \tag{1.38}$$

Now for $n \in \mathbf{N}_0$,

$$\begin{aligned} V(t;A)^{n+1}e_{k^2} &= V(t;A)^n(c_{2,k}e_{k^2} + b_k e_k) \\ &= c_{2,k}V(t;A)^n e_{k^2} + b_k c_{1,k}^n e_k, \end{aligned}$$

from which we have

$$\textstyle\sum_{i=0}^n c_{2,k}^{n-i}\left[V(t;A)^{i+1} - c_{2,k}V(t;A)^i\right]e_{k^2} = \sum_{i=0}^n c_{2,k}^{n-i}b_k c_{1,k}^i e_k = \frac{c_{2,k}^{n+1}-c_{1,k}^{n+1}}{c_{2,k}-c_{1,k}}b_k e_k.$$

But the leftmost side of the above equalities equals

$$c_{2,k}^{n+1}\textstyle\sum_{i=0}^n\left[\frac{V(t;A)^{i+1}}{c_{2,k}^{i+1}} - \frac{V(t;A)^i}{c_{2,k}^{\,i}}\right]e_{k^2} = [V(t;A)^{n+1} - c_{2,k}^{n+1}]e_{k^2}.$$

So,

$$V(t;A)^{n+1}e_{k^2} = c_{2,k}^{n+1}e_{k^2} + \frac{c_{2,k}^{n+1} - c_{1,k}^{n+1}}{c_{2,k} - c_{1,k}}b_k e_k,$$

from which we obtain $\|V(t;A)^{n+1}e_{k^2}\| \geq \left|\frac{c_{2,k}^{n+1}-c_{1,k}^{n+1}}{c_{2,k}-c_{1,k}}b_k\right|$. Thus, to show that $V(t;A)^n$ is not bounded for any $n \in \mathbf{N}$, it suffices to show that

$$\lim_{k\to\infty}\left|\frac{c_{2,k}^{n+1} - c_{1,k}^{n+1}}{c_{2,k} - c_{1,k}}b_k\right| = \infty. \tag{1.39}$$

We will show first that

$$\lim_{k\to\infty}\left|\frac{c_{2,k}^{n+1}-c_{1,k}^{n+1}}{c_{2,k}-c_{1,k}}\right| = n+1. \tag{1.40}$$

Since $\frac{a-1}{a+1} < \frac{b-1}{b+1} < 1$ for $b > a > 1$, we have that $|c_{1,k}| < |c_{2,k}| < 1$. Obviously, $\lim_{k\to\infty} c_{1,k} = \lim_{k\to\infty} c_{2,k} = -1$. So, $|c_{2,k}| > 0$ when k is large enough, and thus

$$\begin{cases} 0 < \frac{c_{1,k}}{c_{2,k}} < 1, & k \text{ large enough}, \\ \lim_{k\to\infty} \frac{c_{1,k}}{c_{2,k}} = 1. \end{cases} \tag{1.41}$$

Therefore, $\left|\frac{c_{2,k}^{n+1}-c_{1,k}^{n+1}}{c_{2,k}-c_{1,k}}\right| = \frac{1-(c_{1,k}/c_{2,k})^{n+1}}{1-(c_{1,k}/c_{2,k})} = \frac{1-x_k^{n+1}}{1-x_k}$, where $x_k := c_{1,k}/c_{2,k}$. Then, by L'Hopital's rule $\lim_{k\to\infty}\left|\frac{c_{2,k}^{n+1}-c_{1,k}^{n+1}}{c_{2,k}-c_{1,k}}\right| = \lim_{k\to\infty}\frac{d(1-x_k^{n+1})}{dx_k}/\frac{d(1-x_k)}{dx_k} = n+1$, which proves (1.40). Obviously, $\lim_{k\to\infty} b_k = -\infty$. Then (1.39) follows immediately from (1.40).

(c) Denote $W(t) := (I+tA_2)(I+\frac{t}{2}A_1)(I-\frac{t}{2}A_1)^{-1}$. Then $\|W(t)\| \le 1+t$ by (1.24) and (1.26), and hence for $(t,n)\in[0,1]\times\mathbf{N}$ and $f\in D(A)$,

$$\begin{aligned} \|V(t;A)^{n+1}f\| &= \|(I-\tfrac{t}{2}A_1)^{-1}W(t)^n(I+tA_2)(I+\tfrac{t}{2}A_1)f\| \\ &\le \|W(t)\|^n\,\|(I-\tfrac{t}{2}A_1)^{-1}\|\cdot\|(I+tA_2)\|\cdot\|(I+\tfrac{t}{2}A_1)f\| \\ &\le (1+t)^{n+1}\|(I+\tfrac{t}{2}A_1)f\|. \end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|V(t;A)^nf\|^{1/n} = \lim_{n\to\infty}(1+t)\left\|(I+\tfrac{t}{2}A_1)f\right\|^{1/n} = 1+t,$$

which shows that the approximation method V satisfies the Von Neumann stability condition (1.21) on $D(A)$.

(d) By definition, the spatially discrete method $\{V(t;A_n) : (t,n)\in[0,1]\times\mathbf{N}\}$ is called unconditionally stable if $\sup_{n\in\mathbf{N}}\left\|V(\frac{t}{m};A_n)^m\right\|_n \le M(t)$ for some positive increasing function $M(\cdot)$. An obvious necessary condition for unconditional stability is

$$\sup_{n\in\mathbf{N}}\left\|V(\tfrac{t}{m};A_n)^m\,P_nf\right\|_n \le M(t)\sup_{n\in\mathbf{N}}\|P_nf\|_n \tag{1.42}$$

for $f\in E$. Thus, to show that $V(t;A_n)$ is not unconditionally stable, it suffices to show that (1.42) does not hold.

Let k be a prime number and $n := k^2+1$. A calculation using (1.28) and (1.30) shows that

$$\begin{cases} V(t;A_n)e_{n,k^2} = \frac{1-\frac{t}{2}a_{n,k^2}}{1+\frac{t}{2}a_{n,k^2}}e_{n,k^2} + \frac{t(1-\frac{t}{2}a_{n,k^2})}{1+\frac{t}{2}a_{n,k}}e_{n,k}, \\ V(t;A_n)e_{n,k} = \frac{1-\frac{t}{2}a_{n,k}}{1+\frac{t}{2}a_{n,k}}e_{n,k}, \end{cases} \tag{1.43}$$

which has the exact the same form as (1.38). Observe that the coefficients before e_{n,k^2} and $e_{n,k}$ have the same properties as those of $c_{1,k}$, $c_{2,k}$ and b_k in (1.38), and that these properties of $c_{1,k}$, $c_{2,k}$ and b_k are used in (c) to show the unboundedness of $\{\|V(t;A)^m e_{k^2}\|_n\}_{k=1}^{\infty}$ for any fixed $m \in \mathbf{N}$ and $t > 0$. By exactly the same argument, we can show that the sequence $\{\|V(t;A_n)^m e_{n,k^2}\|_n\}_{k=1}^{\infty}$ is unbounded for any fixed $m \in \mathbf{N}$ and $t > 0$. This shows that (1.42) does not hold.

(e) Define $W_n(t) := (I+tA_{n,2})(I+\frac{t}{2}A_{n,1})(I-\frac{t}{2}A_{n,1})^{-1}$. Then $\|W_n(t)\|_n \leq 1+t$ by (1.29) and (1.31), and for $m \in \mathbf{N}_0$ and $(t,n) \in [0,\delta] \times \mathbf{N}$,

$$\begin{aligned}
\|V(t;A_n)^{m+1} f_n\|_n &= \|(I-\tfrac{t}{2}A_{n,1})^{-1} W_n(t)^m (I+tA_{n,2})(I+\tfrac{t}{2}A_{n,1}) f_n\|_n \\
&\leq \|W_n(t)\|_n^m \, \|(I-\tfrac{t}{2}A_{n,1})^{-1}\|_n \cdot \|(I+tA_{n,2})\|_n \cdot \|(I+\tfrac{t}{2}A_{n,1}) f_n\|_n \\
&\leq (1+t)^{m+1} \|(I+\tfrac{t}{2}A_1) f_n\|_n,
\end{aligned}$$

where the last inequality is due to (1.29), (1.31) and the inequality $\|W_n(t)\|_n \leq 1+t$. Again, since the discrete matrix $A_{n,1}$ is a bounded linear operator for each fixed $n \in \mathbf{N}$, it follows that $\|V^m(t;A_n)\|_n \leq (1+t)^m \|I+\frac{t}{2}A_{n,1}\|_n$. Therefore,

$$\mathbf{r}(V(t;A_n)) = \lim_{m\to\infty} \sqrt[m]{(1+t)^m \|I+tA_{n,1}\|_n} \;=\; 1+t.$$

This shows that $V(t;A_n)$ satisfies the Von Neumann stability condition. □

1.2.3 Numerical Testings

To examine the numerical stability of the classically unstable method given in (1.32), we choose an initial value condition for the problem with a known solution. The simulation time interval, the initial value condition, and the true solution of the example problem for the initial value condition are:

$$\begin{cases}
\text{Time interval:} & [0,1]; \\
\text{Initial value:} & u(0,x) = \sin(2x)+\sin(3x); \\
\text{True solution:} & u(t,x) = e^{-4t}\sin(2x)+e^{-9t}\sin(3x).
\end{cases}$$

We also choose three initial value errors for the testing problem. They are

$$\begin{cases}
e_1(x) &= & 0, \\
e_2(x) &= & \frac{x(\pi-x)}{10^5}\cos(nx), \\
e_3(x) &= & \sum_{k^2=1}^{k^2<n} \frac{\sin(k^2 x)}{10^4 n},
\end{cases}$$

where $n = \pi/\Delta x$ is the spatial partition size. The second and third initial value errors contain very high frequency components. High frequency errors usually tend to be enlarged considerably for unstable methods like the explicit method, and we choose these two errors in order to see how they will affect the simulation errors for our unstable example method.

We solved the example problem with the three perturbed initial value conditions by the explicit method $V(t)u^{n+1} = (I+tA)u^n$, the implicit method $V(t)u^{n+1} = (I-tA)^{-1}u^n$, the factorized unstable method (1.32) which is listed as FAC in the tables, and the Crank-Nicolson method (1.15) which is listed as C-N in the tables. We have tested these methods

Table 1: Stability testing — Small spatial partition sizes

Δx	Δt	Initial Err	Explicit	Implicit	F A C	C-N
		$e_1(x)$	$2.47e-01$	$4.06e-02$	$3.04e-03$	$3.04e-03$
$\pi/16$	1/5	$e_2(x)$	$7.13e+01$	$4.06e-02$	$3.03e-03$	$3.03e-03$
		$e_3(x)$	$1.31e+00$	$4.06e-02$	$3.04e-03$	$3.03e-03$
		$e_1(x)$	$6.70e-03$	$9.32e-03$	$7.24e-04$	$7.24e-04$
$\pi/16$	1/20	$e_2(x)$	$5.48e+07$	$9.32e-03$	$7.24e-04$	$7.24e-04$
		$e_3(x)$	$1.77e+01$	$9.32e-03$	$7.29e-04$	$7.29e-04$
		$e_1(x)$	$2.01e-03$	$4.18e-03$	$9.45e-04$	$9.45e-04$
$\pi/16$	1/50	$e_2(x)$	$2.27e-03$	$4.18e-03$	$9.45e-04$	$9.45e-04$
		$e_3(x)$	$2.00e-03$	$4.18e-03$	$9.49e-04$	$9.49e-04$
Δx	Δt	Initial Err	Explicit	Implicit	F A C	C-N
		$e_1(x)$	$2.46e+01$	$1.77e-02$	$7.81e-04$	$7.81e-04$
$\pi/32$	1/10	$e_2(x)$	$2.87e+11$	$1.77e-02$	$7.88e-04$	$7.88e-04$
		$e_3(x)$	$1.07e+10$	$1.77e-02$	$7.79e-04$	$7.78e-04$
		$e_1(x)$	$3.76e+34$	$1.78e-03$	$2.32e-04$	$2.32e-04$
$\pi/32$	1/100	$e_2(x)$	$1.26e+45$	$1.78e-03$	$2.32e-04$	$2.32e-04$
		$e_3(x)$	$1.96e+37$	$1.78e-03$	$2.34e-04$	$2.34e-04$
		$e_1(x)$	$5.07e-04$	$1.00e-03$	$2.40e-04$	$2.40e-04$
$\pi/32$	1/200	$e_2(x)$	$1.94e+01$	$1.00e-03$	$2.40e-04$	$2.40e-04$
		$e_3(x)$	$5.05e-04$	$1.01e-03$	$2.42e-04$	$2.42e-04$

with different spatial mesh sizes Δx and time step size Δt, and the computed approximated solutions are compared with the true solution and the maximal errors are listed in the two tables.

For the four methods chosen, the explicit method is not stable, not unconditionally stable, and even not Von Neumann stable. The FAC is also not stable or unconditionally stable, but satisfies the Von Neumann stability condition as shown in the previous section. The implicit method is stable for all well-posed problems by the Widder's theorem [2], and the Crank-Nicolson is stable for quasi-dissipative problems (see Section 2 of Chapter 4) like the example problem given in the previous section.

The test results show that the explicit method is obviously unstable. It has huge numerical errors when time step sizes Δt are large relative to the spatial mesh size Δx. The unstable method FAC exhibits much smaller numerical errors than does the explicit method, even smaller than the errors produced by the always-stable implicit method. Only when compared with the stable Crank-Nicolson method, the FAC method shows slightly larger numerical errors for the cases of simultaneously large Δt and small Δx when the initial value has high frequency errors. The Crank-Nicolson method is a second order method while the FAC method is only first order. The first order accuracy of the FAC method is due to the first order approximation of the semigroup e^{tA_2} by the explicit method $(I+tA_2)$ — the second factor in the FAC method (1.32). And even in the worst tested case for the FAC

Table 2: Stability testing — Large spatial partition sizes

Δx	Δt	Initial Err	Explicit	Implicit	F A C	C-N
$\pi/4096$	1/40	$e_1(x)$	∞	3.91e-03	6.45e-05	6.45e-05
		$e_2(x)$	∞	3.91e-03	8.28e-05	8.28e-05
		$e_3(x)$	∞	3.91e-03	4.33e-04	6.47e-05
$\pi/4096$	1/400	$e_1(x)$	∞	3.77e-04	6.31e-07	6.31e-07
		$e_2(x)$	∞	3.77e-04	2.25e-05	2.25e-05
		$e_3(x)$	∞	3.77e-04	4.04e-06	8.69e-07
$\pi/4096$	1/4000	$e_1(x)$	∞	3.76e-05	8.27e-09	8.27e-09
		$e_2(x)$	∞	3.76e-05	9.81e-09	9.81e-09
		$e_3(x)$	∞	3.76e-05	2.85e-08	2.85e-08
Δx	Δt	Initial Err	Explicit	Implicit	F A C	C-N
$\pi/8192$	1/80	$e_1(x)$	∞	1.92e-03	1.61e-05	1.61e-05
		$e_2(x)$	∞	1.92e-03	3.56e-05	3.56e-05
		$e_3(x)$	∞	1.92e-03	3.46e-04	1.63e-05
$\pi/8192$	1/800	$e_1(x)$	∞	1.88e-04	1.58e-07	1.58e-07
		$e_2(x)$	∞	1.88e-04	2.25e-05	2.25e-05
		$e_3(x)$	∞	1.88e-04	2.96e-06	3.66e-07
$\pi/8192$	1/8000	$e_1(x)$	∞	1.88e-05	2.10e-09	2.10e-09
		$e_2(x)$	∞	1.88e-05	3.69e-09	3.69e-09
		$e_3(x)$	∞	1.88e-05	1.29e-08	1.29e-08

method, when compared with the Crank-Nicolson the errors are still close to those of the C-N method. So it is unclear if these larger errors of the FAC method is caused by the unstability or the first order accuracy.

1.3 Outline of the Dissertation

As mentioned in the beginning of this chapter, the purpose of this dissertation is to study weakened stability conditions and their effects on or benefits to the convergence of approximation methods. Thus, the dissertation is presented in a way toward revealing the convergence properties and possible "benefits" of Von Neumann stable methods like the one presented in Section 1.2. Before addressing convergence of Von Neumann stable approximation methods in Chapter 3, we need to accumulate sufficient preparatory results in Chapter 2. In Chapter 4, the convergence results established in Chapter 3 are applied to the example in Section 1.2, and to two practically useful temporal approximation methods.

2 Spatial Approximations

2.1 Introduction

For an evolutionary system many of its spatial properties are closely related to its temporal properties. For instance, a property of the system at a spatial point will affect the properties of the system at the neighboring spatial points as time progresses. Such tempo-spatial interactions are also true with approximation procedures of an evolutionary system. For instance, an approximation of the operator A (which is a purely spatial approximation involving no time at all) clearly will affect the stability of a temporal approximation method of the spatially approximated system.

A numerical simulation of an evolutionary system consists of spatial and temporal approximations. Because of the influence of spatial approximations on the temporal approximations, we shall examine spatial approximations in this chapter.

A spatial approximation of the evolutionary system (1.1) involves the approximation of the operator A by a sequence of operators A_n, which converges in some sense to A. Thus, with spatial approximation we obtain a sequence of problems

$$\begin{cases} u'(t) = A_n u(t), \quad t \geq 0, \\ u(0) = f \in D(A_n) \subset E. \end{cases} \tag{2.1}$$

In order that the approximating problems converge to the original problem in the sense that the true solutions $u_n(t)$ of (2.1) converge to the true solution $u(t)$ of the original problem (1.1) uniformly on compact time intervals for a sufficiently large set of initial values, some stability condition is usually enforced on the approximations in addition to the condition of the strong convergence of A_n to A. An important sufficient condition for the convergence of spatial approximations is stated in the following result (see J. A. Goldstein [9], pp. 44) derived from the Trotter-Kato theorem [13, 26, 36].

Theorem *Let $A_n (n \in \mathbf{N}_0)$ defined on $D(A_n) \supset D(A_0)$ generate C_0-semigroups S_n satisfying the "stability condition"*

$$\|S_n(t)\| \leq M e^{\omega t}, \ t \geq 0, \tag{2.2}$$

for positive constants M, ω independent of n, t. Assume that for each $f \in D(A_0)$, $\lim_{n\to\infty} A_n f = A_0 f$. Then $\lim_{n\to\infty} S_n(t) f = S_0(t) f$ for each $f \in E$, where the limit is uniform for t in compact intervals of $[0, \infty)$.

The stability condition (2.2) not only assures the strong convergence of the semigroups, but also suffices to absorb small initial value errors. In fact, for any $f \in E$ and any sequence $\{f_n\} \subset E$ with $\lim_{n\to\infty} f_n = f$, the inequality

$$\|S_n(t) f_n - S(t) f\| \leq \|S_n(t)(f_n - f)\| + \|S_n(t) f - S(t) f\|$$

implies that $\lim_{n\to\infty} S_n(t) f_n = f$ when $\|S_n(t)\|$ is uniformly bounded on any compact interval of t. It may happen that the spatial approximations are such that the semigroup sequence is not stable even though the approximating operators converge. The following is such an example.

