Mathematics Comprehensive Examination
Algebra
August, 2000

Directions: Do problem 1, do either problem 2 or problem 3, and do either problem 4 or problem 5. For your fourth problem, do any one of the three problems remaining after your required choices. Please start each problem on a new sheet of paper, with your name and the problem number written at the top of every sheet. Hand in only the four problems you wish to have graded. Naturally, all answers require proof. However, it is perfectly appropriate to quote the standard theorems which you have learned. Good luck!

1. Determine the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and an invertible matrix P such that $P^{-1}AP$ is in Jordan canonical form.
2. Determine, with proof, if the groups $G = \mathbb{Z}_5 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{36} \oplus \mathbb{Z}_{54}$ and $H = \mathbb{Z}_{50} \oplus \mathbb{Z}_{108} \oplus \mathbb{Z}_{450}$ are isomorphic.
3. Let $\mathbb{F}$ be a finite field of order q and let $n \in \mathbb{N}$. As usual $\mathrm{GL}_n(\mathbb{F})$ denotes the group of invertible $n \times n$ matrices with entries in $\mathbb{F}$, and $\mathrm{SL}_n(\mathbb{F})$ denotes the subgroup of $\mathrm{GL}_n(\mathbb{F})$ consisting of matrices of determinant 1.
   (a) Show that $\mathrm{SL}_n(\mathbb{F})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{F})$.
   (b) Show that $[\mathrm{GL}_n(\mathbb{F}) : \mathrm{SL}_n(\mathbb{F})] = q - 1$.
4. Determine all of the ideals in the ring $\mathbb{Z}[X]/\langle 2, X^3 + 1 \rangle$.
5. Let $A = \begin{bmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{bmatrix}$.
   (a) Determine invertible integer matrices P^{-1}, Q which diagonalize A; that is, $QAP^{-1} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \end{bmatrix}$ where $d_1 | d_2$.
   (b) Using your answer to part (a), find all integer solutions to the matrix equation $AX = 0$.
6. Let V be a finite dimensional vector space and let $T : V \to V$ be a linear transformation such that $T^2 = T$.
   (a) Prove that $\mathrm{Im}(T) \cap \mathrm{Ker}(T) = \{0\}$.
   (b) Prove that $V = \mathrm{Im}(T) \oplus \mathrm{Ker}(T)$.
   (c) Prove that there is a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.