Example 2.1 *Let* $Y = \{f \in L^2([0,1], \mathbf{C}) \ : \ f(0) = f(1) = 0\}$. *Let* E *be the product Banach space given by* $E = \{(f,g) : f,g \in Y, \|(f,g)\| := \max(\|f\|_Y, \|g\|_Y) < \infty\}$. *Define an operator* A *by* $A(f,g)(x) := (if''(x), ig''(x))$, *or, in matrix form*

$$A \begin{bmatrix} f \\ g \end{bmatrix}(x) = \begin{bmatrix} \frac{id^2}{dx^2} & 0 \\ 0 & \frac{id^2}{dx^2} \end{bmatrix} \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}.$$

Since the Schrödinger operator $A_s := \frac{id^2}{dx^2}$ *generates a strongly continuous isometric group* $S_s(t)$ *on* Y, *the operator* A *generates a* C_0*-semigroup on* $S(t) := \begin{pmatrix} S_s(t) & 0 \\ 0 & S_s(t) \end{pmatrix}$ *on* E. *For each* $n \in \mathbf{N}$, *define an operator* A_n *on* $D(A)$ *by*

$$A_n := A + \begin{pmatrix} 0 & B_n \\ 0 & 0 \end{pmatrix},$$

where $B_n f := n\pi \langle f, e_n \rangle e_n$ *with* e_n *representing the function* $x \mapsto \sqrt{2}\sin(n\pi x)$. *Then,*

(i) *each* A_n *generates a* C_0*-semigroup* S_n,

(i i) *the operators* A_n *converge strongly to* A *on* $D(A)$, *but*

(iii) $\|S_n(t)\| \geq nt$.

Proof: (i) Define $P_n := \begin{pmatrix} 0 & B_n \\ 0 & 0 \end{pmatrix}$. Since A generates a C_0-semigroup and P_n is bounded for each $n \in \mathbf{N}$, it follows that $A_n = A + P_n$ generates a C_0-semigroup (see [9], pp. 40).

(ii) Let $u = (f,g) \in D(A)$. Then

$$\begin{aligned} \|A_n u - Au\| &= \left\| \begin{pmatrix} 0 & B_n \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \right\| \\ &= n\pi \| \langle g, e_n \rangle e_n \|_Y \\ &= (n\pi)^{-1} \| \langle g'', e_n \rangle e_n \|_Y \\ &\leq (n\pi)^{-1} \|g''\|_Y \ \|e_n\|_Y^2. \end{aligned}$$

Since $u \in D(A)$, it follows that $g'' \in L^2[0,1]$ by the definition of A. Since $\|e_n\|_Y = 1$ we obtain that $\lim_{n\to\infty} \|A_n u - Au\| = 0$.

(iii) To prove the inequality we will derive an expression for the semigroup S_n. For $f \in D(A_s) \subset Y$,

$$\begin{aligned} B_n A_s f &= B_n i f'' \ = \ -iB_n \textstyle\sum_{k=1}^\infty k^2\pi^2 \langle f, e_k \rangle e_k \\ &= -i \textstyle\sum_{k=1}^\infty k^2\pi^2 \langle f, e_k \rangle B_n e_k \\ &= -i(n\pi)^3 \langle f, e_n \rangle e_n. \end{aligned}$$

On the other hand, $A_s B_n f = A_s n\pi \langle f, e_n \rangle e_n = i\, n\pi \langle f, e_n \rangle e_n'' = -i(n\pi)^3 \langle f, e_n \rangle e_n$, which shows that

$$A_s B_n f = B_n A_s f \quad \text{for all } f \in D(A_s). \tag{2.3}$$

Now, for $u=(f,g)\in D(A)$ we have that $f,g\in D(A_s)$. Then,

$$AP_nu = \begin{pmatrix} A_s & \\ & A_s \end{pmatrix}\begin{pmatrix} 0 & B_n \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} A_sB_ng \\ 0 \end{pmatrix} = \begin{pmatrix} B_nA_sg \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & B_n \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_sf \\ A_sg \end{pmatrix} = P_nAu. \tag{2.4}$$

Let $t\geq 0$. Then, $P_n(I-tA)^{-1}=(I-tA)^{-1}P_n$ since for all $u\in E$,

$$\begin{aligned}(I-tA)^{-1}P_nu-P_n(I-tA)^{-1}u &= (I-tA)^{-1}[P_n(I-tA)-(I-tA)P_n](I-tA)^{-1}u \\ &= (I-tA)^{-1}[tP_nA-tAP_n](I-tA)^{-1}u \;=\; 0.\end{aligned}$$

It then follows that

$$\begin{aligned} P_nS(t) &= P_n\lim_{k\to\infty}(I-\tfrac{t}{k}A)^{-k} \\ &= \lim_{k\to\infty}(I-\tfrac{t}{k}A)^{-k}P_n \\ &= S(t)P_n \end{aligned} \tag{2.5}$$

for all $t\geq 0$. Similarly, with (2.3) we can show that

$$B_nS_s(t)=S_s(t)B_n \quad \text{for all } t\geq 0. \tag{2.6}$$

By Trotter's product formula [37] we have that $S_n(t)=\lim\limits_{k\to\infty}\left[e^{\frac{t}{k}P_n}S(\tfrac{t}{k})\right]^k$. Since $P_n^2=0$, it follows that $e^{tP_n}=I+tP_n$ for all $t\geq 0$. Thus,

$$\begin{aligned} S_n(t) &= \lim_{k\to\infty}\left[(I+\tfrac{t}{k}P_n)S(\tfrac{t}{k})\right]^k \\ &= \lim_{k\to\infty}(I+\tfrac{t}{k}P_n)^kS(\tfrac{t}{k})^k, \end{aligned}$$

where the last equality is due to (2.5). Then we obtain

$$S_n(t) \;=\; e^{tP_n}S(t) \;=\; (I+tP_n)S(t) \;=\; \begin{pmatrix} I & tB_n \\ 0 & I \end{pmatrix}S(t).$$

Now let $f_n(x):=(n\pi)^{-1}e_n$ and let $u_n:=\begin{pmatrix} 0 \\ f_n \end{pmatrix}$. Then

$$\begin{aligned} \|S_n(t)u_n\| &= \|e^{tP_n}S(t)u_n\| \\ &= \|\begin{pmatrix} I & tB_n \\ 0 & I \end{pmatrix}\begin{pmatrix} S_s(t) & 0 \\ 0 & S_s(t) \end{pmatrix}\begin{pmatrix} 0 \\ f_n \end{pmatrix}\| \\ &= \|\begin{pmatrix} tB_nS_s(t)f_n \\ S_s(t)f_n \end{pmatrix}\| \\ &= \max\{\|tB_nS_s(t)f_n\|_Y,\ \|S_s(t)f_n\|_Y\} \\ &= \max\{\|tS_s(t)B_nf_n\|_Y,\ \|S_s(t)f_n\|_Y\}, \end{aligned}$$

where the last equality is due to (2.6). It follows from Stone's theorem (see e.g. [9], pp. 32) that $S_s(t)$ is a unitary operator for all $t\geq 0$. Thus,

$$\begin{aligned} \|S_n(t)u_n\| &= \max\{\|tB_nf_n\|_Y,\ \|f_n\|_Y\} \\ &= \max\{tn\pi|\langle f_n,e_n\rangle|,(n\pi)^{-1}\|e_n\|_Y\} \\ &= \max\{t,(n\pi)^{-1}\}. \end{aligned}$$

Since $\|n\pi u_n\|=1$, we obtain that

$$\begin{aligned} \|S_n(t)\| &\geq \|S_n(t)(n\pi u_n)\| &&= n\pi\|S_n(t)u_n\| \\ &= n\pi\max\{t,(n\pi)^{-1}\} &&\geq n\pi t. \end{aligned}$$

$\square$

Although the spatial approximation given in the example above is unstable, it is stabilizable in the following sense. The implications of the stabilizability of S_n with respect to convergence will be described in the next section.

Example 2.2 *The semigroup sequence $\{S_n\}$ in Example 2.1 is* stabilizable *in the sense that there exists a sequence of bounded operators $\{W_n\}$ such that*

(a) $\|W_n S_n(t)\| \le Me^{\omega t}$ *for some constants M, ω independent of n, t, and*

(b) $\lim_{n\to\infty} W_n f = f$ *for all $f \in E$.*

Proof: For $n \in \mathbf{N}$ define operators W_n by $W_n(f,g) := \left((I - \frac{1}{n\pi}\frac{d^2}{dx^2})^{-1} f,\ g\right)$. Then, $\|W_n S_n(t)\| = \|W_n e^{tP_n} S(t)\| \le \|W_n e^{tP_n}\|$, where the last inequality holds because $\|S(t)\| = 1$. Now we will show that $\|W_n e^{tP_n}\| \le e^t$ for all $t \ge 0$ and $n \in \mathbf{N}$.

(a) We introduce the notation $Q_n f := (I - \frac{1}{n\pi}\frac{d^2}{dx^2})^{-1} f$. Then $W_n\binom{f}{g} = \begin{pmatrix} Q_n & 0 \\ 0 & I\end{pmatrix}\binom{f}{g}$. For $(f,g) \in E$,

$$\begin{aligned}
\|W_n e^{tP_n}\binom{f}{g}\| &= \|W_n \begin{pmatrix} I & tB_n \\ 0 & I\end{pmatrix}\binom{f}{g}\| \\
&= \|W_n \binom{f}{g} + W_n \begin{pmatrix} 0 & tB_n \\ 0 & 0\end{pmatrix}\binom{f}{g}\| \\
&\le \|W_n \binom{f}{g}\| + \|\begin{pmatrix} Q_n & 0 \\ 0 & I\end{pmatrix}\begin{pmatrix} 0 & tB_n \\ 0 & 0\end{pmatrix}\binom{f}{g}\| \\
&= \|W_n \binom{f}{g}\| + t\|\binom{Q_n B_n g}{0}\|.
\end{aligned}$$

By definition, $B_n g = n\pi \langle g, e_n\rangle e_n$. Since $\|W_n\| \le 1$, it follows that

$$\|W_n e^{tP_n}(f,g)\| \le \|(f,g)\| + tn\pi|\langle g, e_n\rangle| \cdot \|(Q_n e_n, 0)\|.$$

An easy calculation shows that $Q_n e_n = \frac{e_n}{1+n\pi}$ for all $n \in \mathbf{N}$. Then from the above inequality we obtain $\|W_n e^{tP_n}(f,g)\| \le e^t$ for $t \ge 0$ for all $n \in \mathbf{N}$ since

$$\begin{aligned}
\|W_n e^{tP_n}(f,g)\| &\le \|(f,g)\| + \tfrac{tn\pi}{1+n\pi}|\langle g, e_n\rangle| \cdot \|(e_n, 0)\| \\
&< \|(f,g)\| + t|\langle g, e_n\rangle| \\
&\le \|(f,g)\| + t\|g\|_Y \\
&\le (1+t)\|(f,g)\| \\
&\le e^t\|(f,g)\|.
\end{aligned}$$

(b) For $f \in D(A_s)$,

$$\begin{aligned}
\lim_{n\to\infty} \|Q_n f - f\| &= \lim_{n\to\infty} \|(I - \tfrac{1}{n\pi}\tfrac{d^2}{dx^2})^{-1} f - f\| \\
&= \lim_{n\to\infty} \|(I - \tfrac{1}{n\pi}\tfrac{d^2}{dx^2})^{-1} \tfrac{1}{n\pi}\tfrac{d^2}{dx^2} f\| \\
&= \lim_{n\to\infty} \tfrac{1}{n\pi}\|(I - \tfrac{1}{n\pi}\tfrac{d^2}{dx^2})^{-1} A_s f\| \\
&\le \lim_{n\to\infty} \tfrac{1}{n\pi}\|A_s f\| \;=\; 0,
\end{aligned} \tag{2.7}$$

where the inequality sign "$\le$" holds since $\|(I - \frac{1}{n\pi}\frac{d^2}{dx^2})^{-1}\| \le 1$. Since $W_n = \begin{pmatrix} Q_n & 0 \\ 0 & I\end{pmatrix}$, it follows from (2.7) that $\lim_{n\to\infty} W_n u = u$ for all $u \in D(A)$. Since $\|W_n\| \le 1$ for all $n \in \mathbf{N}$

and $D(A)$ is dense in E, statement (b) follows from Theorem III.4.5 in H. H. Schaefer's [30]. $\square$

One purpose of this chapter is to weaken the stability condition (2.2) to be able to accommodate unstable but stabilizable spatial approximations like the one in the example. In the next section we will investigate the convergence properties of unstable spatial approximations which can be stabilized in the sense of Example 2.2.

One tool we use to establish convergence theorems in this thesis is the Laplace-Stieljes transform and its approximation theory. In the following we collect some definitions and known approximation results for the Laplace-Stieljes transform.

Definition 2.1 *Let Lip_ω denote the Banach space of Lipschitz continuous functions*

$$Lip_\omega := \{v : [0,\infty) \to E : v(0) = 0, \text{ and } \|v\|_{Lip_\omega} < \infty\},$$

where $\|v\|_{Lip_\omega} := \sup\{\frac{\|v(t)-v(t')\|}{|\int_{t'}^{t} e^{\omega\tau} d\tau|} : t, t' \in [0,\infty)\}$. For $v \in Lip_\omega$, the Laplace-Stieljes transform *of v is defined as $(\mathcal{L}_s v)(\lambda) := \int_0^\infty e^{-\lambda t} dv(t)$ for all complex numbers λ with real parts satisfying $Re\lambda > \omega$.*

Remark 2.1 *It is not difficult to verify that for any locally integrable function $u : [0,\infty) \to E$ satisfying $\|u(t)\| \leq Me^{\omega t}$ for some constants $M, \omega \geq 0$, the antiderivative $v : t \mapsto \int_0^t u(s)ds$ is in Lip_ω and $(\mathcal{L}_s v)(\lambda) = \int_0^\infty e^{-\lambda t} u(t)dt = \lambda \int_0^\infty e^{-\lambda t} v(t)dt$.*

The following theorem for approximations of Laplace-Stieljes transforms is taken from Bäumer and Neubrander's paper [2].

Theorem 2.1 *Let $M > 0$, $f_n \in Lip_\omega$ with $\|f_n\|_{Lip_\omega} \leq M$ and $r_n = \mathcal{L}_s f_n$ for all $n \in \mathbf{N}_0$. The following are equivalent.*
(i) $\lim_{n\to\infty} r_n(\lambda) = r_0(\lambda)$ *for all* $\lambda \in (\omega, \infty)$.
(i i) $\lim_{n\to\infty} f_n(t) = f_0(t)$ *uniformly on compact subsets of* $[0,\infty)$.

2.2 Stabilized Spatial Approximations

One important result in the convergence theory for spatial approximations is the following theorem originally due to Trotter (see [13, 26, 36]).

Theorem (Trotter-Kato) *Let A be a generator of a C_0-semigroup S with $\|S(t)\| \leq Me^{\omega t}$ ($M, \omega > 0$, and $t \geq 0$). For $n \in \mathbf{N}$, let A_n with $D(A_n) \supset D(A)$ be generators of C_0-semigroups $S_n(t)$. If the semigroups S_n ($n \in \mathbf{N}$) satisfy the stability condition*

$$\|S_n(t)\| \leq Me^{\omega t}, \tag{2.8}$$

then the following statements are equivalent.

(i) $\lim_{n\to\infty} R(\lambda, A_n)f = R(\lambda, A)f$ *for all $\lambda > \omega$ and $f \in D$, where D is a total subset of E.*

(i i) *For* $f \in E$, $\lim_{n\to\infty} S_n(t)f = S(t)f$ *uniformly for* t *in compact intervals of* $[0, \infty)$.

An important assumption of the Trotter-Kato theorem is the stability condition (2.8) — the uniform boundedness of the semigroups. This condition implies another important but less conspicuous condition: a *non-empty intersection of resolvent sets* which contains a complex right half plane. In this section we modify the Trotter-Kato theorem by requiring only a stabilizability condition that does not imply a non-empty intersection of resolvent sets, and a weaker strong convergence of the resolvents, namely, stabilized strong convergence.

Theorem 2.2 *Suppose that* A *generates a* C_0*-semigroup* S *with* $\|S(t)\| \leq Me^{\omega t}$ $(M, \omega > 0,$ *and* $t \geq 0)$. *For* $n \in \mathbf{N}$, *let* $W_n \in \mathbf{L}(E)$ *and* A_n *with* $D(A_n) \supset D(A)$ *be generators of* C_0*-semigroups* $S_n(t)$. *If*

(a) $\lim_{n\to\infty} W_n f = f$ *for all* $f \in E$,

(b) *For all* $f \in D(A)$, $W_n f \in D(A)$ *and* $W_n A_n f = A_n W_n f$ *for* $f \in D(A)$, *and*

(c) $\|W_n S_n(t)\| \leq Me^{\omega t}$,

then the following statements are equivalent.

(i) $\lim_{n\to\infty} R_n(\lambda)f = R(\lambda, A)f$ *for all* $\lambda > \omega$ *and* $f \in D$, *where* D *is a total subset of* E, *and* $R_n(\lambda)f := \int_0^\infty e^{-\lambda t} W_n S_n(t)f\ dt$ *for* $\lambda > \omega$ *and* $f \in E$.

(i i) *For all* $f \in E$, $\lim\limits_{n\to\infty} W_n S_n(t)f = S(t)f$ *uniformly for* t *in compact intervals of* $[0, \infty)$.

Remark 2.2 *It is possible that the intersection of the resolvent sets of the operators* A_n *is empty. But since* A_n *is a generator of a* C_0*-semigroup for each* $n \in \mathbf{N}$, *there exists* $\omega_n > 0$ *such that* $R(\lambda, A_n)$ *exists for all* $\lambda > \omega_n$. *Hence,* $R_n(\lambda)$ *is an analytic continuation of* $W_n R(\lambda, A_n)$ *from* (ω_n, ∞) *to* (ω, ∞).

Proof of Theorem 2.2: (i)$\Longrightarrow$(ii): By condition (c) and the definition of $R_n(\lambda)$, a simple calculation shows that $\|R_n(\lambda)\| \leq M(\lambda - \omega)^{-1}$ for all $\lambda > \omega$. Then it follows from Banach's Convergence Theorem that statement (i) is equivalent to

$$\lim_{n\to\infty} R_n(\lambda)f \ = \ R(\lambda, A)f \tag{2.9}$$

for all $f \in E$.

Condition (c) implies that the function $t \mapsto \int_0^t W_n S_n(\tau) f d\tau$ is in Lip_ω (see Definition 2.1) and $\| \int_0^t W_n S_n(\tau) f d\tau \|_{Lip_\omega} \leq M$ for all $f \in E$. By Remark 2.1 and the definition of R_n, the Laplace-Stieljes transform of $\int_0^t W_n S_n(\tau) f\, d\tau$ is $R_n(\lambda)f$. Since the Laplace-Stieljes transform of $\int_0^t S(\tau) f\ d\tau$ is $R(\lambda, A)f$, it follows from (2.9) and Theorem 2.1 that

$$\lim_{n\to\infty} \int_0^t W_n S_n(t) f\ dt = \int_0^t S(t) f\ dt \tag{2.10}$$

for all $f \in E$.

Let $f \in E$ and $\lambda > \omega$. By condition (c) and Remark 2.1,

$$\begin{aligned} R_n(\lambda)f &= \lambda \int_0^\infty e^{-\lambda t} \left(\int_0^t W_n S_n(\tau) f \, d\tau \right) dt \\ &= \lambda \int_0^\infty e^{-\lambda t} W_n \left(\int_0^t S_n(\tau) f \, d\tau \right) dt. \end{aligned} \tag{2.11}$$

Since $\int_0^t S_n(\tau) f \, d\tau \in D(A_n)$, the commutativity condition (b) implies that $W_n \int_0^t S_n(\tau) f \, d\tau \in D(A_n)$. Then it follows from (2.11) that $R_n(\lambda) f \in D(A_n)$. Thus, by the closedness of A_n and the commutativity condition (b),

$$\begin{aligned} A_n R_n(\lambda) f &= \lambda \int_0^\infty e^{-\lambda t} A_n W_n \left(\int_0^t S_n(\tau) f \, d\tau \right) dt \\ &= \lambda \int_0^\infty e^{-\lambda t} W_n \left(A_n \int_0^t S_n(\tau) f \, d\tau \right) dt \\ &= \lambda \int_0^\infty e^{-\lambda t} W_n \left(S_n(t) f - f \right) dt \\ &= \lambda R_n(\lambda) f - W_n f. \end{aligned} \tag{2.12}$$

Since $AR(\lambda)f = \lambda R(\lambda, A)f - f$, equation (2.9), condition (a) and equality (2.12) imply that for all $f \in E$,

$$\lim_{n\to\infty} A_n R_n(\lambda) f = AR(\lambda, A) f. \tag{2.13}$$

For $f \in E$, let $V_n(t)f := A_n \int_0^t W_n S_n(\tau) f d\tau = W_n S_n(t) f - W_n f$. Then, condition (c) implies that

$$\|V_n(t)\| \le M e^{\omega t} + M. \tag{2.14}$$

Now for $f \in E$,

$$\begin{aligned} \|W_n S_n(t) f - S(t) f\| &\le \|W_n[S_n(t)f - f] - [S(t)f - f]\| + \|W_n f - f\| \\ &= \|V_n(t) f - A \int_0^t S(\tau) f \, d\tau\| + \|W_n f - f\| \end{aligned}$$

Since $\|W_n f - f\| \to 0$ by condition (a), to show $\lim_{n\to\infty} \|W_n S_n(t) f - S(t) f\| = 0$ it suffices to show that

$$\lim_{n\to\infty} V_n(t) f = \int_0^t AS(\tau) f \, d\tau \tag{2.15}$$

for all $f \in D(A)$ (due to (2.14) and Banach's Convergence Theorem.

For $f \in D(A)$, let $g = (\lambda - A) f$ for some $\lambda > \omega$. Then,

$$\begin{aligned} \|V_n(t) f - \int_0^t AS(\tau) f \, d\tau\| &\le \| \int_0^t W_n A_n S_n(\tau)[R(\lambda, A) - R_n(\lambda)] g d\tau\| \\ &+ \| \int_0^t W_n S_n(\tau)[A_n R_n(\lambda) - AR(\lambda,A)] g d\tau\| \\ &+ \| \int_0^t [W_n S_n(\tau) - S(\tau)] AR(\lambda, A) g d\tau\|. \end{aligned}$$

Denote the three terms on the right hand side of the above inequality by I_1, I_2 and I_3 respectively. Then $\lim_{n\to\infty} I_1 = 0$ by (2.14) and (2.9), $\lim_{n\to\infty} I_2 = 0$ by (2.13) and condition (c), and $\lim_{n\to\infty} I_3 = 0$ by (2.10). Therefore, (2.15) holds for all $f \in D(A)$.

(ii)$\Longrightarrow$(i): It is well known that $R(\lambda,A)f=\int_0^\infty e^{-\lambda t}S(t)f dt$ for $\lambda>\omega$ and $f\in E$. Then with statement (ii) and the stability condition (c), statement (i) follows from the dominated convergence theorem. $\square$

The commutativity condition (b) in Theorem 2.2 is a severe restriction on the stabilizing operators W_n. The stabilizers W_n given in Example 2.2 do not commute with the semigroups they stabilize, so the theorem is not applicable. But if we choose the stabilizers W_n to be $W_n=\begin{pmatrix} Q_n & 0 \\ 0 & Q_n \end{pmatrix}$, where $Q_n=(I-\frac{1}{n\pi}\frac{d^2}{dx^2})^{-1}$, then it is verifiable that W_n and A_n commute. Thus in this case, the unstable semigroup sequence $\{S_n\}$ converges strongly to S under the stabilization of these new stabilizers.

2.3 Unstable Spatial Approximations

In this section we will derive several results with the commutativity condition removed. To establish these convergence result for stabilized semigroups, we need to modify the stabilizability condition accordingly.

Theorem 2.3 *Suppose that A generates a C_0-semigroup S with $\|S(t)\|\le Me^{\omega t}$ ($M,\omega>0$ and $t\ge 0$). For $n\in\mathbf{N}$, let $W_n\in\mathbf{L}(E)$ and A_n be the generators of C_0-semigroups S_n. Assume that*

(a) $f_o\in D:=\cap_{n=1}^\infty D(A_n)\cap D(A)$,

(b) $\lim_{n\to\infty}W_nf=f$ *for all* $f\in D(A)$, *and*

(c) $\|W_nS_n(t)A_nf\|\le M_fe^{\omega_f t}$ *for all* $f\in D(A_n)$ *and for f-dependent constants* $M_f>0,\omega_f\ge\omega$.

Then for each $f\in D$, the integrals $R_n(\lambda)f:=\int_0^\infty e^{-\lambda t}W_nS_n(t)f\ dt$ and $R_n(\lambda)A_nf:=\int_0^\infty e^{-\lambda t}W_nS_n(t)A_nf\ dt$ exist for all $\lambda>\omega_f$ and $n\in\mathbf{N}$; and the following statements are equivalent.

(i) $\lim_{n\to\infty}R_n(\lambda)f_o=R(\lambda,A)f_o$ *for all* $\lambda>\omega_{f_o}$.

(i i) $\lim_{n\to\infty}W_nS_n(t)f_o=S(t)f_o$ *uniformly for t in compact intervals of* $[0,\infty)$.

Proof: Since each operator A_n generates a C_0-semigroup S_n for $n\in\mathbf{N}$, it follows that for $f\in D$

$$S_n(t)f=f+\int_0^t S_n(\tau)A_nf d\tau.$$

From the boundedness of W_n we obtain that

$$W_nS_n(t)f=W_nf+\int_0^t W_nS_n(\tau)A_nf d\tau.$$

Conditions (b) and (c) imply that

$$\|W_nS_n(t)f\|\ \le\ \sup_{n\in\mathbf{N}}\|W_nf\|+M_f\int_0^t e^{\omega_f\tau}d\tau\ =\ G_fe^{\omega_f t},$$

for some constant G_f. Thus the integrals $R_n(\lambda)f$ are well defined for all $\lambda > \omega_f$ and $n \in \mathbf{N}$. Due to condition (c), the integral $R_n(\lambda)A_n f$ exists for $f \in D$, $\lambda > \omega_f$, and $n \in \mathbf{N}$. Thus, for each $f \in D$ the integrals $R_n(\lambda)(\lambda - A_n)f$ also exist for all $\lambda > \omega_f$ and $n \in \mathbf{N}$.

Now, for $f \in D$, $\lambda > \omega_f$, and $n \in \mathbf{N}$,

$$\begin{aligned}
\lambda R_n(\lambda)f &= \lambda \textstyle\int_0^\infty e^{-\lambda t} W_n S_n(t) f dt \\
&= \lambda \textstyle\int_0^\infty e^{-\lambda t} W_n [f + \int_0^t S_n(\tau) A_n f d\tau] dt \\
&= W_n f + \lambda \textstyle\int_0^\infty e^{-\lambda t} \int_0^t W_n S_n(\tau) A_n f \, d\tau \; dt \\
&= W_n f + \textstyle\int_0^\infty e^{-\lambda t} W_n S_n(t) A_n f \, dt \\
&= W_n f + R_n(\lambda) A_n f,
\end{aligned}$$

which is equivalent to

$$R_n(\lambda)(\lambda - A_n)f \;=\; W_n f \tag{2.16}$$

for $f \in D(A_n)$, $\lambda > \omega_f$ and $n \in \mathbf{N}$. Since $R(\lambda, A)(\lambda - A)f = f$ for all $f \in D(A)$ and $\lambda > \omega$, and since $\lim_{n\to\infty} W_n f = f$ for $f \in D(A)$, it follows that statement (i) is equivalent to

(i′) $$\lim_{n\to\infty} R_n(\lambda) A_n f_o = R(\lambda, A) A f_o$$

for all $\lambda > \omega_{f_o}$.

Let $W_0 := I$, $S_0(t) := S(t)$ and $A_0 := A$. Then, $R_0(\lambda) = R(\lambda, A)$, and for all $n \in \mathbf{N}_0$ and $f \in D(A_n)$, $W_n S_n(t) f - W_n f = \int_0^t W_n A_n S_n(\tau) f d\tau$ since A_n generates the C_0-semigroup S_n. Since $\lim_{n\to\infty} W_n f = f$ for all $f \in D(A)$, it follows that statement (ii) is equivalent to

(ii′) $$\lim_{n\to\infty} \int_0^t W_n A_n S_n(\tau) f_o d\tau = \int_0^t A S(\tau) f_o d\tau.$$

For $n \in \mathbf{N}_0$, let $v_n(t) := \int_0^t W_n A_n S_n(\tau) f_o d\tau = \int_0^t W_n S_n(\tau) A_n f_o d\tau$. An easy calculation shows that the Laplace-Stieljes transform of v_n is

$$(\mathcal{L}_s v_n)(\lambda) = R_n(\lambda) A_n f_o,$$

for all $n \in \mathbf{N}_0$. By condition (c), $v_n \in Lip_\omega$ and $\sup_{n\in\mathbf{N}} \|v_n\|_{Lip_\omega} \le M_{f_o}$. It then follows from Theorem 2.1 that statements (i′) and (ii′) are equivalent. □

Theorem 2.3 relates the strong convergence of stabilized "resolvents" to that of the stabilized semigroups. It is common that the strong convergence of generators, rather than resolvents, is given. The following is a result relating the strong convergence of generators to that of their stabilized semigroups.

Corollary 2.1 *Suppose that A generates a C_0-semigroup S with $\|S(t)\| \le Me^{\omega t}$ $(M, \omega, t > 0)$. For $n \in \mathbf{N}$, let $W_n \in \mathbf{L}(E)$ and A_n with $D(A_n) \supset D(A)$ be generators of C_0-semigroups S_n. If*

(a) $\lim_{n\to\infty} W_n f = f$ *for all* $f \in D(A)$,

(b) $\|W_n S_n(t)\| \le Me^{\omega t}$, *and*

(c) $\lim_{n\to\infty} A_n f = Af$ *for all* $f \in D(A^\infty)$,

then, for all $f \in E$, $\lim\limits_{n\to\infty} W_n S_n(t) f = S(t) f$ *uniformly for* t *in compact intervals of* $[0,\infty)$.

Proof: Condition (c) implies that for $f \in D(A^\infty)$, there exists an f-dependent positive constant M_f such that $\|A_n f\| \le M_f$ for all $n \in \mathbf{N}$. Then,

$$\begin{aligned} \|W_n S_n(t) A_n f\| &\le \|W_n S_n(t)\| \cdot \|A_n f\| \\ &\le Me^{\omega t} \|A_n f\| \\ &\le M_f M e^{\omega t}. \end{aligned} \tag{2.17}$$

Now, for $f \in D(A^\infty)$, $\lambda > \omega$, and $n \in \mathbf{N}$, let $R_n(\lambda) f := \int_0^\infty e^{-\lambda t} W_n S_n(t) f\, dt$ and $g := R(\lambda, A) f$. Then

$$\begin{aligned} \|R_n(\lambda) f - R(\lambda, A) f\| &= \|[R_n(\lambda)(\lambda - A) g - g\| \\ &= \|[R_n(\lambda)(\lambda - A) - I]\, g\| \\ &= \|[R_n(\lambda)(\lambda - A_n) + R_n(\lambda)(A_n - A) - I]\, g\|. \end{aligned}$$

Obviously, $g = R(\lambda, A) f \in D(A^\infty) \subset D(A)$. Then (2.16) implies that

$$R_n(\lambda)(\lambda - A_n) g = W_n g.$$

Therefore,

$$\begin{aligned} \|R_n(\lambda) f - R(\lambda, A) f\| &= \|[W_n + R_n(\lambda)(A_n - A) - I]\, g\| \\ &\le \|(W_n - I) g\| + \|R_n(\lambda)(A_n - A) g\|. \end{aligned} \tag{2.18}$$

Using the definition of $R_n(\lambda)$ and condition (b), an easy calculation shows that $\|R_n(\lambda)\| \le M(\lambda - \omega)^{-1}$. This, together with (2.18), implies

$$\|R_n(\lambda) f - R(\lambda, A) f\| \le \|(W_n - I) g\| + \tfrac{M}{\lambda - \omega} \|(A_n - A) g\|. \tag{2.19}$$

Since $g \in D(A^\infty)$, it follows from condition (c) that $\lim\limits_{n\to\infty} (A_n - A) g = 0$. And since $(W_n - I) g \to 0$ by condition (a), it follows from (2.19) that $R_n(\lambda) f$ converges to $R(\lambda, A) f$. Then, we obtain from (2.17) and Theorem 2.3 that for all $f \in D(A^\infty)$,

$$\lim_{n\to\infty} W_n S_n(t) f = S(t) f \tag{2.20}$$

uniformly for t in compact intervals of $[0,\infty)$. By Theorem 2.7 in Pazy's [28], $D(A^\infty)$ is dense in E and hence total in E. Thus we obtain from Banach's Convergence Theorem the convergence in (2.20) for all $f \in E$ uniformly for t in compact intervals of $[0,\infty)$. $\square$

Now, with the above corollary, we re-examine the problem discussed in Examples 2.1 and 2.2 from the viewpoint of convergence.

Example 2.3 *Using Corollary 2.1, statement* (i) *in Example 2.1, and statements* (a) *and* (b) *in Example 2.2, we obtain that the unstable sequence of semigroups* $\{S_n\}$ *converges strongly to* S *under stabilization of* $\{W_n\}$*, namely,* $\lim_{n\to\infty} W_n S_n(t) f = S(t) f$ *for all* $f \in E$. *The stabilizability condition* (a) *proven in Example 2.2 further implies that for any* $f \in E$ *and any sequence* $\{f_n\}$ *convergent to* f*, the limit*

$$\lim_{n\to\infty} W_n S_n(t) f_n = S(t) f$$

holds uniformly for t *in compact intervals, due to the following inequalities*

$$\begin{aligned} \|W_n S_n(t) f_n - S(t) f\| &\leq \|W_n S_n(t)(f_n - f)\| + \|W_n S_n(t) f - S(t) f\| \\ &\leq \|W_n S_n(t)\| \cdot \|f_n - f\| + \|W_n S_n(t) f - S(t) f\|. \end{aligned}$$

□

It is evident from the proof of Theorem 2.3 that we can weaken the C_0-semigroup assumption for S since the uniform norm-boundedness $\|S(t)\| \leq M e^{\omega t}$ is not used at all. And we can also relinquish the C_0-semigroup assumption for the approximating operator families S_n because we only require an initial value dependent uniform boundedness condition, not uniform norm-boundedness. In the following theorem, we will weaken the assumptions of Theorem 2.3 and generalize it, in the case of $W_n = I$, to families of linear, possibly unbounded, and even unclosed operators.

Theorem 2.4 *Let* $\{A_n\}_{n=0}^{\infty}$ *be a sequence of linear operators. Let* $f \in \bigcap_{n=0}^{\infty} D(A_n^{\infty})$. *For each* $n \in \mathbf{N}_0$*, let* $\{S_n(t) : t \geq 0\}$ *be a one parameter family of linear operators with* $S_n(0) = I$ *and* $\bigcap_{t\geq 0} D(S_n(t)) \supset D$ *for all* $n \in \mathbf{N}$. *Suppose that*

(a) *the function* $t \mapsto S_n(t) f$ *is differentiable for all* $t \geq 0$ *and* $\frac{d S_n(t) f}{dt} = S_n(t) A_n f$ *for* $n \in \mathbf{N}_0$*, and*

(b) *there exist constants* $M, \omega \geq 0$ *such that* $\|S_n(t) A_n f\| \leq M e^{\omega t}$ *for all* $t \geq 0$ *and* $n \in \mathbf{N}_0$.

Then the following statements are equivalent.

(i) $\lim_{n\to\infty} R_n(\lambda) f = R_0(\lambda) f$ *for all* $\lambda > \omega$*, where* $R_n(\lambda) f := \int_0^\infty e^{-\lambda t} S_n(t) f dt$ *for* $\lambda > \omega$.

(i i) $\lim_{n\to\infty} S_n(t) f = S_0(t) f$ *uniformly for* t *in compact intervals of* $[0, \infty)$.

Proof: Condition (a) implies that

$$S_n(t) f - f = \int_0^t S_n(\tau) A_n f d\tau \quad \text{for } n \in \mathbf{N}_0. \tag{2.21}$$

Then, condition (b) implies that

$$\|S_n(t) f\| \leq \|f\| + M_f \int_0^t e^{\omega \tau} d\tau = \|f\| + \omega^{-1} M_f \left(e^{\omega t} - 1\right).$$

Then $R_n(\lambda)f$ is well defined for $\lambda > \omega$, and $n \in \mathbf{N}_0$.

Equation (2.21), condition (b) and Remark 2.1 imply that for each $n \in \mathbf{N}_0$, the function $t \mapsto (S_n(t)f - f)$ has a Laplace-Stieljes transform and its Laplace-Stieljes transform satisfies

$$\begin{aligned} [\mathcal{L}_s(S_n(\cdot)f - f)](\lambda) &= \lambda \int_0^\infty e^{-\lambda t} \int_0^t S_n(\tau) A_n f \, d\tau dt \\ &= \lambda \int_0^\infty e^{-\lambda t} (S_n(t)f - f) dt \\ &= \lambda R_n(\lambda) f - f. \end{aligned}$$

for all $\lambda > \omega$. Then, statement (i) is equivalent to

$$\text{(i}'\text{)} \qquad \lim_{n\to\infty} [\mathcal{L}_s(S_n(\cdot)f - f)](\lambda) = [\mathcal{L}_s(S_0(\cdot)f - f)](\lambda) f \quad \text{for } \lambda > \omega.$$

With conditions (a) and (b), a straightforward calculation using (2.21) shows that the Lip_ω norm of $S_n(\cdot)f - f$ as defined in Definition 2.1 satisfies

$$\|S_n(\cdot)f\|_{Lip_\omega} \leq M$$

for all $n \in \mathbf{N}_0$. Then, by Theorem 2.1, (i$'$) is equivalent to

$$\lim_{n\to\infty} S_n(t)f - f = S_0(t)f - f$$

uniformly for t in compact intervals, which is obviously equivalent to (ii). □

Without the C_0-semigroup assumption for the operator family S, Theorem 2.4 aims at ill-posed problems (as opposed to well-posed problems where the spatial operator A generates a C_0-semigroup). We would like to mention that in this dissertation we are not going further toward that direction. Like Theorem 2.3, the main contribution of Theorem 2.4, at least in this dissertation, is to serve as a bridge to the study of other approximation methods. Because A is not required to be a C_0-semigroup, and because of the flexibility of allowing unstable or even unbounded spatial approximation methods, we are able to unify stable temporal approximation methods with a general class of possibly unstable but Von-Neumann stable methods in the next chapter. This is done by bridging the temporal methods to the true solution with a sequence of spatially approximating problems, which are constructed from the approximating temporal methods and have only initial value dependent uniform boundedness.

3 Temporal Approximations

One practically useful convergence theorem for temporal approximation methods is the following result originally due to Chernoff (see [28], pp. 90).

Theorem (Chernoff Product Formula) *Suppose that A is the generator of a C_0-semigroup S with $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ for some constants $M, \omega > 0$. Let $\{V(t) : t \in [0, \infty]\}$ be a temporal approximation method which is consistent on $D(A)$ and satisfies $\|V(t)^n\| \leq Me^{n\omega t}$ for all $(n,t) \in \mathbf{N} \times [0, \delta]$. Then, for each $f \in E$, $\lim_{n\to\infty} V(\frac{t}{n})^n f = S(t)f$ uniformly for t in compact interval $[0,T]$ for any $T > 0$.*

In this chapter, we will generalize Chernoff's product formula to (i) stabilized temporal approximation methods, and (ii) to Von Neumann stable temporal approximations.

3.1 Stabilized Temporal Approximations

As an application of Corollary 2.1, we extend in this section the Chernoff product formula to possibly unstable, but stabilizable approximation methods.

Theorem 3.1 *Suppose that A generates a C_0-semigroup S with $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and some constants $M, \omega \geq 0$ independent of t. Let $\{V(t) : t \in [0,\delta]\}$ and $\{W(t) : t \in [0,\delta]\}$ be two strongly continuous families of bounded linear operators on E satisfying $V(0) = W(0) = I$, $\lim_{t\to 0} \frac{V(t)f - f}{t} = Af$ for all $f \in D(A^\infty)$, and*

$$\|W(t)V(t)^{k-1}\| \leq Me^{\omega tk} \tag{3.1}$$

for all $(k,t) \in \mathbf{N}_0 \times [0,\delta]$. Then, for any $f \in E$, $\lim_{n\to\infty} W(\frac{t}{n})V(\frac{t}{n})^{n-1} f = S(t)f$ uniformly for t in compact intervals.

Lemma 3.1 *Let $W(t), V(t)$ be as in Theorem 3.1. Define $A_s f := \frac{V(s)-I}{s} f$ for all $s \in (0,T], f \in E$. Then for any $\omega_o > \omega$, there exist $s_o > 0$ such that $\|W(s)e^{tA_s}\| < Me^{\omega_o(t+s)}$ whenever $s \in (0, s_o)$.*

Proof: By definition, A_s is bounded. Thus, $e^{tA_s} f = \lim_{m\to\infty} \left(I + \frac{t}{m} A_s\right)^m f$ for all $f \in E$, and therefore

$$\begin{aligned} \|W(s)e^{tA_s} f\| &= \lim_{m\to\infty} \|W(s)\left(I + \tfrac{t}{m} A_s\right)^m f\| \\ &= \lim_{m\to\infty} \|W(s)\left(I + \tfrac{t}{m}\tfrac{V(s)-I}{s}\right)^m f\| \\ &= \lim_{m\to\infty} \|W(s)\left(\tfrac{ms-t}{ms} + \tfrac{t}{ms} V(s)\right)^m f\| \\ &= \lim_{m\to\infty} \|W(s)[\lambda_m + \mu_m V(s)]^m f\|, \end{aligned} \tag{3.2}$$

where $\lambda_m := \frac{ms-t}{ms}$ and $\mu_m := 1 - \lambda_m = \frac{t}{ms}$. Since $\lambda_m > 0$ for m large, it follows that

$$\begin{aligned} \|W(s)\left(\lambda_m + \mu_m V(s)\right)^m f\| &= \|\textstyle\sum_{i=0}^m \binom{m}{i} \lambda_m^{m-i} \mu_m^i W(s)V(s)^i f\| \\ &\leq \textstyle\sum_{i=0}^m \binom{m}{i} \lambda_m^{m-i} \mu_m^i Me^{\omega(1+i)s} \|f\| \\ &= Me^{\omega s}\left(\lambda_m + \mu_m e^{\omega s}\right)^m \|f\|. \end{aligned}$$

It follows from (3.2) that $\|W(s)e^{tA_s}\| \le Me^{\omega s}\lim_{m\to\infty}(1+\mu_m(e^{\omega s}-1))^m$ by noticing that $\lambda_m = 1-\mu_m$. For $\omega_o > \omega$ there exists $s_o > 0$ such that $e^{\omega s}-1 \le \omega_o s$ for all $s \in (0, s_o]$. Then $\|W(s)e^{tA_s}\| \le Me^{\omega s}\lim_{m\to\infty}(1+\mu_m\omega_o s)^m$. Replacing μ_m by $\frac{t}{ms}$, we obtain

$$\|W(s)e^{tA_s}\| \quad = \quad Me^{\omega s}\lim_{m\to\infty}\left(1+\tfrac{\omega_o t}{m}\right)^m \quad \le \quad Me^{\omega_o(s+t)},$$

which proves the lemma. □

The following lemma extends an estimate due to Chernoff [7]. The proof is a modification of Chernoff's original proof for his estimate.

Lemma 3.2 *Let $W, L \in \mathbf{L}(E)$. Suppose there exist constants $M, l \ge 1$ such that for all $f \in E$, $\|WL^n\| \le Ml^{n+1}$ for all $n \ge 0$. Then*

$$\|We^{n(L-I)}f - WL^nf\| \le M\sqrt{n}l^ne^{(l^2-1)n/2}\| \cdot \|Lf-f\|$$

for each $f \in E$ and $n \in \mathbf{N}$.

Proof: Let $f \in E$. Clearly, for $n \in \mathbf{N}$,

$$\|We^{n(L-I)}f - WL^nf\| \le e^{-n}\sum_{k=0}^{\infty}\frac{n^k}{k!}\|WL^kf - WL^nf\|. \tag{3.3}$$

Now for $k \ge n$,

$$\begin{aligned}
\|WL^kf - WL^nf\| &\le \textstyle\sum_{i=n+1}^{k}\|WL^if - WL^{i-1}f\| \\
&\le \textstyle\sum_{i=n+1}^{k}\|WL^{i-1}\| \cdot \|Lf-f\| \\
&\le \textstyle\sum_{i=n+1}^{k}Ml^{i-1}\|Lf-f\| \\
&\le Ml^k\textstyle\sum_{i=n+1}^{k}\|Lf-f\| \\
&\le Ml^k|k-n| \cdot \|Lf-f\| \\
&\le Ml^{k+n}|k-n| \cdot \|Lf-f\|.
\end{aligned}$$

Similarly, we can prove that $\|WL^kf - WL^nf\| \le Ml^{k+n}|k-n| \cdot \|Lf-f\|$ for $k < n$. It follows from (3.3) that

$$\begin{aligned}
\|e^{n(L-I)}f - L^nf\| &\le e^{-n}\textstyle\sum_{k=0}^{\infty}\frac{n^k}{k!}M\,l^{k+n}|k-n| \cdot \|Lf-f\| \\
&\le \frac{M\,l^n}{e^n}\|Lf-f\|\textstyle\sum_{k=0}^{\infty}\frac{n^k}{k!}|n-k|l^k \\
&= \frac{M\,l^n}{e^n}\|Lf-f\|\textstyle\sum_{k=0}^{\infty}\frac{(\sqrt{n}l)^k}{\sqrt{k!}}\frac{(\sqrt{n})^k(n-k)}{\sqrt{k!}}.
\end{aligned}$$

Then by Schwartz inequality for inner products,

$$\begin{aligned}
\|e^{n(L-I)}f - L^nf\| &\le \frac{M\,l^n}{e^n}\|Lf-f\|\left\{\textstyle\sum_{k=0}^{\infty}\frac{(nl^2)^k}{k!}\right\}^{1/2}\left\{\textstyle\sum_{k=0}^{\infty}\frac{n^k}{k!}(n-k)^2\right\}^{1/2} \\
&= \frac{M\,l^n}{e^n}\|Lf-f\|e^{nl^2/2}\left\{\textstyle\sum_{k=0}^{\infty}\frac{n^k}{k!}(n^2-2nk+k^2)\right\}^{1/2}.
\end{aligned}$$

But the summation $\sum_{k=0}^{\infty}\frac{n^k}{k!}(n^2-2nk+k^2)$ can be simplified into

$$\begin{aligned}
\sum_{k=0}^{\infty}\frac{n^k}{k!}(n^2-2nk+k^2) &= n^2e^n - 2n\sum_{k=1}^{\infty}\frac{n^k}{(k-1)!} + \sum_{k=1}^{\infty}\frac{kn^k}{(k-1)!}\\
&= n^2e^n - 2n^2e^n + \sum_{k=0}^{\infty}\frac{(k+1)n^{k+1}}{k!}\\
&= -n^2e^n + n\sum_{k=0}^{\infty}\frac{(k+1)n^k}{k!}\\
&= -n^2e^n + n(ne^n+e^n) \;=\; ne^n.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|e^{n(L-I)}f - L^nf\| &\le \tfrac{M\,l^n}{e^n}\|Lf-f\|e^{nl^2/2}(ne^n)^{1/2}\\
&= M\sqrt{n}\,l^ne^{(l^2-1)n/2}\|Lf-f\|.
\end{aligned}$$

□

Lemma 3.3 *Let $W(t), V(t)$ and A_s be as in Lemma 3.1. Then, for any $T>0$ and $f\in E$,*

$$\lim_{n\to\infty}\|W(t/n)V^n(t/n)f - W(t/n)e^{tA_{t/n}}f\| = 0 \tag{3.4}$$

uniformly for $t\in(0,T]$.

Proof: By the condition (3.1) and Lemma 3.1, $\|W(t/n)V^n(t/n)f - W(t/n)e^{tA_{t/n}}f\|$ is exponentially bounded. Since $D(A^\infty)$ is dense in E it suffices to show that (3.4) holds for $f\in D(A^\infty)$ by Banach's Convergence Theorem.

For $s\in(0,T]$, $\|W(s)V^n(s)\|\le Ml^n$, where $l=e^{\omega s}\ge 1$. Then, by Lemma 3.2,

$$\begin{aligned}
\|W(s)V^n(s)f - W(s)e^{n(V(s)-I)}f\| &\le M\sqrt{n}l^ne^{(l^2-1)n/2}\|V(s)f-f\|\\
&= M\sqrt{n}e^{n\omega s}e^{(e^{2\omega s}-1)n/2}\|V(s)f-f\|.
\end{aligned}$$

By setting $s=\frac{t}{n}$ and noticing that $e^{n(V(\frac{t}{n})-I)}f = e^{tA_{t/n}}f$, we obtain

$$\|W(\tfrac{t}{n})V^n(\tfrac{t}{n})f - W(\tfrac{t}{n})e^{tA_{t/n}}f\| \;\le\; M\sqrt{n}e^{\omega t}e^{(e^{2\omega t/n}-1)n/2}\|V(\tfrac{t}{n})f-f\|.$$

For any $\omega_o>\omega$, there exists n_o (dependent on T) such that for all $t\in(0,T]$, $e^{2\omega t/n}-1 < 2\omega_o t/n$ for all $n>n_o$. So, for all $n>n_o$,

$$\begin{aligned}
\|W(\tfrac{t}{n})V^n(\tfrac{t}{n})f - W(\tfrac{t}{n})e^{tA_{t/n}}f\| &\le M\sqrt{n}e^{\omega t}e^{\omega_o t}\|V(\tfrac{t}{n})f-f\|\\
&\le M\tfrac{t}{\sqrt{n}}e^{2\omega_o t}\tfrac{\|V(t/n)f-f\|}{t/n}.
\end{aligned}$$

Since $\lim_{n\to\infty}\frac{\|V(t/n)f-f\|}{t/n} = Af$ for $f\in\cap_{k=1}^{\infty}D(A^k)$, it follows that

$$\lim_{n\to\infty}\|W(t/n)V^n(t/n)f - W(t/n)e^{tA_{t/n}}f\| = 0$$

uniformly for $t\in(0,T]$. □

Proof of Theorem 3.1: For $f\in E$,

$$\begin{aligned}
\|W(\tfrac{t}{n})V^{n-1}(\tfrac{t}{n})f - S(t)f\| &\le \|W(\tfrac{t}{n})V^{n-1}(\tfrac{t}{n})f - W(\tfrac{t}{n})V^n(\tfrac{t}{n})f\|\\
&+ \|W(\tfrac{t}{n})V^n(\tfrac{t}{n})f - W(\tfrac{t}{n})e^{tA_{t/n}}f\|\\
&+ \|W(\tfrac{t}{n})e^{tA_{t/n}}f - S(t)f\|.
\end{aligned}$$

Let I_1, I_2 and I_3 denote the three components in the right hand side of the above inequality. We already know that $I_2 \to 0$ as $n \to \infty$ uniformly for $t \in [0,T]$ (by Lemma 3.3).

For I_1 we have that

$$\begin{aligned} I_1 &= \|W(\tfrac{t}{n})V^{n-1}(\tfrac{t}{n})\{V(\tfrac{t}{n})f - f\}\| \\ &\leq Me^{\omega t}\|V(\tfrac{t}{n})f - f\|. \end{aligned}$$

where the last inequality is due to the stability assumption (3.1). Since $V(0) = I$, the strong continuity of V implies that $I_1 \to 0$ uniformly for $t \in [0,T]$ as $n \to \infty$.

It remains to be shown that $I_3 \to 0$. By the definition of A_s, $\lim_{s\to 0} A_s f = V'(0)f = Af$ for $f \in \cap_{k=1}^{\infty} D(A^k)$. By Lemma 3.1, there exist $\omega_o, M_o > 0$ such that $\|W(s)e^{tA_s}\| \leq M_o e^{t\omega_o}$. Since W is a strongly continuous family of bounded linear operators and $W(0) = I$, $W(s)$ converges strongly to the identity operator I as $s \to 0$. Then, by Corollary 2.1, $I_3 \to 0$ uniformly for $t \in [0,T]$ as $n \to \infty$. □

3.2 Unstable Temporal Approximations

3.2.1 An Exponential Formula

In this subsection, we will generalize an exponential formula for semigroups generated by bounded linear operators [23, 24, 39] to one-parameter families of linear (possibly unbounded and even not closed) operators.

Lemma 3.4 *Let A be a linear operator and $f \in D(A^\infty)$. Suppose that there exists constants $M > 0$ and $\omega \geq 1$ such that*

$$\|A^n f\| \leq M\omega^n \quad \text{for all } n \in \mathbf{N}, \tag{3.5}$$

Let $t \mapsto e^{tA}f : [0,\infty) \to E$ be defined by $e^{tA}f := \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} f$. Then,

(i) *$e^{tA}f$ is well-defined, entire, and $\frac{d}{dt}e^{tA}f = e^{tA}Af = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} Af$;*

(i i) *$(\mathcal{L}_s e^{tA}f)(\lambda) = \int_0^\infty e^{-\lambda t} e^{tA} Af \; dt$ for all $f \in D(A^\infty)$ and $\lambda > \omega$;*

(iii) *for any $T > 0$, $e^{tA}f = \lim_{n\to\infty} (I + \frac{t}{n}A)^n f$ uniformly for $t \in [0,T]$.*

Proof: (i) It is obvious that the series $\sum_{n=0}^{\infty} \frac{t^n A^n}{n!} f$ is uniformly convergent for all $t \in \mathbf{C}$. Thus $e^{tA}f$ is well-defined, entire, and

$$\begin{aligned} \tfrac{d}{dt}e^{tA}f &= \textstyle\sum_{n=0}^{\infty} \tfrac{d}{dt}\tfrac{t^n A^n}{n!} f \\ &= \textstyle\sum_{n=0}^{\infty} \tfrac{t^n A^n}{n!} Af \;=\; e^{tA}Af. \end{aligned}$$

(ii) It is easily verifiable that condition (3.5) implies that $\|e^{tA}f\| \leq Me^{\omega t}$ and $\|e^{tA}Af\| \leq \omega Me^{\omega t}$ for $t \geq 0$. Then, by statement (i) of this lemma and Remark 2.1, the map $t \mapsto e^{tA}f$ has Laplace-Stieljes transform for $\lambda > \omega$ given by

$$\mathcal{L}_s(e^{tA}f)(\lambda) = \int_0^\infty e^{-\lambda t} e^{tA} Af \; dt.$$

$\square$

To prove statement (iii) of the lemma above, we need the following lemma which extends an estimate due to Chernoff [7].

Lemma 3.5 *Let L be a linear operator defined on $D(L) \subset E$. Suppose that there exists a constant $l \geq 1$ such that for each $f \in D(L^\infty)$, there exists an f-dependent constant M_f such that*

$$\|L^{n+1} f - L^n f\| \leq M_f\, l^n$$

for all $n \geq 0$. Then, for each $t \geq 0$, the operator $e^{t(L-I)} : f \mapsto \sum_{n=0}^{\infty} \frac{t^n (L-I)^n}{n!} f$ is well-defined on $D(L^\infty)$, and $\|e^{n(L-I)} f - L^n f\| \leq M_f \sqrt{n}\, l^n e^{(l^2-1)n/2}$ for $f \in D(L^\infty)$ and $n \in \mathbf{N}_0$.

Proof: For $f \in D(L^\infty)$ and $n \in \mathbf{N}$,

$$\begin{aligned}
\|(L-I)^n f\| &= \|(L-I)^{n-1}(L-I) f\| \\
&= \| \textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^{n-i} L^i (L-I) f\| \\
&\leq \textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} \|L^i (L-I) f\| \\
&\leq \textstyle\sum_{i=0}^{n-1} \binom{n-1}{i} M_f\, l^i \\
&= M_f (1+l)^{n-1}.
\end{aligned}$$

Then, by Lemma 3.4, $e^{t(L-I)} f$ is well-defined for all $t > 0$ and $f \in D(L^\infty)$.

It is well known that the series $\sum_{k=0}^{\infty} \frac{t^k}{k!}$ is absolutely convergent to e^t. Therefore the series $\sum_{k=0}^{\infty} \frac{t^k}{k!} L^k f$ converges to $e^{tL} f$ by the convergence theorem for the Cauchy product $e^t e^{t(L-I)} f$. Thus, we have that $e^{n(L-I)} f = e^{-n} \sum_{k=0}^{\infty} \frac{n^k}{k!} L^k f$ for all $f \in D(L^\infty)$, which implies that

$$\left\| e^{n(L-I)} f - L^n f \right\| = e^{-n} \left\| \sum_{k=0}^{\infty} \frac{n^k}{k!} (L^k - L^n) f \right\|. \tag{3.6}$$

Now, for $k \geq n$ and $f \in D(L^\infty)$,

$$\begin{aligned}
\|L^k f - L^n f\| &\leq \textstyle\sum_{i=n+1}^{k} \|L^i f - L^{i-1} f\| \\
&\leq \textstyle\sum_{i=n+1}^{k} M_f l^i \\
&\leq M_f l^{k+n} |k-n|.
\end{aligned}$$

Similarly we can prove that $\|L^k f - L^n f\| \leq M_f l^{k+n} |k-n|$ for $k < n$. It then follows from (3.6) that

$$\begin{aligned}
\|e^{n(L-I)} f - L^n f\| &\leq e^{-n} \textstyle\sum_{k=0}^{\infty} \frac{n^k}{k!} \|L^k f - L^n f\| \\
&\leq M_f e^{-n} l^n \textstyle\sum_{k=0}^{\infty} \frac{n^k}{k!} |n-k| l^k \\
&= M_f e^{-n} l^n \textstyle\sum_{k=0}^{\infty} \frac{(\sqrt{n} l)^k}{\sqrt{k!}} \frac{(\sqrt{n})^k (n-k)}{\sqrt{k!}}.
\end{aligned}$$

Then, by Schwartz inequality for inner products,

$$\begin{aligned}
\|e^{n(L-I)} f - L^n f\| &\leq \frac{M_f l^n}{e^n} \left\{ \textstyle\sum_{k=0}^{\infty} \frac{(n l^2)^k}{k!} \right\}^{1/2} \left\{ \textstyle\sum_{k=0}^{\infty} \frac{n^k}{k!} (n-k)^2 \right\}^{1/2} \\
&= \frac{M_f l^n}{e^n} e^{n l^2/2} \left\{ \textstyle\sum_{k=0}^{\infty} \frac{n^k}{k!} (n^2 - 2nk + k^2) \right\}^{1/2}.
\end{aligned}$$

But,

$$\begin{aligned}
\sum_{k=0}^{\infty} \tfrac{n^k}{k!}(n^2-2nk+k^2) &= n^2e^n - 2n\sum_{k=1}^{\infty}\tfrac{n^k}{(k-1)!} + \sum_{k=1}^{\infty}\tfrac{kn^k}{(k-1)!} \\
&= n^2e^n - 2n^2e^n + \sum_{k=0}^{\infty}\tfrac{(k+1)n^{k+1}}{k!} \\
&= -n^2e^n + n\sum_{k=0}^{\infty}\tfrac{(k+1)n^k}{k!} \\
&= -n^2e^n + n(ne^n + e^n) \;=\; ne^n.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|e^{n(L-I)}f - L^n f\| &\le \tfrac{M_f l^n}{e^n} e^{nl^2/2}(ne^n)^{1/2} \\
&= M_f\sqrt{n}\, l^n e^{(l^2-1)n/2}.
\end{aligned}$$

□

Proof of Lemma 3.4: (iii) For $t=0$, the limit in statement (iii) holds trivially.

For $s>0$, let $L_s := I+sA$. Then for $f\in D(A^\infty)$,

$$\begin{aligned}
\|L_s^{n+1}f - L_s^n f\| &= \|L_s^n(L_s f - f)\| \\
&= s\|(I+sA)^n Af\| \\
&= s\|\textstyle\sum_{k=0}^{n}\binom{n}{k} s^k A^k Af\| \\
&\le s\textstyle\sum_{k=0}^{n}\binom{n}{k} s^k M_f\omega^{k+1} \\
&= s\,\omega M_f(1+s\omega)^n \\
&\le G_f e^{n\omega s},
\end{aligned}$$

where $G_f = s\omega M_f$. It follows from Lemma 3.5 that

$$\|e^{n(L_s-I)}f - L_s^n f\| \;\le\; G_f\sqrt{n}e^{n\omega s}e^{(e^{2\omega s}-1)n/2} \quad \text{for } f\in D(L^\infty).$$

Replacing L_s by $I+sA$ and G_f by $s\omega M_f$, we obtain

$$\|e^{n\,s\,A}f - (I+sA)^n f\| \;\le\; s\omega M_f\sqrt{n}e^{n\omega s}e^{(e^{2\omega s}-1)n/2}.$$

Setting $s=\frac{t}{n}$ in the above inequality yields

$$\|e^{tA}f - (I+\tfrac{t}{n}A)^n f\| \le \tfrac{t}{\sqrt{n}}M_f e^{\omega t}e^{(e^{2\omega t/n}-1)n/2}.$$

For $t\in[0,T]$, there exists $N_T>0$ such that $e^{2\omega t/n}\le 1+\frac{4\omega t}{n}$ for all $n>N_T$. Then

$$\begin{aligned}
\|e^{tA}f - (I+\tfrac{t}{n}A)^n f\| &\le \tfrac{t}{\sqrt{n}}M_f e^{\omega t}e^{(1+4\omega t/n-1)n/2} \\
&\le \tfrac{t}{\sqrt{n}}M_f e^{3\omega t},
\end{aligned}$$

which implies that $\lim_{n\to\infty}(I+\frac{t}{n}A)^n f = e^{tA}f$. □

3.2.2 Von-Neumann Stable Temporal Approximations

In this section, we extend the Chernoff product formula to possibly unstable approximation methods.

Theorem 3.2 *Suppose that A generates a C_0-semigroup S. Let $V := \{V(t) : t \in [0,\delta]\}$ be a family of linear operators satisfying $D(A) \subset D(V(t)^\infty)$ for all $t \in [0,\delta]$. Suppose that the map $t \mapsto V(t)f$ is continuous for all $f \in D(A)$. If there exists a constant $\omega \geq 0$ such that*

(a) *for all $f \in D(A)$, $t \in [0,\delta]$ and $n \in \mathbf{N}$, $\|V^{n+1}(t)f - V^n(t)f\| \leq tM_f e^{n\omega t}$ for some f-dependent positive number M_f, and*

(b) *there exists a family $\{W(t) : t \in [0,\delta]\}$ of operators defined on $D(A^2)$ such that for each $f \in D(A^2)$ the map $t \mapsto W(t)f$ is continuous, $W(0)f = 0$, and $\|V^{n+1}(t)f - V^n(t)f - t\,V^n(t)Af\| \leq t\,e^{n\omega t}\|W(t)f\|$ for all $t \in [0,\delta]$ and $n \in \mathbf{N}$,*

then $\lim_{n\to\infty} V(\frac{t}{n})^n f = S(t)f$ for each $f \in D(A)$, uniformly for t in compact interval $[0,T]$ for any $T > 0$.

Before proceeding to prove Theorem 3.2, we first discuss connections of this theorem to the Chernoff product formula and to the Von Neumann stability condition.

Remark 3.1 *If a temporal approximation method V is consistent on $D(A)$, and stable in the classical sense that*

$$\|V(t)^n\| \leq Me^{n\omega t}, \quad \text{for all } (t,n) \in [0,\delta] \times \mathbf{N} \tag{3.7}$$

for some positive constant M, ω independent of n and t, then conditions (a) *and* (b) *of Theorem 3.2 hold. Hence, Theorem 3.2 is a generalization of the Chernoff product approximation formula.*

Proof of Remark 3.1: For $f \in D(A)$, $t \in [0,\delta)$ and $n \in \mathbf{N}$,

$$\begin{aligned} \|V(t)^{n+1}f - V(t)^n f\| &\leq \|V(t)^n\| \cdot \|V(t)f - f\| \\ &\leq Me^{n\omega t}\|t \cdot \tfrac{V(t)f - f}{t}\|. \end{aligned} \tag{3.8}$$

Since V is consistent on $D(A)$, it follows that the limit $lim_{t\to 0}\frac{V(t)f-f}{t}$ exists and hence $\|\frac{V(t)f-f}{t}\|$ is bounded on $(0,\delta)$. Without loss of generality, we assume that $\sup\limits_{t\in(0,\delta)}\|\frac{V(t)f-f}{t}\| \leq \frac{M_f}{M}$. Then it follows from (3.8) that $\|V^{n+1}(t)f - V^n(t)f\| \leq tM_f e^{n\omega t}$, which is condition (a) of Theorem 3.2.

For $f \in D(A^2)$, $t \in [0,\delta)$ and $n \in \mathbf{N}$,

$$\begin{aligned} \|V(t)^{n+1}f - V(t)^n f - tV(t)^n Af\| &\leq \|V(t)^n\| \cdot \|V(t)f - f - tAf\| \\ &\leq Me^{n\omega t}\|t \cdot \tfrac{V(t)f-f}{t} - tAf\| \\ &= tMe^{n\omega t}\|\tfrac{V(t)f-f}{t} - Af\|. \end{aligned} \tag{3.9}$$

For $t \in [0,\delta]$, define a family W(t) of operators on $D(A)$ by

$$W(t)f := \begin{cases} M\left(\frac{V(t)-I}{t}f - Af\right) & \text{for } t > 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Then, from (3.9) we obtain that

$$\|V(t)^{n+1}f - V(t)^n f - tV(t)^n Af\| \quad \leq \quad te^{n\omega t}\|W(t)f\|$$

for $f \in D(A^2)$. It is obvious that the function $t \mapsto W(t)f$ is continuous for $t \in (0, \delta]$ for each $f \in D(A)$. Since V is consistent on $D(A)$, it follows that for each $f \in D(A)$, the function $t \mapsto W(t)f$ is continuous at $t = 0$. This shows that V also satisfies condition (b) of Theorem 3.2 □

Remark 3.2 *Condition* (a) *of Theorem 3.2 implies that* V *satisfies the Von Neumann stability condition (1.21) on* $D(A)$, *namely, for all* $f \in D(A)$,

$$\lim_{n\to\infty} (\|V(t)^n f\|)^{1/n} \leq e^{\omega t}. \tag{3.10}$$

Proof: Let $f \in D(A)$. The telescoping equality $V(t)^n f = \sum_{k=0}^{n-1} \left[V(t)^{k+1} f - V(t)^k f\right] + f$ yields that $\|V(t)^n f\| \leq \sum_{k=0}^{n-1} \|V(t)^{k+1} f - V(t)^k f\| + \|f\|$. Then, condition (a) of Theorem 3.2 implies that

$$\begin{aligned}
\|V(t)^n f\| &\leq \sum_{k=0}^{n-1} tM_f e^{k\omega t} + \|f\| \\
&\leq ntM_f e^{n\omega t} + \|f\| \\
&\leq e^{nt} M_f e^{n\omega t} + \|f\| \\
&\leq (M_f + \|f\|)e^{n(\omega+1)t},
\end{aligned}$$

from which (3.10) follows immediately. □

Similarly, we can show that the spatially discretized temporal approximation method $V(t; A_h)$ also satisfies the Von Neumann stability condition (1.20) when condition (a) of Theorem 3.2 is preserved for the method $V(t; A_h)$. Namely, the following remark holds.

Remark 3.3 *If a spatially discretized temporal approximation method* $\{V(t, A_h) : E_h \to E_h : t \in [0, \delta], h \in (0, \varepsilon)\}$ *satisfies*

$$\|V(t; A_h)^{n+1} f_h - V(t; A_h)^n f_h\| \leq tM_{f_h} e^{n\omega t} \tag{3.11}$$

for all $(t, h) \in [0, \delta] \times (0, \varepsilon)$ *and* $n \in \mathbf{N}$. *Then the Von Neumann stability condition (1.20) holds for the approximation method* $\{V(t; A_h) : (t, h) \in [0, \delta] \times (0, \varepsilon)\}$.

Now, we accumulate two lemmas for the proof of Theorem 3.2.

Lemma 3.6 *Let* $V(t)$ *be as in Theorem 3.2. Let* $T > 0$. *For each* $s \in (0, T]$, *define* $A_s f := \frac{V(s)-I}{s} f$ *for* $f \in D(A)$. *Then condition* (a) *of Theorem 3.2 implies the following.*

(i) *For* $f \in D(A)$,

$$\lim_{n\to\infty} \|V^n(t/n)f - e^{tA_{t/n}} f\| = 0 \tag{3.12}$$

uniformly for $t \in (0, T]$.

(i i) *For* $f \in D(A)$ *and* $t \geq 0$*, the function* $t \mapsto e^{tA_s} A_s f := \sum k = 0^{\infty} \frac{t^k A_s^k}{k!} A_s f$ *is well defined, and for any* $\omega_o > \omega$ *there exist* $s_o \in (0, T]$ *such that for each* $s \in (0, s_o)$*,* $\|e^{tA_s} A_s f\| < M_f e^{\omega_o t}$ *for all* $f \in D(A)$ *and* $t \geq 0$*.*

(iii) *For* $f \in D(A)$ *and* $t \geq 0$*, the function* $t \mapsto e^{tA_s} f := \sum_{k=0}^{\infty} \frac{t^k A_s^k}{k!} f$ *is well defined, and for any* $\omega_o > \omega$ *there exist* $s_o \in (0, T]$ *such that for each* $s \in (0, s_o)$*,* $\|e^{tA_s} f\| < (\|f\| + \frac{M_f}{\omega_o}) e^{\omega_o t}$ *for all* $f \in D(A)$ *and* $t \geq 0$*.*

(iv) *For any* $\omega_o > \omega$ *there exist* $s_o \in (0, T]$ *such that for each* $s \in (0, s_o)$ *the integrals* $R_s(\lambda) g := \int_0^{\infty} e^{-\lambda t} e^{tA_s} g \, dt$ *exist for all* $g \in D(A) \cup A_s(D(A))$ *and* $\lambda > \omega_o$*, and the equality* $R_s(\lambda)(\lambda - A_s) f = f$ *holds for all* $\lambda > \omega_o$ *and* $f \in D(A)$*.*

Proof: (i) For $t = 0$, (3.12) is obviously true. For $t \in (0, T]$, condition (a) of Theorem 3.2 implies that

$$\begin{aligned} \|V^n(s) f - V^{n-1}(s) f\| &\leq sM_f e^{ns\omega} \\ &= G_{s,f}\, l^n, \end{aligned}$$

where $G_{s,f} = sM_f$ and $l = e^{s\omega} \geq 1$. Then, by Lemma 3.5,

$$\begin{aligned} \|V^n(s) f - e^{n[V(s)-I]} f\| &\leq G_{s,f} \sqrt{n} l^n e^{(l^2-1)n/2} \\ &= sM_f \sqrt{n} e^{ns\omega} e^{(e^{2s\omega}-1)n/2}. \end{aligned} \tag{3.13}$$

By definition of A_s, $e^{n[V(s)-I]} f = e^{nsA_s} f$. Thus taking $s = t/n$, we have that $e^{n[V(t/n)-I]} f = e^{tA_{t/n}}$. Then it follows from (3.13) that

$$\|V\left(\tfrac{t}{n}\right)^n f - e^{tA_{t/n}} f\| \leq M_f \frac{t}{\sqrt{n}} e^{\omega t} e^{(e^{2\omega t/n}-1)n/2}.$$

For any $\omega_o > \omega$, there exists n_o (dependent on T) such that $e^{2\omega t/n} - 1 < 2\omega_o t/n$ for all $n > n_o$. So for all $n > n_o$,

$$\begin{aligned} \|V^n(\tfrac{t}{n}) f - e^{tA_{t/n}} f\| &\leq M_f \frac{t}{\sqrt{n}} e^{\omega t} e^{\omega_o t} \\ &= M_f \frac{t}{\sqrt{n}} e^{\omega t} e^{\omega_o t}, \end{aligned}$$

from which statement (i) follows.

(ii) For $f \in D(A)$ and $n \in \mathbf{N}$,

$$\begin{aligned} \|A_s^n A_s f\| &= \| \left(\tfrac{V(s)-I}{s}\right)^n \left(\tfrac{V(s)-I}{s}\right) f\| \\ &= s^{-n} \| \textstyle\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} V^i(s) \frac{V(s)-I}{s} f\| \\ &\leq s^{-n} \textstyle\sum_{i=0}^{n} \binom{n}{i} M_f e^{i\omega s} \\ &= M_f \left(\tfrac{1+e^{\omega s}}{s}\right)^n, \end{aligned}$$

where the last inequality is due to condition (a) of Theorem 3.2. Then, by Lemma 3.4, $e^{tA_s} A_s f$ is well-defined for $f \in D(A)$ and $t \geq 0$.

Statement (iii) of Lemma 3.4 implies that

$$\begin{aligned} \|e^{tA_s}A_sf\| &= \|\lim_{m\to\infty}\left(I+\tfrac{t}{m}A_s\right)^m A_sf\| \\ &= \lim_{m\to\infty}\|[\lambda_m+\mu_m V(s)]^m A_sf\|, \end{aligned} \tag{3.14}$$

where $\lambda_m = \frac{ms-t}{ms}$ and $\mu_m = 1-\lambda_m = \frac{t}{ms}$. Since

$$\begin{aligned} \|\left(\lambda_m+\mu_m V(s)\right)^m A_sf\| &= \|\textstyle\sum_{i=0}^m \binom{m}{i}\lambda_m^{m-i}\mu_m^i V^i(s)\frac{V(s)-I}{s}f\| \\ &\le \textstyle\sum_{i=0}^m \binom{m}{i}\lambda_m^{m-i}\mu_m^i M_f e^{i\omega s} \\ &= M_f\left(\lambda_m+\mu_m e^{\omega s}\right)^m, \end{aligned}$$

it follows from (3.14) that $\|e^{tA_s}A_sf\| \le M_f\lim_{m\to\infty}[1+\mu_m(e^{\omega s}-1)]^m$ by noticing that $\lambda_m = 1-\mu_m$. For $\omega_o > \omega$, there exists $s_o > 0$ such that $e^{\omega s}-1 \le \omega_o s$ for all $s\in(0,s_o]$. Then,

$$\begin{aligned} \|e^{tA_s}A_sf\| &\le M_f\lim_{m\to\infty}\left(1+\mu_m\omega_o s\right)^m \\ &= M_f\lim_{m\to\infty}\left(1+\tfrac{\omega_o t}{m}\right)^m \\ &\le M_f e^{\omega_o t}. \end{aligned}$$

(iii) By an argument similar to that in the proof of statement (i), we can show that $e^{tA_s}f$ is well defined for $f\in D(A)$ and $t\ge 0$. Now, by statement (i) of Lemma 3.4, $e^{tA_s}f = f+\int_0^t e^{\tau A_s}A_sf\,d\tau$. Then by (i),

$$\|e^{tA_s}f\| \;\le\; \|f\| + \int_0^t M_f e^{\omega_o\tau}d\tau \;\le\; (\|f\|+\frac{M_f}{\omega_o})e^{\omega_o\tau}.$$

(iv) By the two estimates established in statements (ii) and (iii) for $e^{tA_s}f$ and $e^{tA_s}A_sf$, the integrals $R_s(\lambda)g = \int_0^\infty e^{-\lambda t}e^{tA_s}g\,dt$ exist for all $s\in(0,s_o)$, $\lambda>\omega_o$ and $g\in D(A)\cup A_s(D(A))$.

By statement (i) of Lemma 3.4, we have that $e^{tA_s}f-f = \int_0^t e^{\tau A_s}A_sf d\tau$. Then it follows from Remark 2.1 that for $s\in(0,s_o), \lambda>\omega_o$ and $f\in D(A)$,

$$\begin{aligned} [\mathcal{L}_s(e^{\cdot A_s}f-f)](\lambda) &= \textstyle\int_0^\infty e^{-\lambda t}e^{tA_s}A_sf\,dt \\ &= R_s(\lambda)A_sf. \end{aligned} \tag{3.15}$$

On the other hand, with the estimate established for $\|e^{tA_s}f\|$ in statement (iii), integrating by parts we obtain

$$\begin{aligned} [\mathcal{L}_s(e^{\cdot A_s}f-f)](\lambda) &= \textstyle\int_0^\infty e^{-\lambda t}d(e^{tA_s}f-f) \\ &= \lambda\textstyle\int_0^\infty e^{-\lambda t}(e^{tA_s}f-f)dt \\ &= -f+\lambda R_s(\lambda)f, \end{aligned}$$

which, together with (3.15), implies that

$$R_s(\lambda)(\lambda-A_s)f = f.$$

□

Lemma 3.7 *Let A_s be as defined in Lemma 3.6. Then condition* (b) *of Theorem 3.2 implies that for any $\omega_o > \omega$, there exist $s_o \in (0,T]$ such that for $s \in (0,s_o)$ and $f \in D(A^2)$, $e^{tA_s}(A_s - A)f$ is well-defined and $\|e^{tA_s}(A_s - A)f\| \le e^{\omega_o t}\|W(s)f\|$ for $t \ge 0$, and $\lim_{s\to 0} R_s(\lambda)(A_s - A)f = 0$.*

Proof: For $f \in D(A^2)$ and $n \in \mathbf{N}$,

$$\begin{aligned}
\|A_s^n(A_s - A)f\| &= \| \left(\tfrac{V(s)-I}{s}\right)^n \left(\tfrac{V(s)-I}{s} - A\right) f\| \\
&= s^{-n}\| \textstyle\sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} V^i(s) \left(\tfrac{V(s)-I}{s} - A\right) f\| \\
&\le s^{-n} \textstyle\sum_{i=0}^{n} \binom{n}{i} e^{i\omega s}\|W(s)f\| \\
&= \left(\tfrac{1+e^{\omega s}}{s}\right)^n \|W(s)f\|,
\end{aligned}$$

where the last inequality is due to condition (b) of Theorem 3.2. Then, by Lemma 3.4, $e^{tA_s}(A_s - A)f$ is well-defined for $f \in D(A)$ and $t \ge 0$.

Statement (iii) of Lemma 3.4 implies that

$$\begin{aligned}
\|e^{tA_s}(A_s - A)f\| &= \| \lim_{m\to\infty} \left(I + \tfrac{t}{m}A_s\right)^m (A_s - A)f\| \\
&= \lim_{m\to\infty} \|[\lambda_m + \mu_m V(s)]^m (A_s - A)f\|,
\end{aligned} \tag{3.16}$$

where $\lambda_m = \frac{ms-t}{ms}$ and $\mu_m = 1 - \lambda_m = \frac{t}{ms}$. Since

$$\begin{aligned}
\| \left(\lambda_m + \mu_m V(s)\right)^m A_s f\| &= \| \textstyle\sum_{i=0}^{m} \binom{m}{i} \lambda_m^{m-i}\mu_m^i V^i(s) \left(\tfrac{V(s)-I}{s} - A\right) f\| \\
&\le \textstyle\sum_{i=0}^{m} \binom{m}{i} \lambda_m^{m-i}\mu_m^i e^{i\omega s}\|W(s)f\| \\
&= \left(\lambda_m + \mu_m e^{\omega s}\right)^m \|W(s)f\|,
\end{aligned}$$

it follows from (3.16) that $\|e^{tA_s}(A_s - A)f\| \le \lim_{m\to\infty} [1 + \mu_m(e^{\omega s} - 1)]^m \|W(s)f\|$ by noticing that $\lambda_m = 1 - \mu_m$. For $\omega_o > \omega$, there exists $s_o > 0$ such that $e^{\omega s} - 1 \le \omega_o s$ for all $s \in (0, s_o]$. Then,

$$\begin{aligned}
\|e^{tA_s}(A_s - A)f\| &\le \lim_{m\to\infty} \left(1 + \mu_m \omega_o s\right)^m \|W(s)f\| \\
&= \lim_{m\to\infty} \left(1 + \tfrac{\omega_o t}{m}\right)^m \|W(s)f\| \\
&\le e^{\omega_o t}\|W(s)f\|.
\end{aligned}$$

With the exponential bound established above for $e^{tA_s}(A_s - A)f$, it follows that the integrals $R_s(\lambda)(A_s - A)f$ exist for all $f \in D(A^2)$ and $s \in (0, s_o)$, and

$$\begin{aligned}
\|R_s(\lambda)(A_s - A)f\| &\le \textstyle\int_0^\infty e^{-\lambda t}\|e^{tA_s}(A_s - A)f\|dt \\
&\le \textstyle\int_0^\infty e^{-\lambda t}e^{\omega_o t}\|W(s)f\|dt \\
&= (\lambda - \omega_o)^{-1}\|W(s)f\|.
\end{aligned}$$

Since W is strongly continuous and $W(0)g = 0$ for $g \in D(A)$, we have that $\lim_{s\to 0} R_s(\lambda)(A_s - A)f = 0$. □

Proof of Theorem 3.2: For $f \in D(A)$,

$$\begin{aligned}
\|V^n(\tfrac{t}{n})f - S(t)f\| &\le \|V^n(\tfrac{t}{n})f - e^{tA_{t/n}}f\| \\
&+ \|e^{tA_{t/n}}f - S(t)f\|.
\end{aligned} \tag{3.17}$$

Since $\lim_{n\to\infty} \|V^n(\frac{t}{n})f - e^{tA_{t/n}}f\| = 0$ by Lemma 3.6, it remains to be shown that

$$\lim_{n\to\infty} e^{tA_{t/n}} f = S(t)f. \tag{3.18}$$

Let R_s and ω_o be as in Lemma 3.6. For $f \in D(A)$ and $\lambda > \omega_o$, we obtain from statement (iv) of Lemma 3.6 that $R(\lambda, A)f = R_s(\lambda)(\lambda - A_s)R(\lambda, A)f$. Then,

$$\begin{aligned} R_s(\lambda)f - R(\lambda, A)f &= R_s(\lambda)f - R_s(\lambda)(\lambda - A_s)R(\lambda, A)f \\ &= R_s(\lambda)[(\lambda - A) - (\lambda - A_s)]R(\lambda, A)f \\ &= R_s(\lambda)(A_s - A)R(\lambda, A)f, \end{aligned} \tag{3.19}$$

Since $f \in D(A)$ implies that $R(\lambda, A)f \in D(A^2)$, it follows from Lemma 3.7 and (3.19) that $\lim_{n\to\infty} R_s(\lambda)f = R(\lambda, A)f$ for all $\lambda > \omega_o$. Then, with the boundedness condition for $e^{tA_s}A_s f$ proven in statement (ii) of Lemma 3.6, it follows from Theorem 2.4 and statement (i) of Lemma 3.4 that $\lim_{s\to 0} e^{tA_s} f = S(t)f$ uniformly for $t \in [0, T]$ for all $f \in D(A)$, from which statement (3.18) follows. □

One well-accepted meaning of stability for convergent methods that is also implied by definition (1.4) is that when the initial value has a small error, the approximated solution has a small error too. However, approximation methods satisfying only conditions (i) and (ii) of Theorem 3.2 may not necessarily be stable in this sense, but are stable in a sense very close to this.

Theorem 3.3 *Suppose that condition* (a) *of Theorem 3.2 is strengthened to the following. There exists a constant $M > 0$ such that for all $f \in D(A), t \in [0, \delta]$ and $n \in \mathbf{N}$,*

$$\|V^{n+1}(t)f - V^n(t)f\| \le tMe^{n\omega t}(\|f\| + \|Af\|). \tag{3.20}$$

Then for each $f \in D(A)$ and each sequence $\{f_n\} \subset D(A)$ with

$$\lim_{n\to\infty} \|f_n - f\| + \|A(f_n - f)\| = 0, \tag{3.21}$$

$\lim_{n\to\infty} V(\frac{t}{n})^n f_n = S(t)f$ *uniformly for t in compact interval $[0, T]$ for any $T > 0$.*

Proof: For $f \in D(A)$, Theorem 3.2 implies that $\lim_{n\to\infty} V(\frac{t}{n})^n f = S(t)f$ uniformly for $t \in [0, T]$. It then suffices to show that for any sequence $\{f_n\}$ satisfying (3.21),

$$\lim_{n\to\infty} V\left(\tfrac{t}{n}\right)^n (f_n - f) = 0 \tag{3.22}$$

uniformly for $t \in [0, T]$.

For $f \in D(A)$ and a sequence $\{f_n\}$ satisfying (3.21), we have that

$$\begin{aligned} \|V(\tfrac{t}{n})^n(f_n - f)\| &= \|\textstyle\sum_{k=1}^n \left[V(\frac{t}{n})^k(f_n - f) - V(\frac{t}{n})^{k-1}(f_n - f)\right] + (f_n - f)\| \\ &\le \|f_n - f\| + \textstyle\sum_{k=1}^n \|V(\frac{t}{n})^k(f_n - f) - V(\frac{t}{n})^{k-1}(f_n - f)\|., \end{aligned}$$

It then follows from (3.20) that

$$\begin{array}{rcl}
\|V(\frac{t}{n})^n(f_n-f)\| & \leq & \|f_n-f\| + \sum_{k=1}^n \frac{t}{n} M e^{k\omega t/n}\,[\|f_n-f\| + \|A(f_n-f)\|] \\
& \leq & \|f_n-f\| + \sum_{k=1}^n \frac{t}{n} M e^{\omega t}\,[\|f_n-f\| + \|A(f_n-f)\|] \\
& = & \|f_n-f\| + tMe^{\omega t}\,[\|f_n-f\| + \|A(f_n-f)\|] \\
& \leq & (1+tMe^{\omega t})\,[\|f_n-f\| + \|A(f_n-f)\|]\,,
\end{array}$$

Then statement (3.22) follows from (3.21). □

4 Applications

In this chapter, we apply Theorem 3.3 to the unstable but Von-Neumann stable temporal approximation method given in Chapter 1, an ADI type factorized temporal discretization method, and a domain decomposition based factorized temporal approximation method.

4.1 The Unstable but Von Neumann Stable Method

In Section 1.2, we have considered the evolutionary system

$$\begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = f \in D(A), \end{cases}$$

on the Hilbert space $E := \{f \in L^2[0,\pi] : f(0) = f(\pi) = 0\}$ with $A + A_1 + A_2$, where $A_1 f = f''$, $A_2 f = \sum_{k=1}^{\infty} \langle f, e_{k^2} \rangle e_k$. We also investigated the temporal approximation method

$$V(t)f := (I - \frac{t}{2}A_1)^{-1}(I + tA_2)(I + \frac{t}{2}A_1)f,$$

and showed that it is unstable in the classical sense but satisfies the Von Neumann stability condition (1.21). Now as an application of Theorem 3.3, we shall prove that this temporal discretization method satisfies the following statement.

Statement 4.1 *For $f \in D(A)$, and any sequence $\{f_n\}_{n=1}^{\infty} \subset D(A)$ with*

$$\lim_{n \to \infty} \|f_n - f\| + \|f_n'' - f''\| = 0,$$

$\lim_{n\to\infty} V(\frac{t}{n})^n f_n = S(t)f$ *uniformly for $t \in [0,T]$ for any $T \geq 0$, where S is the semigroup generated by A.*

Proof: Since A_2 is bounded, $D(A) = D(A_1)$. Thus $D(A) \subset D(V(t)^\infty)$. Since A_1 generates a contraction semigroup and A_2 is bounded, it follows immediately that the map $t \mapsto V(t)f$ is continuous for all $f \in D(A)$.

The three inequalities (1.23), (1.24) and (1.26) are combined into

$$\begin{cases} \|(I - \frac{t}{2}A_1)^{-1}\| \leq 1, \\ \|(I + \frac{t}{2}A_1)(I - \frac{t}{2}A_1)^{-1}\| \leq 1, \\ \|I + tA_2\| \leq 1 + t. \end{cases} \tag{4.1}$$

For $t \geq 0$, denote $Y(t) := (I + tA_2)(I + \frac{t}{2}A_1)(I - \frac{t}{2}A_1)^{-1}$, that is,

$$V(t) = (I - \frac{t}{2}A_1)^{-1} Y(t)(I - \frac{t}{2}A_1).$$

Then (4.1) implies that $\|Y(t)\| \leq 1 + t$. Now for $f \in D(A)$ and $n \in \mathbf{N}$,

$$\begin{aligned} \|V(t)^{n+1}f - V(t)^n f\| &= \|V(t)^n [V(t) - I] f\| \\ &= \|(I - \tfrac{t}{2}A_1)^{-1} Y(t)^n \left[(I + tA_2)(I + \tfrac{t}{2}A_1) - (I - \tfrac{t}{2}A_1)\right] f\| \\ &= t\|(I - \tfrac{t}{2}A_1)^{-1} Y(t)^n \left[(I + \tfrac{t}{2}A_2)A - \tfrac{t}{2}A_2^2\right] f\| \\ &\leq t\|Y(t)^n \left[(I + \tfrac{t}{2}A_2)A - \tfrac{t}{2}A_2^2\right] f\|, \end{aligned}$$

where the last inequality is du to (4.1). From the inequality $\|Y(t)\| \le 1+t$ we have that

$$\begin{aligned}\|V(t)^{n+1}f - V(t)^n f\| &\le t(1+t)^n \| \left[(I+\tfrac{t}{2}A_2)A - \tfrac{t}{2}A_2^2\right] f\| \\ &\le t\, e^{nt} \left[\|(I+\tfrac{t}{2}A_2)Af\| + \|\tfrac{t}{2}A_2^2 f\|\right] \\ &\le t\, e^{nt} \left[(1+\tfrac{t}{2})\|Af\| + \tfrac{t}{2}\|f\|\right] \\ &\le t\, e^{nt} \left[e^t\|Af\| + e^t\|f\|\right] \\ &\le t\, e^{2nt} \left[\|Af\| + \|f\|\right].\end{aligned}$$

Now, for $f \in D(A^2)$ and $n \in \mathbf{N}$,

$$\begin{aligned}\|V(t)^{n+1}f - V(t)^n f - tV(t)^n Af\| &= \|V(t)^n \left[V(t) - I - tA\right] f\| \\ &= \|(I-\tfrac{t}{2}A_1)^{-1}Y(t)^n[(I+tA_2)(I+\tfrac{t}{2}A_1)-(I-\tfrac{t}{2}A_1)(I-tA)]f\| \\ &= t\|(I-\tfrac{t}{2}A_1)^{-1}Y(t)^n[\tfrac{t}{2}A^2 - \tfrac{t}{2}A_2^2]f\| \\ &\le t(1+t)^n\|\tfrac{t}{2}(A^2 - A_2^2)f\| \\ &\le t\, e^{2nt}\|\tfrac{t}{2}(A^2 - A_2^2)f\|.\end{aligned}$$

Set $W(t) := \frac{t}{2}(A^2 - A_2^2)$. Obviously, the map $t \mapsto W(t)f$ is continuous and $W(0)f = 0$ for all $f \in D(A^2)$. Then, by Theorem 3.3, $\lim_{n\to\infty} V\left(\frac{t}{n}\right)^n f_n = S(t)f$ uniformly for $t \in [0,T]$ for $T \ge 0$. □

4.2 ADI-type Factorized Methods

In this section, we use Theorem 3.3 to exmaine the convergence of the popular ADI-type temporal approximation methods for quasi-dissipative operators in a Hilbert space.

Lemma 4.1 *Let E be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let A be a C_0-semigroup generator on E satisfying the quasi-dissipative conditon*

$$Re\,\langle Af, f\rangle \le \omega\|f\|^2, \tag{4.2}$$

for some constant $\omega \ge 0$. Then the Crank-Nicolson method $V(t) = \left(I - \frac{t}{2}A\right)^{-1}\left(I + \frac{t}{2}A\right)$ satisfies the stability condition

$$\|V(t)\| \le e^{2\omega t} \quad \textit{for all } t \in [0, \tfrac{1}{\omega}]. \tag{4.3}$$

Proof: Since A generates a C_0-semigroup, it follows that $(I - tA)^{-1} \in \mathbf{L}(E)$ for $t \in [0, \frac{1}{\omega})$. Since $(I - \frac{t}{2}A)^{-1}$ and $(I + \frac{t}{2}A)$ commute on $D(A)$ for $t \in [0, \frac{1}{\omega}]$, it follows that the stability conclusion (4.3) is equivalent to that $\|(I + \frac{t}{2}A)(I - \frac{t}{2}A)^{-1}g\| \le e^{2\omega t}\|g\|$ for all $g \in E$. Replacing $(I - \frac{t}{2}A)^{-1}g$ by f, this is equivalent to

$$\|(I + \tfrac{t}{2}A)f\| \le e^{2\omega t}\|(I - \tfrac{t}{2}A)f\| \quad \text{for } t \in [0, \tfrac{1}{\omega}], f \in D(A). \tag{4.4}$$

Squaring the left hand side of the above inequality, we obtain

$$\begin{aligned}\|(I + \tfrac{t}{2}A)f\|^2 &= \|(1 + \tfrac{\omega t}{2})f + \tfrac{t}{2}(A-\omega)f\|^2 \\ &= (1+\tfrac{\omega t}{2})^2\|f\|^2 + t(1+\tfrac{\omega t}{2})\, Re\,\langle (A-\omega)f, f\rangle + \tfrac{t^2}{4}\|(A-\omega)f\|^2.\end{aligned}$$

Squaring the right hand side of inequality (4.4), we obtain

$$\begin{aligned} e^{4\omega t}\|(I-\tfrac{t}{2}A)f\|^2 &= e^{4\omega t}\|(1-\tfrac{\omega t}{2})I-\tfrac{t}{2}(A-\omega)f\|^2 \\ &= e^{4\omega t}\left[(1-\tfrac{\omega t}{2})^2\|f\|^2-t(1-\tfrac{\omega t}{2})Re\,\langle (A-\omega)f,f\rangle+\tfrac{t^2\|(A-\omega)f\|^2}{4}\right]. \end{aligned}$$

Therefore, (4.4) holds as long as

$$t[(1+\tfrac{\omega t}{2})+(1-\tfrac{\omega t}{2})e^{4\omega t}]Re\,\langle (A-\omega)f,f\rangle \le \left[(1-\tfrac{\omega t}{2})^2e^{4\omega t}-(1+\tfrac{\omega t}{2})^2\right]\|f\|^2 \tag{4.5}$$

for all $t\in[0,\frac{1}{\omega}]$ and $f\in D(A)$. Since $t[(1+\frac{\omega t}{2})+(1-\frac{\omega t}{2})e^{4\omega t}]>0$ for all $t\in[0,\frac{1}{\omega}]$, and $Re\,\langle (A-\omega)f,f\rangle \le 0$ by the quasi-dissipativity assumption, it follows that (4.5) holds if $(1-\frac{\omega t}{2})^2e^{4\omega t}-(1+\frac{\omega t}{2})^2\ge 0$ for $t\in[0,\frac{1}{\omega}]$, which is obviously equivalent to

$$(1-\tfrac{\omega t}{2})e^{2\omega t}-(1+\tfrac{\omega t}{2})\ge 0 \quad \text{for } t\in[0,\tfrac{1}{\omega}], \tag{4.6}$$

Let $h(t):=(1-\frac{t}{2})e^{2t}-(1+\frac{t}{2})$ for $t\in[0,1]$. Obviously h is differentiable on $[0,1]$ and $h'(t)=(\frac{3}{2}-t)e^{2t}-\frac{1}{2}$. Since $h'(t)\ge(1-t)e^{2t}\ge 0$ for $t\in[0,1]$, it follows that the function $t\mapsto h(t)$ is increasing on $[0,1]$. Again since $h(0)=0$, it follows that $h(t)\ge 0$ for $t\in[0,1]$. Therefore, $h(\omega t)\ge 0$ for $t\in[0,\frac{1}{\omega}]$, which is exactly (4.6). □

Theorem 4.1 *Suppose that $A=A_1+A_2$ with $D(A)\subset D(A_1)\cap D(A_2)$ generates a C_0-semigroup S, and there exists a constant $\omega\ge 0$ such that*

$$\mathrm{Re}\langle A_if,f\rangle \le \omega\|f\|^2 \quad \textit{for all } f\in D(A) \tag{4.7}$$

for $i=1,2$. Let $\{V(t):t\in[0,\frac{1}{\omega}]$ be a family of linear operators given by

$$V(t)=\left(I-\tfrac{t}{2}A_1\right)^{-1}\left(I-\tfrac{t}{2}A_2\right)^{-1}\left(I+\tfrac{t}{2}A_2\right)\left(I+\tfrac{t}{2}A_1\right).$$

Then, for $f\in D(A)$ and $\{f_n\}_{n=1}^{\infty}\subset D(A)$ with $\lim_{n\to\infty}\|f-f_n\|+\|A(f-f_n)\|=0$, the limit $\lim_{n\to\infty}V^n(\frac{t}{n})f_n=S(t)f$ exists uniformly for t in compact subsets of $[0,\infty)$, where S is the C_0-semigroup generated by A.

Proof: By the Lumer-Phillips theorem, condition (4.7) implies that $\|(\lambda-\omega)(\lambda-A_i)^{-1}\|\le 1$ for $\lambda>\omega$ for $i=1,2$. This is equivalent to $\|(I-tA_i)^{-1}\|\le(1-\omega t)^{-1}$ for $t\in[0,\frac{1}{\omega})$ and $i=1,2$. Since $(1-\omega t)^{-1}\le e^{2\omega t}$ for $t\in[0,\frac{1}{2\omega}]$, we obtain that

$$\|(I-\tfrac{t}{2}A_i)^{-1}\|\le e^{\omega t} \tag{4.8}$$

for $t\in[0,\frac{1}{\omega}]$ and $i=1,2$. For $t\in[0,\frac{1}{\omega}]$, let

$$Y(t):=(I-\tfrac{t}{2}A_2)^{-1}(I+\tfrac{t}{2}A_2)(I+\tfrac{t}{2}A_1)(I-\tfrac{t}{2}A_1)^{-1}.$$

By Lemma 4.1, the quasi-dissipativity condition (4.7) implies that

$$\|Y(t)\|\le e^{4\omega t} \tag{4.9}$$

for $t\in[0,\frac{1}{\omega}]$ and $i=1,2$.

Let $t\in[0,\frac{1}{\omega}]$. Then $D(A)\subset D(V(t)^\infty)$ and the map $t\mapsto V(t)f$ is continuous for $f\in D(A)$. For $f\in D(A)$ and $n\in\mathbf{N}$,

$$\begin{aligned}\|V(t)^{n+1}f-V(t)^nf\| &= t\|V^n(t)(I-\tfrac{t}{2}A_1)^{-1}(I-\tfrac{t}{2}A_2)^{-1}Af\|\\ &= t\|(I-\tfrac{t}{2}A_1)^{-1}Y(t)^n(I-\tfrac{t}{2}A_2)^{-1}Af\|.\end{aligned}$$

Then, by (4.8) and (4.9),

$$\begin{aligned}\|V(t)^{n+1}f-V(t)^nf\| &\le t\|Af\|\,e^{\omega t}(e^{4\omega t})^n e^{\omega t}\\ &\le t\|Af\|\,e^{6\omega nt}.\end{aligned}$$

Now for $f\in D(A^2)$ and $n\in\mathbf{N}$,

$$\begin{aligned}\|V(t)^{n+1}f-V(t)^nf-tV(t)^nAf\| &= t\|V^n(t)\left[(I-\tfrac{t}{2}A_1)^{-1}(I-\tfrac{t}{2}A_2)^{-1}-I\right]Af\|\\ &= t\|(I-\tfrac{t}{2}A_1)^{-1}Y(t)^n\left[(I-\tfrac{t}{2}A_2)^{-1}-(I-\tfrac{t}{2}A_1)\right]Af\|\\ &\le t\,e^{\omega t}(e^{4\omega t})^n\|\left[(I-\tfrac{t}{2}A_2)^{-1}-(I-\tfrac{t}{2}A_1)\right]Af\|\\ &\le t\,e^{6\omega nt}\|\left[(I-\tfrac{t}{2}A_2)^{-1}-(I-\tfrac{t}{2}A_1)\right]Af\|.\end{aligned}$$

Let $W(t)=(I-\frac{t}{2}A_2)^{-1}-(I-\frac{t}{2}A_1)$ for $t\in[0,\frac{1}{\omega}]$. Clearly, the map $t\mapsto W(t)f$ is continuous and $W(0)f=0$ for all $f\in D(A_1)$. Since $D(A)\subset D(A_1)\cap D(A_2)$, it follows that $Af\in D(A_1)$ for all $f\in D(A^2)$, and hence the map $t\mapsto W(t)Af$ is continuous and $W(0)Af=0$ for $f\in D(A^2)$. Then by Theorem 3.3, for any sequence $\{f_n\}_{n=1}^\infty\subset D(A)$ with $\lim_{n\to\infty}\|f-f_n\|+\|A(f-f_n)\|=0$, the limit $\lim_{n\to\infty}V\left(\frac{t}{n}\right)^n f_n=S(t)f$ exists uniformly for t in compact subsets of $[0,\infty)$. □

4.3 A Domain Decomposition Method

In this section, we use Theorem 3.3 to analyze the convergence of an operator splitting based domain decomposition method (see [40, 41]). A description of the method is given below.

We consider the homogeneous initial-boundary value problem

$$\begin{cases}\frac{\partial}{\partial t}u(t,x,y) &= Au(t,x,y) & t\ge 0, & (x,y)\in\Omega,\\ u(t,x,y) &= \quad 0 & t\ge 0, & (x,y)\in\partial\Omega,\\ u(0,x,y) &= f(x,y) & f\in D(A), & (x,y)\in\Omega\end{cases}\tag{4.10}$$

where A is a closed self-adjoint linear operator in the Hilbert space

$$L_0^2(\Omega)=\left\{f\in L^2(\Omega):f(x,y)=0\text{ for }(x,y)\in\partial\Omega\right\},$$

and Ω is a two dimensional rectangular domain. Suppose that A satisfy the quasi-dissipative condition

$$Re\,\langle Af,f\rangle<\omega\,\langle f,f\rangle\,.\tag{4.11}$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product. We define a bilinear form $\langle\cdot,\cdot\rangle_E$ from $D(A)\times D(A)\to L_0^2(\Omega)$ by

$$\langle f,g\rangle_E:=2\omega\,\langle f,g\rangle-\langle f,Ag\rangle\,.\tag{4.12}$$

Since A is self-adjoint and quasi-dissipative with the L^2 inner product, it is easy to verify that $\langle \cdot, \cdot \rangle_E$ is an inner product. Define a new space

$$E := \{f \in D(A) : \|f\|_E := \sqrt{\langle f, f \rangle_E} < \infty\}.$$

Using the closedness of A, it is verifiable that E is a Hilbert space.

The domain Ω is divided into p open subdomains Ω_1, Ω_2, $\cdot$, Ω_p (e.g. as in Figure 1), where these subdomains do not contain any part of the interface boundary and thus are disjoint. And the interface boundaries are defined as the set of points which are on the boundary of the subdomains and are in the interior of Ω. We denote the interface boundaries by B. Then the complement of the interface boundary is $B^c = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_p$, and then $\Omega = B^c \cup B$.

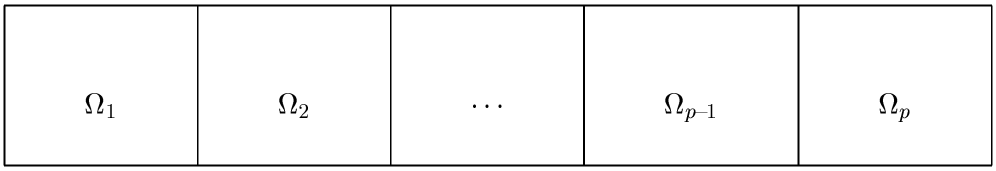

Figure 1

We discretize the domain Ω uniformly in each dimension with mesh size $h = \max(h_x, h_y)$, obtaining a discrete grid

$$\Omega_h = \{(x_i, y_j) : i = 0, 1, \cdots, m, \text{ and } j = 0, 1, \cdots, n\},$$

where x_{i-1} and x_i are neighboring coordinates in the x-dimension, and y_{j-1} and y_j are neighboring coordinates in the y-dimension with (x_0, y_j), (x_m, y_j), (x_i, y_0), (x_i, y_n) on the boundary $\partial\Omega$. We define a finite dimensional function space E_h on the discrete domain by

$$E_h = \left\{ f \in L^2(\Omega_h) : f(x, y) = 0 \text{ for } (x, y) \in \partial\Omega \cap \Omega_h \right\}.$$

Let A_h be the discrete spatial approximation of A. If A_h retains the self-adjointness and the quasi-dissipative condition

$$Re \langle A_h f, f \rangle < \omega \langle f, f \rangle, \tag{4.13}$$

with respect to the finite dimensional L^2 inner product

$$\langle f, g \rangle := \frac{l\, w}{mn} \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} f(x_i, y_j) g(x_i, y_j),$$

where l and w respectively are the length ($|x_0 - x_m|$) and width ($|y_0 - y_n|$) of the rectangular domain Ω, then E_h is a Hilbert space under the inner product

$$\langle f, g \rangle_h := 2\omega \langle f, g \rangle - \langle f, A_h g \rangle. \tag{4.14}$$

With both the decomposition and discretization of the domain Ω ready, we define the decomposition of the discrete domain Ω_h simply by inheriting the decomposition of the original domain:

$$\begin{cases} \Omega_{h,i} = \Omega_h \cap \Omega_i \quad \text{for } \ i = 1,\ 2,\ \cdots,\ p, \\ B_h = \Omega_h \cap B. \end{cases}$$

Then our domain decomposition algorithm for computing the solution u_h^{n+1} at the $(n{+}1)$-th time step from the current n-th time step is as follows.

1. Compute u_h^{n+1} at B_h using an explicit scheme. These computed data provide the interface boundary conditions.
2. Compute u_h^{n+1} on the subdomains B_h^c using any unconditional stable scheme with the interface boundary conditions computed at step 1.
3. Throw away the interface boundary condition computed at step 1, bring back u^n on B_h, and then implicitly recompute u_h^{n+1} on B_h.

There are several choices for the explicit operator in step 1, for the implicit subdomain solvers in step 2, and for the implicit replacement operator in step 3. For the convenience of discussion, we choose the explicit scheme in step 1, and the implicit scheme in both steps 2 and 3.

We first introduce some notations for the representation of our algorithm. Define two operators $A_{h,1} := \chi_{B_h} A_h$, $A_{h,2} := \chi_{B_h^c} A_h$, where χ_S is the characteristic matrix for a subset S of Ω_h defined by

$$(\chi_S\, u_h)\,(x) = \begin{cases} u_h(x), & x \in S, \\ 0, & x \notin S. \end{cases}$$

Thus χ_S is a diagonal matrix that has 1 at the positions corresponding to the grid points in the subset S and 0 everywhere else. The two matrices $A_{h,1}$ and $A_{h,2}$ are hence "restrictions" of A_h on B_h and B_h^c respectively. They form a splitting of matrix A_h, namely, $A_h = A_{h,1} + A_{h,2}$. It is easy to see that the matrices $A_{h,1}$ and $A_{h,2}$ have several rows whose entries are all zero. For the matrix $A_{h,1}$, the non-zero rows correspond to the grid points on the interface boundary, and the non-zero rows of the matrix $A_{h,2}$ correspond to the grid points on the subdomains. If we order the grid points in such a way that the grid points on the interface boundaries are listed before the grid points in the subdomains $\Omega_{B_h^c}$ as in the ordering $u_h = (u_{B_h}\ u_{B_h^c})^T$, then with any spatial approximation method, the matrix A_h has the form $A_h = \left(\begin{smallmatrix} D_1 & U_1 \\ L_2 & D_2 \end{smallmatrix}\right)$, where

$$A_{h,1} = \begin{pmatrix} D_1 & U_1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{h,2} = \begin{pmatrix} 0 & 0 \\ L_2 & D_2 \end{pmatrix}. \tag{4.15}$$

We first establish some properties about these two component matrices.

Lemma 4.2 *If the matrix A_h is symmetric and satisfies the quasi-dissipative condition (4.13) for each $h \in (0,1)$, then for all $h \in (0,1)$, the component matrices $A_{h,1}$ and $A_{h,2}$*

satisfy the quasi-dissipative condition $Re\,\langle A_{h,i}f,f\rangle_h \le \omega\,\langle f,f\rangle_h$, *with respect to the inner product given by (4.14).*

Proof: We prove only the quasi-dissipativity of $A_{h,1}$. The proof for $A_{h,2}$ is exactly the same and thus omitted.

Let $f\in E_h$. By definition,

$$\langle A_{h,1}f,f\rangle_h = 2\omega\,\langle A_{h,1}f,f\rangle - \langle A_{h,1}f,A_hf\rangle\,. \tag{4.16}$$

Since $A_{h,1}=\chi_{B_h}\,A_h$, we have that

$$\langle A_{h,1}f,A_hf\rangle = \langle \chi_{B_h}\,A_hf,A_hf\rangle = \langle \chi_{B_h}\,A_hf,\chi_{B_h}\,A_hf\rangle = \langle A_{h,1}f,A_{h,1}f\rangle\,.$$

Then from (4.16), we obtain that

$$\begin{array}{rcl} Re\,\langle A_{h,1}f,f\rangle_h &=& 2\omega Re\,\langle A_{h,1}f,f\rangle - \langle A_{h,1}f,A_{h,1}f\rangle \\ &=& \omega(\langle A_{h,1}f,f\rangle + \langle f,A_{h,1}f\rangle) - \langle A_{h,1}f,A_{h,1}f\rangle \\ &=& \omega^2\,\langle f,f\rangle - \langle(\omega-A_{h,1})f,(\omega-A_{h,1})f\rangle \\ &\le& \omega^2\,\langle f,f\rangle\,. \end{array} \tag{4.17}$$

The quasi-dissipative condition (4.13) is obviously equivalent to $0<\omega\,\langle f,f\rangle - \langle A_hf,f\rangle$, which implies that $\omega\,\langle f,f\rangle < 2\omega\,\langle f,f\rangle - \langle A_hf,f\rangle$. But the right hand side is exactly $\langle f,f\rangle_h$. Then from (4.17), we have that $Re\,\langle A_{h,1}f,f\rangle_h < \omega\,\langle f,f\rangle_h$. □

Lemma 4.3 *Suppose A_h satisfies the quasi-dissipative condition (4.13) for all $h\in(0,1)$. Then for all $t\in[0,\frac{1}{8\omega}]$, $h\in(0,1)$ and $f\in E_h$,*

$$\|\,\chi_{B_h}(I-tA_{h,1})^{-1}f\|_h^2 \le e^{4\omega t}\|\,\chi_{B_h}f\|_h^2 + (e^{4\omega t}-1)\|\,\chi_{B_h^c}f\|_h^2. \tag{4.18}$$

$$\|\,\chi_{B_h^c}(I-tA_{h,2})^{-1}f\|_h^2 \le e^{4\omega t}\|\,\chi_{B_h^c}f\|_h^2 + (e^{4\omega t}-1)\|\,\chi_{B_h}f\|_h^2. \tag{4.19}$$

Proof: We will prove only (4.18). The proof for (4.19) is the same and thus omitted.

Inequality (4.18) is obviously equivalent to

$$\|\,\chi_{B_h}\,g\|_h^2 \le e^{4\omega t}\|\,\chi_{B_h}(I-tA_{h,1})g\|_h^2 + (e^{4\omega t}-1)\|\,\chi_{B_h^c}(I-tA_{h,1})g\|_h^2 \tag{4.20}$$

for all $g\in E_h$. Since $\chi_{B_h^c}A_{h,1}=0$ and $\chi_{B_h}A_{h,1}=A_{h,1}$, the above inequality is equivalent to

$$\|\,\chi_{B_h}\,g\|_h^2 \le e^{4\omega t}\|\,\chi_{B_h}g - tA_{h,1}g\|_h^2 + (e^{4\omega t}-1)\|\,\chi_{B_h^c}\,g\|_h^2. \tag{4.21}$$

The right hand side of the above inequality is equal to

$$e^{4\omega t}\left[\|\,\chi_{B_h}g\|_h^2 - 2t\,Re\,\langle A_{h,1}g,g\rangle_h + t^2\|A_{h,1}g\|_h^2\right] + (e^{4\omega t}-1)\|\,\chi_{B_h^c}g\|_h^2.$$

Thus inequality (4.21) is equivalent to

$$0\le (e^{4\omega t}-1)\left[\|\,\chi_{B_h}g\|_h^2 + \|\,\chi_{B_h^c}g\|_h^2\right] + e^{4\omega t}\left[t^2\|A_{h,1}g\|_h^2 - 2t\,Re\,\langle A_{h,1}g,g\rangle_h\right].$$

But $\| \chi_{B_h} g\|_h^2 + \| \chi_{B_h^c} g\|_h^2 = \|g\|_h^2$, the inequality above is again equivalent to

$$2t\, e^{4\omega t} \left\langle A_{h,1} g, g\right\rangle_h \le (e^{4\omega t} - 1)\|g\|_h^2 + e^{4\omega t} t^2 \|A_{h,1} g\|_h^2.$$

Since $e^{4\omega t} t^2 \|A_{h,1} g\|_h^2 \ge 0$, the above inequality holds if

$$e^{4\omega t} Re \left\langle A_{h,1} g, g\right\rangle_h \;\le\; \frac{e^{4\omega t} - 1}{2t} \|g\|_h^2. \tag{4.22}$$

So to show (4.18), it suffices to show that the inequality (4.22) holds for all $g \in E_h$ and $t \in [0, \frac{1}{2\omega_0}]$.

Lemma 4.2 states that for all $g \in E_h$, $Re \left\langle A_{h,1} g, g\right\rangle_h \le \omega \left\langle g, g\right\rangle_h$, which, by multipying both sides by $e^{4\omega t}$, is equivalent to

$$e^{4\omega t} Re \left\langle A_{h,1} g, g\right\rangle_h \le e^{4\omega t} \omega \left\langle g, g\right\rangle_h \tag{4.23}$$

for all $t \ge 0$. Since $e^{4\omega t} < 2$ for $t \in [0, \frac{1}{8\omega}]$, it follows from (4.23) that $e^{4\omega t} Re \left\langle A_{h,1} g, g\right\rangle_h < 2\omega \left\langle g, g\right\rangle_h$ for $t \in [0, \frac{1}{8\omega}]$. But $2\omega < \frac{e^{4\omega t} - 1}{2t}$, then (4.22) follows immediately from the above inequality. $\square$

With the domain decomposition based matrix splitting given, the domain decomposition method is representable by (see [41] for its derivation)

$$u_h^{n+1} = (I - \Delta t A_{h,1})^{-1} \left[\chi_{B_h} + \chi_{B_h^c} (I - \Delta t A_{h,2})^{-1} (I + \Delta t A_{h,1})\right] u_h^n. \tag{4.24}$$

Theorem 4.2 *Let $V(t,h)$ denote the error amplification matrix of method (4.24) for $(t,h) \in [0,1] \times (0,1)$, namely,*

$$V(t,h) = (I - t A_{h,1})^{-1} \left[\chi_{B_h} + \chi_{B_h^c} (I - t A_{h,2})^{-1} (I + t A_{h,1})\right].$$

Suppose that for each $h \in (0,1)$, A_h is self adjoint with respect to the finite dimensional L^2 space inner product and satisfies the quasi-dissipative condition (4.13) with $\omega = 0$.[1] *For each $h \in (0,1)$, let E_h be the Hilbert space with inner product (4.14). If for each $h \in (0,1)$, there exists a projection operators $P_h : E \to E_h$ such that for each $f \in D(A)$,*

$$\begin{cases} \sup_{h \in (0,1)} \|P_h f\|_h \le M \|f\|_E, \\ \sup_{h \in (0,1)} \|A_h P_h f\|_h \le M(f) \quad \text{for some } M(f) > 0, \end{cases} \tag{4.25}$$

and

$$\lim_{h \to 0} \|A_h P_h f - P_h A f\|_h = 0. \tag{4.26}$$

then the limit

$$\lim_{\substack{h \to 0 \\ n \to \infty}} \|V(\tfrac{t}{n}, h)^n P_h f - P_h S(t) f\|_h = 0$$

holds uniformly for t in compact intervals, where $S(t)$ is the semigroup generated by A.

[1] The case of $\omega > 0$ can be proven similarly but with more messy calculations. The two lemmas Lemma 4.2 and Lemma 4.3 are sufficient for establishing estimates for the proof of the case $\omega > 0$.

To prove this theorm, we need the following estimates.

Lemma 4.4 *The following inequalities hold for* $t\in[0,\frac{1}{2\omega_0}]$, $h\in(0,\varepsilon)$ *and* $f\in E_h$.

(i) $\|V(t,h)^n f\|_h \;\leq\; \|(I-tA_{h,1})f\|_h$ *for* $n\in\mathbf{N}$*;*

(i i) $\|V(t,h)^{n+1}f - V(t,h)^n f\|_h \;\leq\; t\|A_h f\|_h$ *for* $n\in\mathbf{N}$*;*

(iii) $\|V(t,h)^{n+1}f - V(t,h)^n f - tV(t,h)^n A_h f\|_h \leq t^2\|A_h^2 f\|_h$ *for* $n\in\mathbf{N}$.

Proof: (i) Let $\widetilde{V}(t,h) = (I-tA_{h,1})\; V(t)\; (I-tA_{h,1})^{-1}$, namely,

$$\widetilde{V} = \chi_{B_h}\,(I-tA_{h,1})^{-1} + \chi_{B_h^c}\,(I-tA_{h,2})^{-1}(I+tA_{h,1})(I-tA_{h,1})^{-1}. \tag{4.27}$$

Then $\|\widetilde{V}f\|_h^2 = \|\chi_{B_h}(I-tA_{h,1})^{-1}f\|_h^2 + \left\|\chi_{B_h^c}(I-tA_{h,2})^{-1}(I+tA_{h,1})(I-tA_{h,1})^{-1}f\right\|_h^2$. It follows from (4.18) and (4.19) that

$$\|\widetilde{V}f\|_h^2 \;\leq\; \|\chi_{B_h} f\|_h^2 + \left\|\chi_{B_h^c}\,(I+tA_{h,1})(I-tA_{h,1})^{-1}f\right\|_h^2. \tag{4.28}$$

A straightforward calculations using the matrix form (4.15) of $A_{h,1}$ and $A_{h,2}$ reveals that

$$\begin{cases} \chi_{B_h^c}\,(I+tA_{h,1})f = \chi_{B_h^c}\,f, \\ \chi_{B_h^c}\,(I-tA_{h,1})^{-1}f = \chi_{B_h^c}\,f, \\ \chi_{B_h^c}\,(I+tA_{h,1})(I-tA_{h,1})^{-1}f = \chi_{B_h^c}\,f. \end{cases} \tag{4.29}$$

for all $f\in E_h$. Then it follows from (4.28) that

$$\begin{aligned} \|\widetilde{V}f\|_h^2 &\leq \|\chi_{B_h} f\|_h^2 + \left\|\chi_{B_h^c} f\right\|_h^2 \\ &= \|f\|_h^2. \end{aligned} \tag{4.30}$$

Now for $f\in E_h$,

$$\begin{aligned} \|V(t,h)^n f\|_h &= \|(I-tA_{h,1})^{-1}\,\widetilde{V}^n\,(I-tA_{h,1})f\|_h \\ &\leq \|(I-tA_{h,1})^{-1}\|\cdot\|\widetilde{V}\|^n\cdot\|(I-tA_{h,1})f\|_h \\ &\leq \|(I-tA_{h,1})^{-1}\|\cdot\|(I-tA_{h,1})f\|_h, \end{aligned} \tag{4.31}$$

where the last inequality is due to (4.30). By Lemma 4.2, $A_{h,1}$ is dissipative since $\omega=0$. Hence $\|(I-tA_{h,1})^{-1}\|_h \leq 1$. Then from (4.31) we obtain

$$\|V(t,h)^n f\|_h \;\leq\; \|(I-tA_{h,1})f\|_h.$$

(ii) For $f\in E_h$,

$$\begin{aligned} \|V(t,h)^{n+1}f - V(t,h)^n f\|_h &= \|V(t,h)^n[V(t,h)f - f]\|_h \\ &= \|(I-tA_{h,1})^{-1}\,\widetilde{V}^n\,(I-tA_{h,1})[V(t,h)f-f]\|_h \\ &\leq \|(I-tA_{h,1})[V(t,h)f-f]\|_h, \end{aligned} \tag{4.32}$$

where the last inequality is due to the (4.30) and the inequality $\|(I-tA_{h,1})^{-1}\|_h \leq 1$ implied by the dissipativity of $A_{h,1}$. And

$$\begin{aligned}(I-tA_{h,1})[V(t,h)f-f] &= (I-tA_{h,1})V(t,h)f-(I-tA_{h,1})f \\ &= [\chi_{B_h}+\chi_{B_h^c}(I-tA_{h,2})^{-1}(I+tA_{h,1})]f-(I-tA_{h,1})f \\ &= \chi_{B_h^c}\left[(I-tA_{h,2})^{-1}(I+tA_{h,1})-I\right]f+tA_{h,1}f \\ &= \chi_{B_h^c}(I-tA_{h,2})^{-1}tA_hf+tA_{h,1}f \\ &= [\chi_{B_h^c}(I-tA_{h,2})^{-1}+\chi_{B_h}]tA_hf.\end{aligned} \tag{4.33}$$

Then it follows from (4.32) that

$$\begin{aligned}\|V(t,h)^{n+1}f-V(t,h)^nf\|_h^2 &\leq \|\chi_{B_h^c}(I-tA_{h,2})^{-1}tA_hf\|_h^2+\|\chi_{B_h}\,tA_hf\|_h^2 \\ &\leq \|\chi_{B_h^c}\,tA_hf\|_h^2+\|\chi_{B_h}\,tA_hf\|_h^2 \\ &= \|tA_hf\|_h^2,\end{aligned}$$

where the second inequality is due to (4.19).

(iii) For $f\in E_h$,

$$\begin{aligned}&\|V(t,h)^{n+1}f-V(t,h)^nf-tV(t,h)^nA_hf\|_h \\ &= \|V(t,h)^n[V(t,h)f-f-tA_hf]\|_h \\ &= \|(I-tA_{h,1})^{-1}\widetilde{V}^n(I-tA_{h,1})[V(t,h)f-f-tA_hf]\|_h \\ &\leq \|(I-tA_{h,1})[V(t,h)f-f-tA_hf]\|_h.\end{aligned} \tag{4.34}$$

And from (4.33) we have that

$$\begin{aligned}(I-tA_{h,1})[V(t,h)f-f-tA_hf] &= [\chi_{B_h^c}(I-tA_{h,2})^{-1}+\chi_{B_h}]\,tA_hf-(I-tA_{h,1})tA_hf \\ &= [\chi_{B_h^c}(I-tA_{h,2})^{-1}-\chi_{B_h^c}+tA_{h,1}]tA_hf \\ &= [\chi_{B_h^c}\{(I-tA_{h,2})^{-1}-I\}+tA_{h,1}]tA_hf \\ &= [\chi_{B_h^c}(I-tA_{h,2})^{-1}tA_{h,2}+tA_{h,1}]tA_hf \\ &= [\chi_{B_h^c}(I-tA_{h,2})^{-1}\,\chi_{B_h^c}+\chi_{B_h}]t^2A_h^2f,\end{aligned}$$

which, together with (4.34), implies that

$$\begin{aligned}&\|V(t,h)^{n+1}f-V(t,h)^nf-tV(t,h)^nA_hf\|_h^2 \\ &\leq \|\chi_{B_h^c}(I-tA_{h,2})^{-1}t^2A_{h,2}A_hf\|_h^2+\|\chi_{B_h}t^2A_h^2f\|_h^2 \\ &\leq \|\chi_{B_h^c}\,t^2A_{h,2}A_hf\|_h^2+\|\chi_{B_h}t^2A_h^2f\|_h^2 \\ &= \|\chi_{B_h^c}\,t^2A_h^2f\|_h^2+\|\chi_{B_h}t^2A_h^2f\|_h^2 \\ &= \|t^2A_h^2f\|_h^2,\end{aligned}$$

where the second inequality is due to (4.19). □

Proof of Theorem 4.2: Define a product space

$$\mathbf{E}=\left\{\{f_h\}:h\in(0,\delta),f_h\in E_h,\ \|\{f_h\}\|=\sup_{h\in(0,1)}\|f_h\|_h<\infty\right\},$$

and define an operator $\mathcal{A}$ on $\mathbf{E}$ by $\mathcal{A}\{f_h\} = \{A_h f_h\}$. Since $D(A_h) = E_h$, the domain of $\mathcal{A}$ is obviously $D(\mathcal{A}) = \Big\{\{f_h\}\in\mathbf{E} : \sup_{h\in(0,1)} \|A_h f_h\|_h < \infty\Big\}$. It is easy to verify that $\mathcal{A}$ is closed. And since $A_h = A_{h,1} + A_{h,2}$ and $A_{h,1}, A_{h,2}$ are dissipative, so is the operator $\mathcal{A}$. Then $\mathcal{A}$ generates a C_0-semigroup $\mathcal{S}$. Again define two operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on $D(\mathcal{A})$ by

$$\mathcal{A}_1\{f_h\} := \{A_{h,1}f_h\}, \qquad\qquad \mathcal{A}_2\{f_h\} := \{A_{h,2}f_h\}.$$

Then similarly we can show that $\mathcal{A}_1$ and $\mathcal{A}_2$ generate C_0-semigroups $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively. It is easy to verify that for $t > 0$,

$$\begin{cases} (I - t\mathcal{A}_1)^{-1} = \{(I - tA_{h,1})^{-1}\}, \\ (I - t\mathcal{A}_2)^{-1} = \{(I - tA_{h,2})^{-1}\}. \end{cases} \tag{4.35}$$

Now, we define a temporal method to approximate the semigroup $\mathcal{S}$. For $t\in[0,1]$, define $W(t)$ by $W(t)\{f_h\} = \{V(t,h)f_h\}$ for $t\in[0,1]$. It is easily verifiable that for each $t\in[0,1]$, $W(t)$ is linear and

$$W(t)f = (I - t\mathcal{A}_1)^{-1}[W_1 + W_2(I - t\mathcal{A}_2)^{-1}U(t)]f \tag{4.36}$$

for $f\in D(\mathcal{A})$, where

$$\begin{cases} W_1 &= \{\chi_{B_h}\}, \\ W_2 &= \{\chi_{B_h^c}\}, \\ U(t) &= \{(I+tA_{h,1})\}. \end{cases} \tag{4.37}$$

To see that $D(\mathcal{A}) \subset D(W(t)^\infty)$ for all $t\in[0,1]$, let $f = \{f_h\}\in D(\mathcal{A})$. Then by statement (i) of Lemma 4.4, for any $k\in\mathbf{N}$ and $t\in[0,1]$,

$$\begin{aligned} \|V(t,h)^k f_h\|_h &\le \|(I - tA_{h,1})f_h\|_h \\ &\le \|f_h\|_h + t\|\chi_{B_h} A_h f_h\|_h \\ &\le \|f_h\|_h + t\|A_h f_h\|_h. \end{aligned}$$

Since $\{f_h\}\in D(\mathcal{A})$, we have that $\sup_{h\in(0,1)}\{\|f_h\|_h, \|A_h f_h\|_h\} < \infty$, so

$$\begin{aligned} \|\!|W(t)^k f\|\!| &= \sup_{h\in(0,1)} \|V(t,h)^k f_h\|_h \\ &\le \sup_{h\in(0,1)}\{\|f_h\|_h + t\|A_h f_h\|_h\} \\ &< \infty, \end{aligned}$$

which means that $f\in D(W(t)^\infty)$.

Next, we shall show that for each $f\in D(\mathcal{A})$, $W(t)f$ is continuous in t. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are dissipative, it follows that for $f\in\mathbf{E}$, the maps $t\mapsto (I-t\mathcal{A}_1)^{-1}f$ and $t\mapsto (I-t\mathcal{A}_2)^{-1}f$ are continuous. Since W_1 and W_2 are bounded operators on $\mathbf{E}$, to show that $W(t)f$ is continuous in t for $f\in D(\mathcal{A})$, it suffices to show that $t\mapsto U(t)f$ is continuous for $f\in D(\mathcal{A})$. Now let $f = \{f_h\} \in D(\mathcal{A})$, and let $t,t' \in [0,1]$. Then, $\|\!|U(t)f - U(t')f\|\!| = \|\!|\{(t-t')A_{h,1}f_h\}\|\!| \le (t-t')\|\!|\mathcal{A}f\|\!|$, from which it follows that $t\mapsto U(t)f$ is continuous for each $f\in D(\mathcal{A})$.

For $f \in D(\mathcal{A})$, by definition of $\|\cdot\|$ we have that

$$\begin{aligned}
\|W(t)^{n+1}f - W^n(t)f\| &= \sup_{h\in(0,1)} \|V(t,h)^{n+1}f_h - V(t,h)^n f_h\|_h \\
&\leq \sup_{h\in(0,1)} \|t\, A_h f_h\|_h \\
&= t\|\mathcal{A}f\|,
\end{aligned}$$

where the inequality above is due to Lemma 4.4. And for $f \in D(\mathcal{A}^2)$, we can also obtain the following immediately from Lemma 4.4.

$$\|W(t)^{n+1}f - W(t)^n f - tW(t)^n \mathcal{A}f\| \leq t^2\|\mathcal{A}^2 f\|.$$

Then it follows from Theorem 3.2 that for $f \in D(\mathcal{A})$,

$$\lim_{n\to\infty} W(\tfrac{t}{n})^n f = \mathcal{S}(t)f \tag{4.38}$$

uniformly in compact intervals. Let $f \in D(A)$. Then (4.25) implies that $\{P_h f\} \in D(\mathcal{A})$. It is easy to see that $\mathcal{S}(t) = \{e^{tA_h}\}$. Then we obtain from (4.38) that

$$\lim_{n\to\infty} \sup_{h\in(0,1)} \|V(t/n,h)^n P_h f - e^{tA_h}P_h f\|_h = 0 \tag{4.39}$$

uniformly for t in compact intervals for each $f \in D(A)$. Since A_h is dissipative for each $h \in (0,1)$, the semigroups $\{e^{tA_h}\}_{h\in(0,1)}$ are uniformly norm-bounded by 1. Since $e^{tA_h} = (e^{\frac{t}{n}A_h})^n$, the assumption (4.26) and the spatially discrete version of the Chernoff's product formula (see [28], pp 96) imply that for $f \in D(A)$,

$$\lim_{h\to 0} \|e^{tA_h}P_h f - P_h S(t)f\|_h = 0 \tag{4.40}$$

uniformly for t in compact intervals. Then, with (4.39) and (4.40), the conclusion of this theorem follows from the inequality

$$\|V(\tfrac{t}{n},h)^n P_h f - S(t)f\|_h \leq \sup_{h\in(0,1)} \|V(\tfrac{t}{n},h)^n P_h f - e^{tA_h}P_h f\|_h + \|e^{tA_h}P_h f - S(t)f\|_h.$$

□

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Vita

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